On the Area-Universality of Triangulations

Linda Kleist(✉)

Technische Universität Berlin, Berlin, Germany
kleist@math.tu-berlin.de

Abstract. We study straight-line drawings of planar graphs with prescribed face areas. A plane graph is *area-universal* if for every area assignment on the inner faces, there exists a straight-line drawing realizing the prescribed areas.

For triangulations with a special vertex order, we present a sufficient criterion for area-universality that only requires the investigation of one area assignment. Moreover, if the sufficient criterion applies to one plane triangulation, then all embeddings of the underlying planar graph are also area-universal. To date, it is open whether area-universality is a property of a plane or planar graph.

We use the developed machinery to present area-universal families of triangulations. Among them we characterize area-universality of accordion graphs showing that area-universal and non-area-universal graphs may be structural very similar.

Keywords: Area-universality · Triangulation · Planar graph
Face area

1 Introduction

By Fary's theorem [11,20,22], every plane graph has a straight-line drawing. We are interested in straight-line drawings with the additional property that the face areas correspond to prescribed values. Particularly, we study *area-universal* graphs for which all prescribed face areas can be realized by a straight-line drawing. Usually, in a *planar* drawing, no two edges intersect except in common vertices. It is worthwhile to be slightly more generous and allow *crossing-free* drawings, i.e., drawings that can be obtained as the limit of a sequence of planar straight-line drawings. Note that a crossing-free drawing of a triangulation is not planar (*degenerate*) if and only if the area of at least one face vanishes. Moreover, we consider two crossing-free drawings of a plane graph as *equivalent* if the cyclic order of the incident edges at each vertex and the outer face coincide.

For a plane graph G, we denote the set of faces by F, and the set of inner faces by F'. An *area assignment* is a function $\mathcal{A}\colon F' \to \mathbb{R}_{\geq 0}$. We say G is *area-universal* if for every area assignment $\mathcal{A}$ there exists an equivalent crossing-free

T. Biedl and A. Kerren (Eds.): GD 2018, LNCS 11282, pp. 333–346, 2018.
https://doi.org/10.1007/978-3-030-04414-5_23

drawing where every inner face $f \in F'$ has area $\mathcal{A}(f)$. We call such a drawing $\mathcal{A}$-*realizing* and the area assignment $\mathcal{A}$ *realizable*.

Related Work. Biedl and Ruiz Velázquez [6] showed that planar partial 3-trees, also known as subgraphs of *stacked triangulations* or *Apollonian networks*, are area-universal. In fact, every subgraph of a plane area-universal graph is area-universal. Ringel [19] gave two examples of graphs that have drawings where all face areas are of equal size, namely the octahedron graph and the icosahedron graph. Thomassen [21] proved that plane 3-regular graphs are area-universal. Moreover, Ringel [19] showed that the octahedron graph is not area-universal. Kleist [15] generalized this result by introducing a simple counting argument which shows that no Eulerian triangulation, different from K_3, is area-universal. Moreover, it is shown in [15] that every 1-subdivision of a plane graphs is area-universal; that is, every area assignment of a plane graph has a realizing polyline drawing where each edge has at most one bend. Evans et al. [10,17] present classes of area-universal plane quadrangulations. In particular, they verify the conjecture that plane bipartite graphs are area-universal for quadrangulations with up to 13 vertices. Particular graphs have also been studied: It is known that the square grid [9] and the unique triangulation on seven vertices [4] are area-universal. Moreover, non-area-universal triangulations on up to ten vertices have been investigated in [13].

The computational complexity of the decision problem of area-universality for a given graph was studied by Dobbins et al. [7]. The authors show that this decision problem belongs to Universal Existential Theory of the Reals ($\forall\exists\mathbb{R}$), a natural generalization of the class Existential Theory of the Reals ($\exists\mathbb{R}$), and conjecture that this problem is also $\forall\exists\mathbb{R}$-complete. They show hardness of several variants, e.g., the analogue problem of volume universality of simplicial complexes in three dimensions.

In a broader sense, drawings of planar graphs with prescribed face areas can be understood as *cartograms*. Cartograms have been intensely studied for duals of triangulations [1,3,5,14] and in the context of rectangular layouts, dissections of a rectangle into rectangles [8,12,23]. For a detailed survey of the cartogram literature, we refer to [18].

Our Contribution. In this work we present three characterizations of area-universal triangulations. We use these characterizations for proving area-universality of certain triangulations. Specifically, we consider triangulations with a vertex order, where (most) vertices have at least three neighbors with smaller index, called *predecessors*. We call such an order a *p-order*. For triangulations with a p-order, the realizability of an area assignment reduces to finding a real root of a univariate polynomial. If the polynomial is surjective, we can guarantee area-universality. In fact, this is the only known method to prove the area-universality of a triangulation besides the simple argument for plane 3-trees relying on K_4.

We discover several interesting facts. First, to guarantee area-universality it is enough to investigate one area assignment. Second, if the polynomial is surjective for one plane graph, then it is for every embedding of the underlying planar graph. Consequently, the properties of one area assignment can imply the area-universality of all embeddings of a planar graph. This may indicate that area-universality is a property of planar graphs.

We use the method to prove area-universality for several graph families including accordion graphs. To obtain an *accordion graph* from the plane octahedron graph, we introduce new vertices of degree 4 by subdividing an edge of the central triangle. Figure 1 presents four examples of accordion graphs. Surprisingly, the insertion of an even number of vertices yields a non-area-universal graph while the insertion of an odd number of vertices yields an area-universal graph. Accordions with an even number of vertices are Eulerian and thus not area-universal [15]. Consequently, area-universal and non-area-universal graphs may have a very similar structure. (In [17], we use the method to classify small triangulations with p-orders on up to ten vertices.)

