Realization and Connectivity of the Graphs of Origami Flat Foldings

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Abstract. We investigate the graphs formed from the vertices and creases of an origami pattern that can be folded flat along all of its creases. As we show, this is possible for a tree if and only if the internal vertices of the tree all have even degree greater than two. However, we prove that (for unbounded sheets of paper, with a vertex at infinity representing a shared endpoint of all creased rays) the graph of a folding pattern must be 2-vertex-connected and 4-edge-connected.

1 Introduction

This work concerns the following question: Which graphs can be drawn as the graphs of origami flat folding patterns?

In origami and other forms of paper folding, a *flat folding* is a type of construction in which an initially-flat piece of paper is folded so that the resulting folded shape lies flat in a plane and has a desired shape or visible pattern. This style of folding may be used as the initial base from which a three-dimensional origami figure is modeled, or it may be an end on its own. Flat foldings have been extensively studied in research on the mathematics of paper folding. The folding patterns that can fold flat with only a single vertex have been completely characterized, for standard models of origami [1–8], for *rigid origami* in which the paper must continuously move from its unfolded state to its folded state without bending anywhere except at its given creases [9], and even for single-vertex folding patterns whose paper does not form a single flat sheet [10]. However, the combinatorics of multi-vertex flat folding patterns is much less well understood, and testing whether a multi-vertex pattern folds flat is NP-hard [11].

From the point of view of graph drawing, origami folding patterns can be thought of as planar graphs, drawn with straight line edges in the Euclidean plane, with each edge representing a crease that must be folded. For instance, the familiar bird base, a starting point for the classic three-dimensional origami crane, can be thought of as a graph drawing of a planar graph with 13 vertices (Fig. 1). This naturally raises the question (analogous to similar questions for other types of geometric graphs such as Voronoi diagrams [12]): which graphs can be drawn this way? The NP-completeness of recognizing multi-vertex flat folding patterns does not extend to this question, because the completeness result is for folding patterns that have already been embedded with a given geometry

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and its proof depends on the specific geometry of the embedding. Here, instead, we ask whether an embedding exists. We do not resolve this question, but we provide partial answers to it in two different directions.

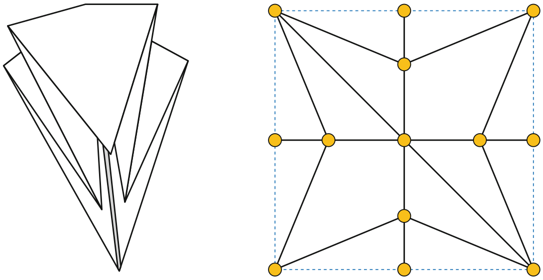

Fig. 1. Origami bird base (as illustrated by Fred the Oyster at https://commons.wikimedia.org/wiki/File:Bird_base.svg) and the corresponding folding pattern, interpreted as a graph drawing. The black lines indicate the final creases of the bird base. Temporary creases made while folding the base but later flattened out are not included. Blue dashed lines indicate the boundary of the sheet of paper; these lines are not considered as edges of the graph because they are not creased. (Color figure online)

First, we investigate the trees that may be drawn as flat folding patterns. For this problem, we make the simplifying assumption that the sheet of paper to be folded is infinite, with internal vertices of the tree at points where multiple creases come together, and with the leaves of the tree corresponding to creases along infinite rays. Cutting the infinite paper of such a drawing along a square that surrounds all the internal vertices would produce a finite representation of the same tree with its leaves on the boundary of the square, like the representation of a non-tree graph in Fig. 1. Similar tree-drawing styles, with infinite rays for the leaves of the trees, have been used in past work on drawings of trees as Voronoi diagrams [12], straight skeletons [13],[1] or with optimized angular resolution [15]. For this model of origami folding and tree realization, we provide a complete characterization: a tree may be drawn in this way if and only if all of its internal vertices have even degree greater than two.

Second, we investigate the connectivity restrictions on the graphs that may be drawn as flat folding patterns. This type of constraint has proven very fruitful in past questions about the geometric realizations of planar graphs, providing complete characterizations of the graphs of convex polyhedra (Steinitz's theorem) [16], drawings with rectangular faces ("rectangular duals") [17–20], orthogonal polyhedra [21], and two-dimensional soap bubble clusters [22].