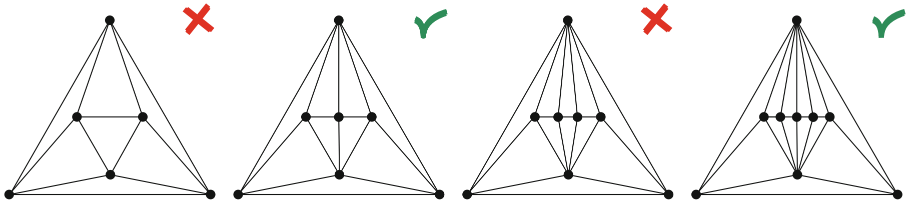


Fig. 1. Examples of accordion graphs. A checkmark indicates area-universality and a cross non-area-universality.

Organization. We start by presenting three characterizations of area-universality of triangulations in Sect. 2. In Sect. 3, we turn our attention to triangulations with p-orders and show how the analysis of one area assignment can be sufficient to prove area-universality of all embeddings of the given triangulation. Then, in Sect. 4, we apply the developed method to prove area-universality for certain graph families; among them we characterize the area-universality of accordion graphs. We end with a discussion and a list of open problems in Sect. 5. For omitted proofs consider the appendices of the full version [16].

2 Characterizations of Area-Universal Triangulations

Throughout this section, let T be a plane triangulation on n vertices. A straight-line drawing of T can be encoded by the $2n$ vertex coordinates, and hence, by a point in the Euclidean space $\mathbb{R}^{2n}$. We call such a vector of coordinates a *vertex placement* and denote the set of all vertex placements encoding crossing-free drawings by $\mathcal{D}(T)$; we also write $\mathcal{D}$ if T is clear from the context.

It is easy to see that an $\mathcal{A}$-realizing drawing of a triangulation can be transformed by an affine linear map into an $\mathcal{A}$-realizing drawing where the outer face corresponds to any given triangle of correct total area $\Sigma\mathcal{A} := \sum_{f\in F'} \mathcal{A}(f)$, where F' denotes the set of inner faces as before.

Lemma 1. [15, Obs. 2] *A plane triangulation T with a realizable area assignment $\mathcal{A}$, has an $\mathcal{A}$-realizing drawing within every given outer face of area $\Sigma\mathcal{A}$.*

Likewise, affine linear maps can be used to scale realizing drawings by any factor. For any positive real number $\alpha \in \mathbb{R}$ and area assignment $\mathcal{A}$, let $\alpha\mathcal{A}$ denote the *scaled area assignment* of $\mathcal{A}$ where $\alpha\mathcal{A}(f) := \alpha \cdot \mathcal{A}(f)$ for all $f \in F'$.

Lemma 2. *Let $\mathcal{A}$ be an area assignment of a plane graph and $\alpha > 0$. The scaled area assignment $\alpha\mathcal{A}$ is realizable if and only if $\mathcal{A}$ is realizable.*

For a plane graph and $c > 0$, let $\mathbb{A}^c$ denote the set of area assignments with a total area of c. Lemma 2 directly implies the following property.

Lemma 3. *Let $c > 0$. A plane graph is area universal if all area assignments in $\mathbb{A}^c$ are realizable.*

2.1 Closedness of Realizable Area Assignments

In [15, Lemma 4], it is shown for triangulations that $\mathcal{A} \in \mathbb{A}^c$ is realizable if and only if in every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^c$ there exists a realizable area assignment. For our purposes, we need a stronger version. Let $\mathbb{A}^{\leq c}$ denote the set of area assignments of T with a total area of at most c. For a fixed face f of T, $\mathbb{A}^{\leq c}|_{f \to a}$ denotes the subset of $\mathbb{A}^{\leq c}$ where f is assigned to a fixed $a > 0$.

Proposition 1. *Let T be a plane triangulation and $c > 0$. Then $\mathcal{A} \in \mathbb{A}^c$ is realizable if and only if for some face f with $\mathcal{A}(f) > 0$ every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^{\leq 2c}|_{f \to \mathcal{A}(f)}$ contains a realizable area assignment.*

Intuitively, Proposition 1 enables us not to worry about area assignments with bad but unlikely properties. In particular, area-universality is guaranteed by the realizability of a dense subset of $\mathbb{A}^c$. Moreover, this stronger version allows to certify the realizability of an area assignment by realizable area assignments with slightly different total areas. The proof of Proposition 1 goes along the same lines as in [15, Lemma 4]; it is based on the fact that the set of drawings of T with a fixed face f and a total area of at most $2c$ is compact.

2.2 Characterization by 4-Connected Components

For a plane triangulation T, a 4-*connected component* is a maximal 4-connected subgraph of T. Moreover, we call a triangle t of T *separating* if at least one vertex of T lies inside t and at least one vertex lies outside t; in other words, t is not a face of T.

Proposition 2. *A plane triangulation T is area-universal if and only if every 4-connected component of T is area-universal.*

Proof (Sketch). The proof is based on the fact that a plane graph G with a separating triangle t is area-universal if and only if G_{E}, the induced graph by t and its exterior, and G_{I}, the induced graph by t and its interior, are area-universal. In particular, Lemma 1 allows us to combine realizing drawings of G_{E} and G_{I} to a drawing of G.

Remark. Note that a plane 3-tree has no 4-connected component. (Recall that K_4 is 3-connected and a graph on $n > 4$ vertices is 4-connected if and only if it has no separating triangle.) This is another way to see their area-universality.

2.3 Characterization by Polynomial Equation System

Dobbins et al. [7, Proposition 1] show a close connection of area-universality and equation systems: For every plane graph G with area assignment $\mathcal{A}$ there exists a polynomial equation system $\mathcal{E}$ such that $\mathcal{A}$ is realizable if and only if $\mathcal{E}$ has a real solution. Here we strengthen the statement for triangulations, namely it suffices to guarantee the face areas; these imply all further properties such as planarity and the equivalent embedding. To do so, we introduce some notation.

A plane graph G induces an orientation of the vertices of each face. For a face f given by the vertices $v_1, \ldots, v_k$, we say f is *counter clockwise (ccw)* if the vertices $v_1, \ldots, v_k$ appear in ccw direction on a walk on the boundary of f; otherwise f is *clockwise (cw)*. Moreover, the function $\text{AREA}(f, D)$ measures the area of a face f in a drawing D. For a ccw triangle t with vertices v_1, v_2, v_3, we denote the coordinates of v_i by (x_i, y_i). Its area in D is given by the determinant

$$\text{Det}(v_1, v_2, v_3) := \det\big(c(v_1), c(v_2), c(v_3)\big) = 2 \cdot \text{AREA}(t, D), \tag{1}$$

where $c(v_i) := (x_i, y_i, 1)$. Since the (complement of the) outer face f_o has area $\Sigma\mathcal{A}$ in an $\mathcal{A}$-realizing drawing, we define $\mathcal{A}(f_o) := \Sigma\mathcal{A}$. For a set of faces $\tilde{F} \subset F$, we define the *area equation system of* $\tilde{F}$ as

$$\text{AEQ}(T, \mathcal{A}, \tilde{F}) := \{\text{Det}(v_i, v_j, v_k) = \mathcal{A}(f) \mid f \in \tilde{F}, f =: (v_i, v_j, v_k) \text{ ccw}\}.$$

For convenience, we omit the factor of 2 in each *area equation*. Therefore, without mentioning it any further, we usually certify the realizability of $\mathcal{A}$ by a $1/2\mathcal{A}$-realizing drawing. That is, if we say a triangle has area a, it may have area $1/2a$. Recall that, by Lemma 2, consistent scaling has no further implications.

Proposition 3. *Let T be a triangulation, $\mathcal{A}$ an area assignment, and f a face of T. Then $\mathcal{A}$ is realizable if and only if $\text{AEQ}(T, \mathcal{A}, F \setminus \{f\})$ has a real solution.*

The key idea is that a (scaled) vertex placement of an $\mathcal{A}$-realizing drawing is a real solution of $\text{AEQ}(T, \mathcal{A}, F \setminus \{f\})$ and vice versa. The main task is to guarantee crossing-freeness of the induced drawing; it follows from the following neat fact.

Lemma 4. *Let D be a vertex placement of a triangulation T where the orientation of each inner face in D coincides with the orientation in T. Then D represents a crossing-free straight-line drawing of T.*

A proof of Lemma 4 can be found in [2, in the end of the proof of Lemma 4.2]. An alternative proof relies on the properties of the determinant, in particular, on the fact that for any vertex placement D the area of the triangle formed by its outer vertices evaluates to

$$\text{AREA}(f_o, D) = \sum_{f \in F'} \text{AREA}(f, D). \tag{2}$$

Equation (2) shows that for every face $f \in F'$, the equation systems $\textsc{aeq}(T, \mathcal{A}, F')$ and $\textsc{aeq}(T, \mathcal{A}, F \setminus \{f\})$ are equivalent. This fact is also used for Proposition 3.

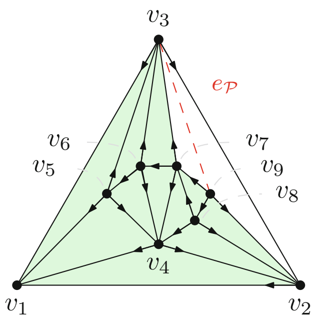
Remark 1. In fact, Lemma 4 and Proposition 3 generalize to *inner triangulations*, i.e., 2-connected plane graphs where every inner face is a triangle.

3 Area-Universality of Triangulations with p-orders

We consider planar triangulations with the following property: An order of the vertices $(v_1, v_2, \ldots, v_n)$, together with a set of *predecessors* $\text{pred}(v_i) \subset N(v_i)$ for each vertex v_i, is a *p-order* if the following conditions are satisfied:

- $\text{pred}(v_i) \subseteq \{v_1, v_2, \ldots, v_{i-1}\}$, i.e., the predecessors of v_i have an index $< i$,
- $\text{pred}(v_1) = \emptyset$, $\text{pred}(v_2) = \{v_1\}$, $\text{pred}(v_3) = \text{pred}(v_4) = \{v_1, v_2\}$, and
- for all $i > 4$: $|\text{pred}(v_i)| = 3$, i.e., v_i has exactly three predecessors.

Note that $\text{pred}(v_i)$ specifies a subset of preceding neighbors. Moreover, a p-order is defined for a planar graph independent of a drawing. We usually denote a p-order by $\mathcal{P}$ and state the order of the vertices; the predecessors are then implicitly given by $\text{pred}(v_i)$. Figure 2 illustrates a p-order.



i	$\text{pred}(v_i)$
5	$\{v_1, v_3, v_4\}$
6	$\{v_3, v_4, v_5\}$
7	$\{v_3, v_4, v_6\}$
8	$\{v_2, v_4, v_7\}$
9	$\{v_2, v_7, v_8\}$

Fig. 2. A plane 4-connected triangulation with a p-order $\mathcal{P}$. In an almost realizing vertex placement constructed with $\mathcal{P}$, all face areas are realized except for the two faces incident to the unoriented (dashed) edge $e_{\mathcal{P}}$ of $\mathcal{O}_{\mathcal{P}}$ (Lemma 8).