Trees are not highly connected, and may be drawn as flat folding diagrams, but it turns out that these diagrams remain highly connected through the bound-

[1] Straight skeletons have also been used to construct folding patterns [14]. However, this technique adds extra folds to the skeleton, so the realizations of trees as straight skeletons do not yield realizations of the same trees as flat folding patterns.

ary of the drawing. To capture this boundary connectivity, we modify our mathematical model of flat folding. We again assume an infinite sheet of paper, but we treat creases along infinite rays as all having a single shared endpoint at infinity, which forms another vertex of the graph. In this model, the tree foldings of the other model become series-parallel graphs, in which all the leaves of the tree have been merged into a single supervertex.

We prove that, for this model of graphs as folding patterns, the graphs that may be realized are highly restricted, beyond even the graphs of polyhedra and beyond the immediate restriction (from the one-vertex case) that all vertices have even degree. In particular, they are necessarily 2-vertex-connected and 4-edge-connected. More strongly, the vertex at infinity is not an articulation vertex, and any subset of vertices that separates the graph and does not include the vertex at infinity must include at least four other vertices. These connectivity restrictions hold even for a weaker model of *local flat foldability* in which we seek a piecewise linear map from the folding pattern to its folded state in the plane without regard to whether this folding can be embedded without self-intersections into three-dimensional space. Our realizations of trees as flat folding patterns show that the 2-vertex-connectivity and 4-edge-connectivity conditions are both tight: no higher restriction on connectivity is possible.

2 Preliminaries

2.1 Mathematical Model of Folding

Departing from the usual square-paper model of origami in order to avoid complications from its boundary conditions, we model the sheet of paper to be folded as the entire Euclidean plane. We first define a *local flat folding*. This is a highly simplified model of how a piece of paper might be folded that only takes into account local constraints (the paper can only be folded, not stretched, sheared, or crumpled), does not prevent self-intersections, and does not even represent the most basic information about how the folding might occur in three dimensions, such as whether a given fold is a mountain fold or a valley fold.

Definition 1. *We define a continuous function φ from the plane to itself to be a local flat folding if every point p of the plane has one of the following three types:*

- *An* unfolded point *of a local flat folding is a point p such that φ is a* local isometry*: there is a neighborhood of p that is mapped by φ in a distance-preserving way (necessarily a combination of translation, rotation, or reflection of the plane).*
- *A* crease point *of a local flat folding is a point p that has a neighborhood N that can be covered by two subsets, each containing p and each mapped by φ in a different distance-preserving way. Necessarily, the boundary between these two subsets must be a line containing p. To preserve continuity of the mapping, the two distinct isometric mappings for the two subsets must be*

reflections of each other across the image of this line. The points within N that belong to this fold line are also crease points, and the other points within N are unfolded points.

– *A* vertex point *of a local flat folding is a point p that has a neighborhood N that can be covered by finitely many (and at least three) subsets, each containing p and each mapped by φ in a distance-preserving way so that there are at least three distinct isometric mappings among these subsets. Necessarily, each subset must be a wedge. The points within N that belong to the rays between pairs of wedges are crease points, and the points within N that do not belong to these rays are unfolded points.*

Then, as stated above, a local flat folding is a continuous function ϕ such that all points of the plane are unfolded points, crease points, and vertex points. We add one more restriction: we consider only local flat foldings that have at least one vertex point. We do not require the number of vertex points to be finite.

As a simple example, consider the function $\varphi : (x, y) \mapsto (f(x), f(y))$ where $f(x) = |(x \bmod 2) - 1|$. Here f is a continuous function that maps the intervals $[2i, 2i + 1]$ to $[0, 1]$ in reverse order, and that maps the intervals $[2i + 1, 2i + 2]$ to $[0, 1]$ linearly. φ corresponds to a folding pattern in which we *pleat* the plane along the integer-coordinate vertical lines (that is, we create a sequence of folds that alternates between mountain and valley folds, like an accordion; see [23, p. 31]), and then we pleat it again along the integer-coordinate horizontal lines, so that the whole plane is mapped to the unit square. Its folding pattern has vertex points at points of the plane where both coordinates are integers, crease points at points with one integer coordinate, and unfolded points everywhere else. That is, it is a drawing of the infinite square grid graph.