We pursue the following one-degree-of-freedom mechanism to construct realizing drawings for a plane triangulation T with a p-order $(v_1, v_2, \ldots, v_n)$ and an area assignment $\mathcal{A}$:

- Place the vertices v_1, v_2, v_3 at positions realizing the area equation of the face $v_1v_2v_3$. Without loss of generality, we set $v_1 = (0, 0)$ and $v_2 = (1, 0)$.
- Insert v_4 such that the area equation of face $v_1v_2v_4$ is realized; this is fulfilled if y_4 equals $\mathcal{A}(v_1v_2v_4)$ while $x_4 \in \mathbb{R}$ is arbitrary. The value x_4 is our variable.

- Place each remaining vertex v_i with respect to its predecessors $\text{pred}(v_i)$ such that the area equations of the two incident face areas are respected; the coordinates of v_i are *rational* functions of x_4.
- Finally, all area equations are realized except for two special faces f_a and f_b. Moreover, the face area of f_a is a *rational* function $\mathfrak{f}$ of x_4.
- If $\mathfrak{f}$ is *almost surjective*, then there is a vertex placement D respecting all face areas and orientations, i.e., D is a real solution of $\text{AEQ}(T, \mathcal{A}, F)$.
- By Proposition 3, D guarantees the realizability of $\mathcal{A}$.
- If this holds for *enough* area assignments, then T is area-universal.

3.1 Properties of p-orders

A p-order $\mathcal{P}$ of a plane triangulation T induces an orientation $\mathcal{O}_\mathcal{P}$ of the edges: For $w \in \text{pred}(v_i)$, we orient the edge from v_i to w, see also Fig. 2. By Proposition 2, we may restrict our attention to 4-connected triangulations. We note that 4-connectedness is not essential for our method but yields a cleaner picture.

Lemma 5. *Let T be a planar 4-connected triangulation with a p-order $\mathcal{P}$. Then $\mathcal{O}_\mathcal{P}$ is acyclic, $\mathcal{O}_\mathcal{P}$ has a unique unoriented edge $e_\mathcal{P}$, and $e_\mathcal{P}$ is incident to v_n.*

It follows that the p-order encodes all but one edge which is easy to recover. Therefore, the p-order of a planar triangulation T encodes T. In fact, T has a p-order if and only if there exists an edge e such that $T - e$ is 3-degenerate.

Convention. Recall that a drawing induces an orientation of each face. We follow the convention of stating the vertices of inner faces ccw and of the outer face in cw direction. This convention enables us to switch between different plane graphs of the same planar graph without changing the order of the vertices. To account for our convention, we redefine $\mathcal{A}(f_o) := -\Sigma\mathcal{A}$ for the outer face f_o. Then, for different embeddings, only the right sides of the AEQs change.

The next properties can be proved by induction and are shown in Fig. 3.

Lemma 6. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$ specified by $(v_1, v_2, \ldots, v_n)$ and let T_i denote the subgraph of T induced by $\{v_1, v_2, \ldots, v_i\}$. For $i \geq 4$,*

- *T_i has one 4-face and otherwise only triangles,*
- *T_{i+1} can be constructed from T_i by inserting v_{i+1} in the 4-face of T_i, and*
- *the three predecessors of v_i can be named $(p_\text{F}, p_\text{M}, p_\text{L})$ such that $p_\text{F}p_\text{M}v_i$ and $p_\text{M}p_\text{L}v_i$ are (ccw inner and cw outer) faces of T_i.*

Remark 2. For every (non-equivalent) plane graph T' of T, the three predecessors $(p_\text{F}, p_\text{M}, p_\text{L})$ of v_i in T' and T coincide.

Remark 3. Lemma 6 can be used to show that the number of 4-connected planar triangulations on n vertices with a p-order is $\Omega(2^n/n)$.

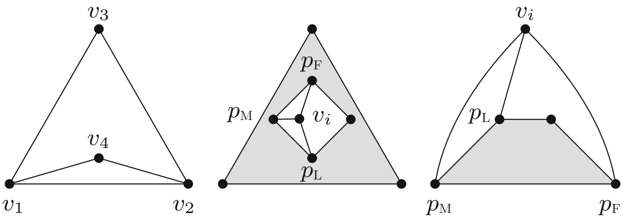


Fig. 3. Illustration of Lemma 6: (a) T_4, (b) v_i is inserted in an inner 4-face, (c) v_i is inserted in outer 4-face.

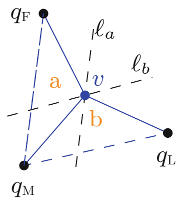


Fig. 4. Illustration of Lemma 7.

3.2 Constructing Almost Realizing Vertex Placements

Let T be a plane triangulation with an area assignment $\mathcal{A}$. We call a vertex placement D of T *almost $\mathcal{A}$-realizing* if there exist two faces f_a and f_b such that D is a real solution of the equation system $\text{AEQ}(T, \mathcal{A}, \tilde{F})$ with $\tilde{F} := F \setminus \{f_a, f_b\}$. In particular, we insist that the orientation and area of each face, except for f_a and f_b be correct, i.e., the area equations are fulfilled. Note that an almost realizing vertex placement does not necessarily correspond to a crossing-free drawing.

Observation. An almost $\mathcal{A}$-realizing vertex placement D fulfilling the area equations of all faces except for f_a and f_b, certifies the realizability of $\mathcal{A}$ if additionally the area equation of f_a is satisfied.

We construct almost realizing vertex placements with the following lemma.

Lemma 7. *Let $a, b \geq 0$ and let $q_{\mathrm{F}}, q_{\mathrm{M}}, q_{\mathrm{L}}$ be three vertices with a non-collinear placement in the plane. Then there exists a unique placement for vertex v such the ccw triangles $q_{\mathrm{F}}q_{\mathrm{M}}v$ and $q_{\mathrm{M}}q_{\mathrm{L}}v$ fulfill the area equations for a and b, respectively.*

Proof. Consider Fig. 4. To realize the areas, v must be placed on a specific line ℓ_a and ℓ_b, respectively. Note that ℓ_a is parallel to the segment $q_{\mathrm{F}}, q_{\mathrm{M}}$ and ℓ_b is parallel to the segment $q_{\mathrm{M}}, q_{\mathrm{L}}$. Consequently, ℓ_a and ℓ_b are not parallel and their intersection point yields the unique position for vertex v. The coordinates of v are specified by the two equations $\text{Det}(q_{\mathrm{F}}, q_{\mathrm{M}}, v) \stackrel{!}{=} a$ and $\text{Det}(q_{\mathrm{M}}, q_{\mathrm{L}}, v) \stackrel{!}{=} b$.

Note that if ℓ_a and ℓ_b are parallel and do not coincide, then there is no position for v realizing the area equations of the two triangles. Based on Lemma 7, we obtain our key lemma.

Lemma 8. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$ specified by $(v_1, v_2, \ldots, v_n)$. Let f_a, f_b be the faces incident to $e_{\mathcal{P}}$ and $f_0 := v_1v_2v_3$. Then there exists a constant $c > 0$ such that for a dense subset $\mathbb{A}_D$ of $\mathbb{A}^c$, every $\mathcal{A} \in \mathbb{A}_D$ has a finite set $\mathcal{B}(\mathcal{A}) \subset \mathbb{R}$, rational functions $x_i(\cdot, \mathcal{A})$, $y_i(\cdot, \mathcal{A})$, $\mathfrak{f}(\cdot, \mathcal{A})$ and a triangle $\triangle$, such that for all $x_4 \in \mathbb{R} \setminus \mathcal{B}(\mathcal{A})$, there exists a vertex placement $D(x_4)$ with the following properties:*

(i) *f_0 coincides with the triangle $\triangle$,*
(ii) *$D(x_4)$ is almost realizing, i.e., a real solution of* $\text{AEQ}(T, \mathcal{A}, F \setminus \{f_a, f_b\})$,

(iii) every vertex v_i is placed at the point $\big(x_i(x_4,\mathcal{A}), y_i(x_4,\mathcal{A})\big)$, and
(iv) the area of face f_a in $D(x_4)$ is given by $\mathfrak{f}(x_4,\mathcal{A})$.

The idea of the proof is to use Lemma 7 in order to construct $D(x_4)$ inductively. Therefore, given a vertex placement $v_1,\ldots,v_{i-1}$, we have to ensure that the vertices of $\operatorname{pred}(v_i)$ are not collinear. To do so, we consider algebraically independent area assignments. We say an area assignment $\mathcal{A}$ of T is *algebraically independent* if the set $\{\mathcal{A}(f)|f\in F'\}$ is algebraically independent over $\mathbb{Q}$. In fact, the subset of algebraically independent area assignments $\mathbb{A}_I$ of $\mathbb{A}^c$ is dense when c is transcendental.

We call the function $\mathfrak{f}$, constructed in the proof of Lemma 8, the *last face function* of T and interpret it as a function in x_4 whose coefficients depend on $\mathcal{A}$.

3.3 Almost Surjectivity and Area-Universality

In the following, we show that almost surjectivity of the last face function implies area-universality. Let A and B be sets. A function $f\colon A\to B$ is *almost surjective* if f attains all but finitely many values of B, i.e., $B\setminus f(A)$ is finite.

Theorem 1. *Let T be a 4-connected plane triangulation with a p-order $\mathcal{P}$ and let $\mathbb{A}_D, \mathbb{A}^c, \mathfrak{f}$ be obtained by Lemma 8. If the last face function $\mathfrak{f}$ is almost surjective for all area assignments in $\mathbb{A}_D$, then T is area-universal.*

Proof. By Lemma 3, it suffices to show that every $\mathcal{A}\in\mathbb{A}_D$ is realizable. Let f_0 be the triangle formed by v_1,v_2,v_3 and $\mathbb{A}^+ := \mathbb{A}^{\leq 2c}|_{f_0\to\mathcal{A}(f_0)}$. By Proposition 1, $\mathcal{A}$ is realizable if every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^+$ contains a realizable area assignment. Let f_a and f_b denote the faces incident to $e_{\mathcal{P}}$ and $a := \mathcal{A}(f_a)$. Lemma 8 guarantees the existence of a finite set $\mathcal{B}$ such that for all $x_4\in\mathbb{R}\setminus\mathcal{B}$, there exists an almost $\mathcal{A}$-realizing vertex placement $D(x_4)$. Since $\mathcal{B}$ is finite and $\mathfrak{f}$ is almost surjective, for every ε with $0<\varepsilon<c$, there exists $\tilde{x}\in\mathbb{R}\setminus\mathcal{B}$ such that $a\leq\mathfrak{f}(\tilde{x})\leq a+\varepsilon$, i.e., the area of face f_a in $D(\tilde{x})$ is between a and $a+\varepsilon$. (If f_a and f_b are both inner faces, then the face f_b has an area between $b-\varepsilon$ and b, where $b:=\mathcal{A}(f_b)$. Otherwise, if f_a or f_b is the outer face, then the total area changes and face f_b has area between b and $b+\varepsilon$.) Consequently, for some $\mathcal{A}'$ in the ε-neighborhood of $\mathcal{A}$ in $\mathbb{A}^+$, $D(\tilde{x})$ is a real solution of $\textsc{aeq}(T,\mathcal{A}',F\setminus\{f_b\})$ and Proposition 3 ensures that $\mathcal{A}'$ is realizable. By Proposition 1, $\mathcal{A}$ is realizable. Thus, T is area-universal.