In general, the graph of a local flat folding is almost a graph drawing, in that its vertex points form a discrete set, connected in pairs by line segments consisting of crease points. For the grid example, it is a graph drawing. However, for other local flat foldings, some of the crease points may belong to semi-infinite rays rather than forming bounded line segments. To make a graph that also includes these rays as edges, we add a special vertex ∞ that is not represented by any geometric point, and we treat this special vertex as an endpoint of each ray of crease points.

Definition 2. *We define the* graph of a local flat folding *φ to be a graph G that has a vertex for each vertex point of φ and (if φ includes any infinite rays of crease points) another special vertex ∞. Two vertex points form adjacent vertices in G when the line segment between them consists only of crease points. A vertex point p and the special vertex ∞ are adjacent when there exists a ray with apex p consisting only (other than at its apex) of crease points. This graph may have multiple adjacencies between ∞ and other vertices (for instance, it will do so in any one-vertex flat folding pattern) but it can have at most one edge between any two vertex points.*

The folding pattern provides a topological planar embedding for the whole graph G, and a geometric straight-line planar embedding for all vertices except ∞. As usual, we call the maximal regions of the plane that are disjoint from the vertices and edges of the embedding (the vertex and crease points of φ) the *faces* of the embedding. These are possibly-unbounded polygonal regions, the connected components of the unfolded points of φ. Because the action of φ on each face of the graph is determined from its action on adjacent faces, the embedding of G completely determines the mapping of φ, up to a congruence transformation of the whole plane.

For our realizations of trees, we will use a slightly different graph, that can be derived from the graph of the folding. (It will not be interesting to study the graph connectivity of this graph, because it will have many degree-one vertices.)

Definition 3. *We define the* truncated graph of a local flat folding *to be the graph formed in either of the following two equivalent ways:*

- *From the graph of the folding, subdivide each edge incident to ∞, and then delete vertex ∞.*
- *Form a graph with a vertex for each vertex point of the folding and another vertex for each ray of crease points of the folding. Connect two vertex points by an edge if the line segment between them consists only of crease points. Add an edge for each ray of crease points, connecting the vertex point at the apex of the ray to the additional vertex for the same ray.*

Truncated graphs of local flat foldings can also be interpreted as the type of graph drawn in Fig. 1 for a folding pattern on a sheet of square paper with the additional property that the creases reaching the boundary form diverging rays. However, the folding pattern in Fig. 1 has creases that instead meet at the boundary, and it is also possible to form converging pairs of rays. Therefore the type of graph shown in the figure, of a folding pattern on a bounded square of paper, is somewhat more general. However, for the purposes for which we use truncated graphs (realization of trees), a less general model is better, as any realization in such a model will also be a realization for the more general model.

It remains to define a mathematical model of foldings as global structures, accounting for how paper can fold in three dimensions and how some parts of the paper can block other parts of paper from passing through them (disallowing self-intersections). It is possible to model precisely the above-below relation of the faces of φ, and the nesting structure of the folding at the creases of φ; see, for instance, [10] for a similar model of lower-dimensional flat-folded structures. However, we will forgo the complexity of such a model in favor of the following simpler topological approach.

Definition 4. *A* global flat folding *is a local flat folding φ with the additional property that, for every $\epsilon > 0$, there exists a topological embedding $\varphi_\epsilon : \mathbb{R}^2 \to \mathbb{R}^3$ (without self-intersections) such that composing φ_ϵ with the coordinatewise vertical projection from $\mathbb{R}^3$ to $\mathbb{R}^2$ produces a mapping that, for every point p, is within distance ϵ of the mapping given by φ.*

Intuitively, a global flat folding is a local flat-folding that, for every $\epsilon > 0$, is ϵ-close to a topological embedding of the plane into three-dimensional space.

2.2 Single-Vertex Restrictions

The geometry of single-vertex folding patterns, such as the one in Fig. 2, is characterized by Maekawa's theorem and Kawasaki's theorem [1–8]. These apply as well to each vertex of a multi-vertex folding pattern.