To prove area-universality, we use the following sufficient condition for almost surjectivity. We say two real polynomials p and q are *crr-free* if they do not have common real roots. For a rational function $f:=\frac{p}{q}$, we define the *max-degree* of f as $\max\{|p|,|q|\}$, where $|p|$ denotes the degree of p. Moreover, we say f is crr-free if p and q are. The following property follows from the fact that polynomials of odd degree are surjective.

Lemma 9. *Let $p,q\colon\mathbb{R}\to\mathbb{R}$ be polynomials and let Q be the set of the real roots of q. If the polynomials p and q are crr-free and have odd max-degree, then the function $f\colon\mathbb{R}\backslash Q\to\mathbb{R},\ f(x)=\frac{p(x)}{q(x)}$ is almost surjective.*

For the final result, we make use of several convenient properties of algebraically independent area assignments. For $\mathcal{A}$, let $\mathfrak{f}_{\mathcal{A}}$ denote the last face function and $d_1(\mathfrak{f}_{\mathcal{A}})$ and $d_2(\mathfrak{f}_{\mathcal{A}})$ the degree of the numerator and denominator polynomial of $\mathfrak{f}_{\mathcal{A}}$ in x_4, respectively. Since $\mathfrak{f}_{\mathcal{A}}$ is a function in x_4 whose coefficients depend on $\mathcal{A}$, algebraic independence directly yields the following property.

Claim 1. *For two algebraically independent area assignments $\mathcal{A}, \mathcal{A}' \in \mathbb{A}_I$ of a 4-connected triangulation with a p−order $\mathcal{P}$, the degrees of the last face functions $\mathfrak{f}_{\mathcal{A}}$ and $\mathfrak{f}_{\mathcal{A}'}$ with respect to $\mathcal{P}$ coincide, i.e., $d_i(\mathfrak{f}_{\mathcal{A}}) = d_i(\mathfrak{f}_{\mathcal{A}'})$ for $i \in [2]$.*

In fact, the degrees do not only coincide for all algebraically independent area assignments, but also for different embeddings of the plane graph. For a plane triangulation T, let T^* denote the corresponding planar graph and $[T]$ the set (of equivalence classes) of all plane graphs of T^*.

Claim 2. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$. Then for every plane graph $T' \in [T]$, and algebraically independent area assignments $\mathcal{A}$ of T and $\mathcal{A}'$ of T', the last face functions $\mathfrak{f}_{\mathcal{A}}$ and $\mathfrak{f}'_{\mathcal{A}'}$ with respect to $\mathcal{P}$ have the same degrees, i.e., $d_i(\mathfrak{f}_{\mathcal{A}}) = d_i(\mathfrak{f}'_{\mathcal{A}'})$ for $i \in [2]$.*

This implies our final result:

Corollary 1. *Let T be a plane triangulation with a p-order $\mathcal{P}$. If the last face function $\mathfrak{f}$ of T is crr-free and has odd max-degree for one algebraically independent area assignment, then every plane graph in $[T]$ is area-universal.*

4 Applications

We now use Theorem 1 and Corollary 1 to prove area-universality of some classes of triangulations. The considered graphs rely on an operation that we call *diamond addition*. Consider the left image of Fig. 5. Let G be a plane graph and let e be an inner edge incident to two triangular faces that consist of e and the vertices u_1 and u_2, respectively. Applying a *diamond addition of order k* on e results in the graph G' which is obtained from G by subdividing edge e with k vertices, $v_1, \ldots, v_k$, and inserting the edges v_iu_j for all pairs $i \in [k]$ and $j \in [2]$. Figure 5 illustrates a diamond addition on e of order 3.

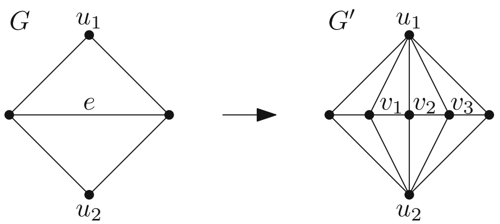


Fig. 5. Obtaining G' from G by a diamond addition of order 3 on edge e.

4.1 Accordion Graphs

An accordion graph can be obtained from the plane octahedron graph $\mathcal{G}$ by a diamond addition: Choose one edge of the central triangle of $\mathcal{G}$ as the *special edge*. The *accordion* graph $\mathcal{K}_\ell$ is the plane graph obtained by a diamond addition of order ℓ on the special edge of $\mathcal{G}$. Consequently, $\mathcal{K}_\ell$ has $\ell+6$ vertices. We speak of an *even accordion* if ℓ is even and of an *odd accordion* if ℓ is odd. Figure 1 illustrates the accordion graphs $\mathcal{K}_i$ for $i \leq 3$. Note that $\mathcal{K}_0$ is $\mathcal{G}$ itself and $\mathcal{K}_1$ is the unique 4-connected plane triangulation on seven vertices. Due to its symmetry, it holds that $[\mathcal{K}_\ell] = \{\mathcal{K}_\ell\}$.

Theorem 2. *The accordion graph $\mathcal{K}_\ell$ is area-universal if and only if ℓ is odd.*

Proof (Sketch). Performing a diamond addition of order ℓ on some plane graph changes the degree of exactly two vertices by ℓ while all other vertex degrees remain the same. Consequently, if ℓ is even, all vertices of $\mathcal{K}_\ell$ have even degree, and hence, $\mathcal{K}_\ell$ as an Eulerian triangulation is not area-universal as shown by the author in [15, Theorem 1].

It remains to prove the area-universality of odd accordion graphs with the help of Theorem 1. Consider an arbitrary but fixed algebraically independent area assignment $\mathcal{A}$. We use the p-order depicted in Fig. 6 to construct an almost realizing vertex placement. We place the vertices v_1 at $(0,0)$, v_2 at $(1,0)$, v_3 at $(1, \Sigma\mathcal{A})$, and v_4 at (x_4, a) with $a := \mathcal{A}(v_1v_2v_4)$. Consider also Fig. 6.

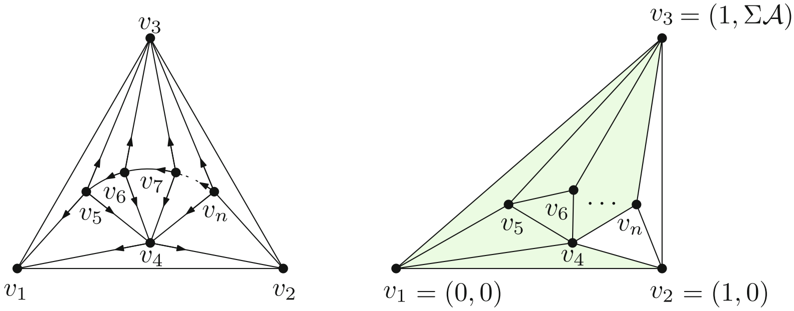


Fig. 6. A p-order of an accordion graph (left) and an almost realizing vertex placement (right), where the shaded faces are realized.

We use Lemma 8 to construct an almost realizing vertex placement. Note that for all vertices v_i with $i > 5$, the three predecessors of v_i are $p_{\mathrm{F}} = v_3$, $p_{\mathrm{M}} = v_{i-1}$ and $p_{\mathrm{L}} = v_4$. One can show that the vertex coordinates of v_i can be expressed as $x_i = {}^{\mathcal{N}_i^x}/_{\mathcal{D}_i}$ and $y_i = {}^{\mathcal{N}_i^y}/_{\mathcal{D}_i}$, where $\mathcal{N}_i^x, \mathcal{N}_i^y, \mathcal{D}_i$ are polynomials in x_4. Moreover, the polynomials fulfill the following crucial properties.

Lemma 10. *For all $i \geq 5$, it holds that $|\mathcal{D}_5| = 1$ and*

$$|\mathcal{N}_{i+1}^x| = |\mathcal{D}_{i+1}| = |\mathcal{N}_{i+1}^y| + 1 = |\mathcal{D}_i| + 1.$$

Consequently, $|\mathcal{N}_i^x| = |\mathcal{D}_i|$ is odd if and only if i is odd. In particular, for odd ℓ, $|\mathcal{N}_n^x| = |\mathcal{D}_n|$ is odd since the number of vertices $n = \ell + 6$ is odd.

Lemma 11. *For all $i \geq 5$ and $\circ \in \{x, y\}$, it holds that $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ are crr-free.*

Consequently, the area of the ccw triangle $v_2v_3v_n$ in $D(x_4)$ is given by the crr-free last face function

$$\mathfrak{f}(x) := \mathrm{Det}(v_2, v_3, v_n) = \Sigma\mathcal{A}(1 - x_n) = \Sigma\mathcal{A}\left(1 - \frac{\mathcal{N}_n^x}{\mathcal{D}_n}\right).$$

Since $|\mathcal{N}_n^x|$ and $|\mathcal{D}_n|$ are odd, the max-degree of $\mathfrak{f}$ is odd. Thus, Lemma 9 ensures that $\mathfrak{f}$ is almost surjective. By Theorem 1, $\mathcal{K}_\ell$ is area-universal for odd ℓ.

This result can be generalized to double stacking graphs.

4.2 Double Stacking Graphs

Denote the vertices of the plane octahedron $\mathcal{G}$ by ABC and uvw as depicted in Fig. 7. The *double stacking graph* $\mathcal{H}_{\ell,k}$ is the plane graph obtained from $\mathcal{G}$ by applying a diamond addition of order $\ell - 1$ on Au and a diamond addition of order $k - 1$ on vw. Note that $\mathcal{H}_{\ell,k}$ has $(\ell + k + 4)$ vertices. Moreover, $\mathcal{H}_{\ell,1}$ is isomorphic to $\mathcal{K}_{\ell-1}$; in particular, $\mathcal{H}_{1,1}$ equals $\mathcal{G}$. Note that $[\mathcal{H}_{\ell,k}]$ usually contains several (equivalence classes of) plane graphs.

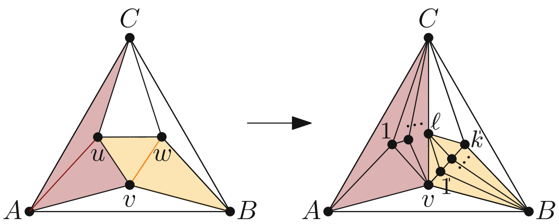


Fig. 7. A double stacking graph $\mathcal{H}_{\ell,k}$.

Theorem 3. *A plane graph in $[\mathcal{H}_{\ell,k}]$ is area-universal if and only if $\ell \cdot k$ is even.*

If $\ell \cdot k$ is odd, every plane graph in $[\mathcal{H}_{\ell,k}]$ is Eulerian and hence not area-universal by [15, Theorem 1]. If $\ell \cdot k$ is even, we consider an algebraically independent area assignment of $\mathcal{H}_{\ell,k}$, show that its last face function is crr-free and has odd max-degree. Then we apply Corollary 1.