Theorem 1 (Maekawa's theorem for one-vertex folding patterns without mountain-valley assignments). *Each vertex point of a folding pattern must be incident to an even number of creases.*

This follows easily from the observation that, at each crease, the paper alternates between having its top side up (a region within which φ is an orientation-preserving isometric mapping) and having its bottom side up (a region within which φ is an orientation-reversing isometric mapping).

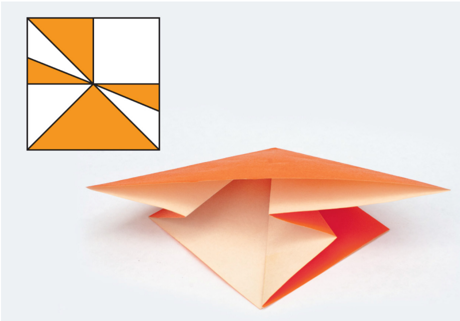

Fig. 2. A single-vertex flat folding and its pattern, demonstrating Maekawa's theorem (the number of folds is even) and Kawasaki's theorem (the face-up orange total angle equals the bottom-up white total angle). Image by the author for Wikipedia, 2011. (Color figure online)

Theorem 2 (Kawasaki's theorem). *At each vertex point of a folding pattern, the alternating sum of wedge angles totals to zero.*

This again follows from the fact that, near the vertex in the flat-folded state of the pattern, each point is covered by equal numbers of upward-facing and downward-facing regions, so the total amount of upward-facing paper must equal the amount of downward-facing paper.

Corollary 1. *Each wedge of a vertex point of a flat folding has angle strictly less than π. Therefore, each face of a flat folding pattern is a (possibly unbounded) convex polygon.*

3 Realization of Trees

Let T be any plane tree. Then by Maekawa's theorem, if T is to be realized as the truncated graph of a local flat folding, its internal vertices must have even degree greater than two. Our purpose in this section is to prove that this condition is necessary as well as sufficient.

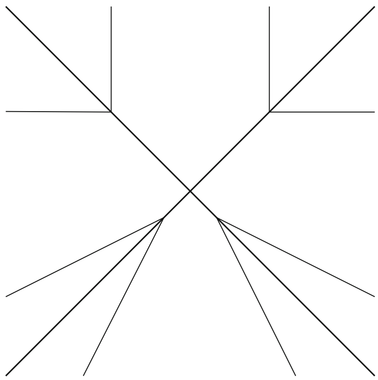

Fig. 3. A tree folding pattern that can be locally flat folded, but not globally flat folded.

We are interested here in global flat foldings, not just local flat foldings, and for this reason some care must be taken. It is not sufficient merely to embed T as a graph in the plane, with its leaf edges drawn as rays, and with each internal vertex meeting the angle sum condition of Kawasaki's theorem. Figure 3 depicts a counterexample. It obeys Kawasaki's theorem, and can be locally flat folded, but not globally flat folded. The four heavier diagonal lines of the figure can be flat folded in only one way up to combinatorial equivalence. Their folding is obtained by first folding along one diagonal line, and then along the other. The four creases of this fold are then modified by subsidiary folds that are each individually possible. But one of the four heavier creases must be nested tightly within another one. The two subsidiary creases of these two nested creases are arranged in such a way that, no matter which crease is nested within the other, the subsidiary crease of one will be blocked by the paper from the other nested crease. (Try it!)

To evade this problem, we seek a stronger type of realization, one in which each crease is "protected" by a wedge surrounding it, within which we can add modifications (such as the subsidiary wedges of Fig. 3) without interfering with other parts of the folding.

Theorem 3. *Let T be any finite tree with all internal vertices having even degree greater than two. Then T can be realized as the truncated graph of a global flat folding.*

Proof. We use induction on the number of internal nodes of T to prove a stronger statement: that T can be realized in such a way that each ray r of T is associated with a wedge W_r, satisfying the following properties:

- Ray r and wedge W_r have the same apex, and r is the median ray of its wedge.
- Each two rays have interior-disjoint wedges. Each edge of T that is not a ray is disjoint from all of the wedges.
- There exists a three-dimensional folded state such that the two halves of each wedge W_r are placed touching each other, with no other paper between them.

The third property above is phrased informally, so let us relate it to our earlier topological definition of a global flat folding. Recall that, in order to formalize the notion of a "three-dimensional folded state" we really have a parameterized family of three-dimensional embeddings. That is, we have both a folding map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ and, for each $\epsilon > 0$, a topological embedding $\varphi_\epsilon : \mathbb{R}^2 \to \mathbb{R}^3$ whose vertical projection to $\mathbb{R}^2$ is ϵ-close to φ. We formalize the "no other paper between them" constraint, again up to ϵ-closeness: for each point $p \in \mathbb{R}^2$ at a distance of ϵ or more from the boundary of $\varphi(W_r)$, the preimage of p (according to the vertical projection) in $\varphi_\epsilon(\mathbb{R}^2)$ should have two points from the two sides of W_r consecutive with each other in the vertical ordering of the points.

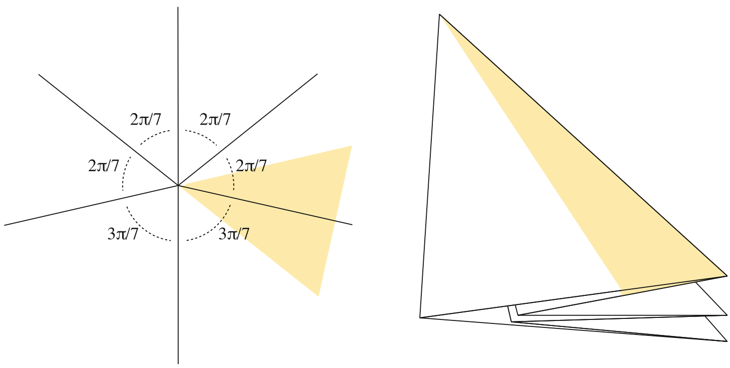

Fig. 4. The base case for realizing a one-internal-vertex tree (here with degree $d = 6$), showing the wedge W_r for one of the rays r both in the folding pattern and in the folded state.

The base case of the induction is a tree T with one internal node v of even degree d greater than four. In this case, we let $\theta = \pi/(d+1)$. We draw T as a set of d rays, all meeting at a common point. We make two of the angles between consecutive rays of T equal to 3θ, and all remaining angles equal to 2θ. For instance, when $d = 7$, we get $\theta = \pi/7$ and six rays separated by angles of $3\pi/7, 3\pi/7, 2\pi/7, 2\pi/7, 2\pi/7, 2\pi/7$. We fold this in three dimensions by placing the two wider wedges on the top and bottom of the folded pattern, and pleating the remaining wedges between them. For this fold, we make each wedge W_r for a ray r of the folding pattern be the wedge centered on that ray with opening angle 2θ. This opening angle is sufficient to make all the wedges interior-disjoint, and it is straightforward to verify that the 3D realization of this fold places no paper between the two halves of any wedge. This case is depicted in Fig. 4.

Otherwise, if T has more than one internal vertex, let v be any internal vertex that has only a single non-leaf neighbor. (For instance, v may be found by choosing any vertex u arbitrarily and letting v be an internal vertex that is maximally far from u.) Let T' be the tree formed from T by removing the leaf neighbors of v, so that v itself becomes a leaf. Then by the induction hypothesis, T' can be realized by a global flat folding, with a ray r that is associated with its leaf v and that is surrounded by a wedge W_r, whose two halves touch each other without being blocked by other paper in the folding. Let θ denote the opening angle of wedge W_r. Suppose also that, in T, v has degree d, and therefore it also has $d-1$ leaf children.

Then we modify the folding that represents T' to form a folding representing T, as follows. We place v at an arbitrarily chosen point along r (for instance, at the point a unit distance away from the apex of ray r). Then, we form $d-1$ creases, along $d-1$ rays with v as apex, to represent the $d-1$ leaf children of r. We choose the angles of these rays so that they are separated from each other and from the two boundary rays of W_r by an angle of θ/d. Finally, we assign each of these rays its own wedge, with v as its apex and with opening angle θ/d. (See Fig. 5.)