Theorem 3 implies that

Corollary 2. *For every $n \geq 7$, there exists a 4-connected triangulation on n vertices that is area-universal.*

5 Discussion and Open Problems

For triangulations with p-orders, we introduced a sufficient criterion to prove area-universality of all embeddings of a planar graph which relies on checking properties of one area assignments of one plane graph. We used the criterion to present two families of area-universal triangulations. Since area-universality is maintained by taking subgraphs, area-universal triangulations are of special interest. For instance, the area-universal double stacking graphs are used in [10,17] to show that all plane quadrangulations with at most 13 vertices are area-universal. The analysis of accordion graphs showns that area-universal and non-area-universal graphs can be structural very similar. The class of accordion graphs gives a hint why understanding area-universality seems to be a difficult problem. In conclusion, we pose the following open questions:

– Is area-universality a property of plane or planar graphs?
– What is the complexity of deciding the area-universality of triangulations?
– Can area-universal graphs be characterized by *local* properties?

Acknowledgements. I thank Udo Hoffmann and Sven Jäger for helpful comments.

References

1. Alam, M.J., Biedl, T.C., Felsner, S., Kaufmann, M., Kobourov, S.G., Ueckerdt, T.: Computing cartograms with optimal complexity. Discrete Comput. Geom. **50**(3), 784–810 (2013). https://doi.org/10.1007/s00454-013-9521-1
2. Angelini, P., et al.: Windrose planarity: embedding graphs with direction-constrained edges. In: Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 985–996 (2016). https://doi.org/10.1137/1.9781611974331
3. de Berg, M., Mumford, E., Speckmann, B.: On rectilinear duals for vertex-weighted plane graphs. Discrete Math. **309**(7), 1794–1812 (2009). https://doi.org/10.1016/j.disc.2007.12.087
4. Bernáthová, A.: Kreslení grafů s podmínkami na velikosti stěn, Bachelor thesis. Charles University Prague (2009). https://is.cuni.cz/webapps/zzp/detail/63479/20281489/
5. Biedl, T.C., Velázquez, L.E.R.: Orthogonal cartograms with few corners per face. In: Algorithms and Data Structures Symposium (WADS), pp. 98–109 (2011). https://doi.org/10.1007/978-3-642-22300-6_9
6. Biedl, T.C., Velázquez, L.E.R.: Drawing planar 3-trees with given face areas. Comput. Geom. **46**(3), 276–285 (2013). https://doi.org/10.1016/j.comgeo.2012.09.004
7. Dobbins, M.G., Kleist, L., Miltzow, T., Rzążewski, P.: Is area universality $\forall\exists\mathbb{R}$-complete? In: Brandstädt, A., KöhlerKlaus, E., Meer, K. (eds.) Graph-Theoretic Concepts in Computer Science, WG 2018. LNCS, vol. 11159. Springer, Heidelberg (2018). https://doi.org/10.1007/978-3-030-00256-5. Accepted
8. Eppstein, D., Mumford, E., Speckmann, B., Verbeek, K.: Area-universal and constrained rectangular layouts. SIAM J. Comput. **41**(3), 537–564 (2012). https://doi.org/10.1137/110834032

9. Evans, W., et al.: Table cartogram. Comput. Geom. **68**, 174–185 (2017). https://doi.org/10.1016/j.comgeo.2017.06.010. Special Issue in Memory of Ferran Hurtado
10. Evans, W., Felsner, S., Kleist, L., Kobourov, S.G.: On area-universal quadrangulations. In preparation (2018)
11. Fáry, I.: On straight line representations of planar graphs. Acta Scientiarum Mathematicarum **11**, 229–233 (1948)
12. Felsner, S.: Exploiting air-pressure to map floorplans on point sets. J. Graph. Algorithms Appl. **18**(2), 233–252 (2014). https://doi.org/10.7155/jgaa.00320
13. Heinrich, H.: Ansätze zur Entscheidung von Flächenuniversalität. Master thesis, Technische Universität Berlin (2018)
14. Kawaguchi, A., Nagamochi, H.: Orthogonal drawings for plane graphs with specified face areas. In: Theory and Applications of Model of Computation (TAMC), pp. 584–594 (2007). https://doi.org/10.1007/978-3-540-72504-6_53
15. Kleist, L.: Drawing planar graphs with prescribed face areas. J. Comput. Geom. (JoCG) **9**(1), 290–311 (2018). https://doi.org/10.20382/jocg.v9i1a9
16. Kleist, L.: On the area-universality of triangulations. preprint, ArXiv:1808.10864v2 (2018). http://arxiv.org/abs/1808.10864v2
17. Kleist, L.: Planar graphs and face areas - area-universality. PhD thesis, Technische Universität Berlin (2018)
18. Nusrat, S., Kobourov, S.: The state of the art in cartograms. In: Computer Graphics Forum, vol. 35, pp. 619–642. Wiley Online Library (2016). https://doi.org/10.1111/cgf.12932
19. Ringel, G.: Equiareal graphs. In: Bodendiek, R. (ed.) Contemporary Methods in Graph Theory, pp. 503–505. BI Wissenschaftsverlag Mannheim (1990). In honour of Prof. Dr. K. Wagner
20. Stein, S.K.: Convex maps. Proc. Am. Math. Soc **2**(3), 464 (1951). https://doi.org/10.1090/S0002-9939-1951-0041425-5
21. Thomassen, C.: Plane cubic graphs with prescribed face areas. Comb. Probab. Comput. **1**(4), 371–381 (1992). https://doi.org/10.1017/S0963548300000407
22. Wagner, K.: Bemerkungen zum Vierfarbenproblem. Jahresbericht der Deutschen Mathematiker-Vereinigung **46**, 26–32 (1936). https://gdz.sub.uni-goettingen.de/id/PPN37721857X_0046
23. Wimer, S., Koren, I., Cederbaum, I.: Floorplans, planar graphs, and layouts. IEEE Trans. Circuits Syst. **35**, 267–278 (1988). https://doi.org/10.1109/31.1739