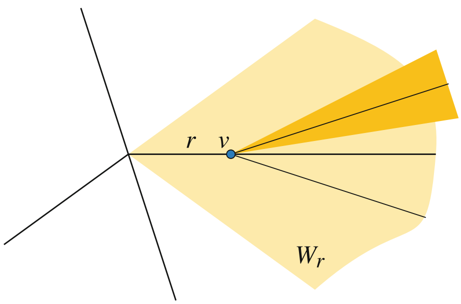

Fig. 5. Adding a vertex v to the folding of T' to create a folding for T. We choose the angles of the new rays incident to v so that they and the two boundary rays of the outer wedge W_r are equally spaced. The wedge surrounding each new ray has opening angle equal to the spacing of the rays. The crease pattern of the figure corresponds to a tree with two degree-four internal nodes.

The 3D folding of the crease pattern for T' can also be modified in the same way to form a 3D folding for the crease pattern for T. At v, the rays and segments representing incident edges of T form d wedges, two of which have opening angle greater than π and the rest of which have opening angle θ/d. As before, we fold this part of the paper so that the two large wedges are outermost and the other wedges are pleated between them. The angles of the creased rays are chosen so that, after this pleat, the creases that are folded to become the closest to the boundary rays of W_r (such as the middle ray of the figure) become parallel to these boundary rays. Because of this, the folded state stays within the region of $\mathbb{R}^3$ previously occupied by the paper for wedge W_r, and the empty space between the two sides of that wedge, so it does not interfere with any other part of the global flat folding. Each of the wedges of opening angle θ/d surrounding the new rays of the folding has its two sides mapped directly above and below each other in the pleating, maintaining the invariant of the induction. □

We remark that, because the pleating pattern used for this realization does not ever tightly nest one crease inside another, it is possible to find a 3D realization that projects exactly to the two-dimensional local flat folding, rather than approaching it through ϵ-approximations.

4 Connectivity

Although we have seen that truncated graphs of flat foldings may be trees (graphs that are not very highly connected), we now show that the full graph, including the special vertex ∞, is (when finite) always well connected. We assume throughout this section that the full graph has at least one finite vertex; otherwise, as a one-vertex graph, the full graph is trivially k-vertex-connected and k-edge-connected for all k.

Lemma 1. *Let G be the graph of a local flat folding. Then the special vertex ∞ is not an articulation vertex of G.*

Proof. If it were, some two components of $G - \infty$ would necessarily be separated by an infinite face of the folding pattern. However, because all faces are convex each connected component of the boundary of an infinite face forms a convex polygonal chain, ending in two rays that span an angle (within the face) of less than π with each other. It is not possible for two such chains to bound a single face without crossing each other, so the boundary of the face can have only one connected component. □

Lemma 2. *Let u and v be two vertex points of a local flat folding φ that belong to the same face of φ and let d denote Euclidean distance. Then $d(u, v) = d(\varphi(u), \varphi(v))$.*

Proof. Because the faces of φ are strictly convex, the line segment between u and v must either consist entirely of crease points (on an edge of the graph of the folding) or unfolded points (if u and v are not consecutive on their shared face). In either case this line segment is mapped to an equal-length line segment by φ. □

Lemma 3. *Let G be the finite graph of a local flat folding. Then removing up to three of the vertex points of the folding from G cannot cause the remaining graph to become disconnected.*

Proof. Suppose for a contradiction that S is a set of at most three vertex points whose removal disconnects G. Since G is a plane graph, there must exist a simple closed curve C in the plane that passes through S and is otherwise disjoint from the vertices and edges of G, with at least one vertex inside the curve and at least one vertex outside the curve. (For folding patterns that include a ray of crease points, we count ∞ as being outside all such curves.) But as we show in the case analysis below, this is not possible:

- If $|S| = 1$, any curve C through the single vertex of S that is otherwise disjoint from G must remain within a single convex face of G, and cannot enclose anything.
- If S consists of two non-adjacent vertices, they can only have one face of G in common. Any curve C through these two vertices that is otherwise disjoint from G must remain within that face, and cannot enclose anything.
- If S consists of two adjacent vertices, then a curve C through the two vertices u and v of S that is otherwise disjoint from G can either stay within one of the two faces incident to edge uv (not enclosing anything) or have one arc in one of these two faces and one arc in the other of the two faces, enclosing edge uv but not enclosing any vertices.
- If S consists of three collinear vertex points, then curve C must visit each of these three points in turn. But the outermost of these two vertex points cannot belong to any convex face of the folding pattern (because this face would also contain the middle point), and cannot be connected by an arc of C.
- If S consists of three non-collinear vertex points u, v, and w, then C can only enclose any vertex points that might lie interior to triangle uvw. However, triangle uvw is mapped by the local flat folding map φ to a congruent triangle, by Lemma 2 and by the fact that there is only one Euclidean triangle (up to congruence) for any triple of distances between its vertices. In order to avoid stretching, every line segment formed by intersecting a line with triangle uvw must be mapped by φ to the corresponding line segment of the image triangle. In particular, there can be no creases within triangle uvw, because whenever a line segment properly crosses a crease of a local flat folding, it is not mapped to a congruent line segment. Therefore, every point inside triangle uvw must be an unfolded point, and C cannot contain a vertex point.

Because there is no way to construct curve C, the hypothesized set S cannot exist. □

The assumption that G is finite is used in the existence of C. If G could be infinite, our tree realization construction could be used to construct a realization of an infinite tree in which ∞ is a degree-one leaf. This does not have the connectivity described by the lemma, but this is not a contradiction because it does not meet the assumptions of the lemma.

Theorem 4. *If G is the finite graph of a local flat folding φ, then G is 2-vertex-connected and 4-edge-connected.*

Proof. G can have no articulation vertex, because neither ∞ nor any vertex point of φ can be an articulation vertex (Lemma 1 and Lemma 3 respectively).

Assume for a contradiction that G could have three edges e_1, e_2, and e_3 whose removal disconnects G. Choose a vertex point v_i as one of the two endpoints of each of these edges (as each edge in G has at least one vertex point as its endpoint). The separation of G caused by the removal of the edges e_i cannot separate any subset of the three vertices v_i from the rest of G, because G has minimum degree four and, in a graph of this degree, any set of up to three vertices is connected to the rest of the graph by at least four incident edges. Therefore, there must be at least one vertex of G on each side of the separation that is not one of the three chosen vertices v_i. However, this implies that these three vertices also separate G, contradicting Lemma 3. This contradiction implies that our assumption is false, and therefore that G is 4-edge-connected. □

We remark that our realizations of 4-regular trees show that both 2-vertex-connectivity and 4-edge-connectivity are tight: some graphs that can be realized as global flat foldings are neither 3-vertex-connected nor 5-edge-connected.

5 Conclusions

We have shown that trees can be realized as the (truncated) graphs of flat folding patterns, and that despite this the (non-truncated) graphs of flat folding patterns must be highly connected. However we have not succeeded in completely characterizing the graphs of flat folding patterns. We leave the following questions as open for future research:

- Which plane graphs (with specified vertex ∞) are the graphs of global flat foldings?
- What is the computational complexity of recognizing and realizing these graphs?
- Is there any graph-theoretic difference between the graphs of global flat foldings and the graphs of local flat foldings? In particular does the folding-assignment version of Maekawa's theorem, that each vertex must have two more mountain folds than valley folds or vice versa, impose any nontrivial constraints on the graphs of flat foldings?
- In the full version of this paper (arXiv:1808.06013) we describe another class of graphs, the *dual orthotrees*, that can always be realized as the graphs of local flat foldings. Can they always be realized as the graphs of global flat foldings?
- What (if anything) changes when we consider folding patterns on a square sheet of paper (or other bounded shape) rather than on an infinite sheet? In the full version we begin a preliminary investigation of this case, in the special case where we restrict the vertex points to the boundary of the paper. On

circular paper, all outerplanar graphs are possible, but on square paper, not even all trees can be folded; we find an exact characterization of the foldable trees, different from the characterization in Sect. 3. However, similar questions without the restriction to boundary points remain open.

– Previously we studied algorithms for realizing trees as convex subdivisions of the plane while optimizing the angular resolution of the resulting tree drawing [15]. Can we use similar ideas to optimize the angular resolution of a folding pattern realization of a tree?

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