Chapter 7

Hardy–Rellich Inequalities and Fundamental Solutions

In this chapter, we describe the Hardy and other inequalities on stratified groups with the $\mathcal{L}$-gauge weights. The appearance of such weights has been discussed in the beginning of Chapter 6. The literature on inequalities with such weights is rather substantial. Apart from describing new results and methods we will be making relevant references to the results existing in the earlier literature.

While horizontal estimates in Chapter 6 can be established on general stratified groups, the picture is not so complete if one is working with the $\mathcal{L}$-gauge weights. We recall that the $\mathcal{L}$-gauge $d(x)$ is a homogeneous quasi-norm arising from the fundamental solution of the sub-Laplacian $\mathcal{L}$ by the condition (1.75), namely, that $d(x)^{2-Q}$ is a constant multiple of Folland's [Fol75] fundamental solution of the sub-Laplacian $\mathcal{L}$, with Q being the homogeneous dimension of the stratified group $\mathbb{G}$.

Using the $\mathcal{L}$-gauge as a weight, the classical Hardy inequality on the Euclidean space $\mathbb{R}^n$,

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|\phi(x)|^p}{|x|_E^p} dx \leq \int_{\mathbb{R}^n} |\nabla \phi(x)|^p dx, \tag{7.1}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$ if $1 \leq p < n$, and for all $\phi \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ if $n < p < \infty$, is replaced by inequalities involving powers of $d(x)$. For instance, D'Ambrosio in [D'A05] and Goldstein and Kombe in [GK08] established the following L^p-Hardy type inequality on polarizable Carnot groups $\mathbb{G}$,

$$\left(\frac{Q-p}{p}\right)^p \int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p} |\phi|^p dx \leq \int_{\mathbb{G}} |\nabla_H \phi|^p dx, \tag{7.2}$$

for all $\phi \in C_0^\infty(\mathbb{G} \backslash \{0\})$, provided that $Q \geq 3$ and $1 < p < Q$. Here, as usual, Q is the homogeneous dimension of $\mathbb{G}$.

In such inequalities the explicit formula (1.103) relating the $\mathcal{L}$-gauge to the fundamental solution of the p-sub-Laplacian often plays an important role. In

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the case $p = 2$, since the p-sub-Laplacian is the usual sub-Laplacian, formula (1.103) just reduces to the definition of the fundamental solution holding on general stratified groups. Consequently, in the case $p = 2$ the version of (7.2) holds on any stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$ and all $\phi \in C_0^\infty(\mathbb{G}\backslash\{0\})$:

$$\left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|\nabla_H d|^2}{d^2}|\phi|^2 dx \leq \int_{\mathbb{G}} |\nabla_H \phi|^2 dx. \tag{7.3}$$

It was shown in [Kom10] (see also [GK08]) that the Hardy inequality (7.3) on general stratified groups of homogeneous dimension $Q \geq 3$ also holds in its weighted form

$$\int_{\mathbb{G}} d^\alpha |\nabla_H \phi|^2 dx \geq \left(\frac{Q+\alpha-2}{2}\right) \int_{\mathbb{G}} d^\alpha \frac{|\nabla_H d|^2}{d^2}|\phi|^2 dx, \quad \alpha > 2-Q, \tag{7.4}$$

for all $\phi \in C_0^\infty(\mathbb{G}\backslash\{0\})$. It can be noted that the constants appearing in (7.2) and (7.4) are sharp but are never achieved.

The aim of this chapter is to discuss these and other related inequalities, and their further extensions. In Remark 7.1.2 we provide a more extensive historical perspective on these inequalities.

7.1 Weighted L^p-Hardy inequalities

We start with a general version of a weighted Hardy inequality on general stratified groups. Subsequently, in the following sections, we consider further extensions from the point of view of the weights in the setting of polarizable Carnot groups. Here we will be mostly working with the $\mathcal{L}$-gauge defined in (1.75), namely, with

$$d(x) := \begin{cases} \varepsilon(x)^{\frac{1}{2-Q}}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases} \tag{7.5}$$

where ε is the fundamental solution of the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$.

Theorem 7.1.1 (Weighted L^p-Hardy inequalities with $\mathcal{L}$-gauge)**.** *Let $\mathbb{G}$ be a stratified group of homogeneous dimension $Q \geq 3$. Let $\alpha \in \mathbb{R}$ and let $1 < p < Q - \alpha$. Then for all complex-valued functions $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\int_{\mathbb{G}} \frac{1}{d^\alpha |\nabla_H d|^{p-2}} |\nabla_H u|^p dx \geq \left(\frac{Q-p-\alpha}{p}\right)^p \int_{\mathbb{G}} \frac{|\nabla_H d|^2}{d^{\alpha+p}} |u|^p dx, \tag{7.6}$$

and the constant $\left(\frac{Q-p-\alpha}{p}\right)^p$ in inequality (7.6) is sharp.

Remark 7.1.2.

1. The inequality (7.6) in the setting of stratified groups of different types has a long history. For $p = 2$ and $\alpha = 0$, on the Heisenberg group it was proved by Garofalo and Lanconelli in [GL90] with an explicit expression for d being the Koranyi norm. Still on the Heisenberg group, it was shown in [NZW01] for $\alpha = 0$ and $1 < p < Q$. The weighted inequality for $p = 2$ was obtained by Kombe in [Kom10]. Different further unweighted versions for $p \neq 2$ in the settings related to those of polarizable Carnot groups were obtained by D'Ambrosio [D'A05], Goldstein and Kombe [GK08], and Danielli, Garofalo and Phuc [DGP11]. The weighted L^p inequality on general stratified groups by using a special class of weighted p-sub-Laplacians and the corresponding fundamental solutions was obtained by Jin and Shen [JS11]. More recently, in [Lia13] Lian has also obtained a similar result but with a sharp constant. In the proof below we follow Lian's arguments, as well as Lian's proof [Lia13] of Theorem 7.2.1.
2. Different formulations are also possible in the setting of polarizable Carnot groups. We present them in Theorem 7.1.3 and in Theorem 7.2.2 following [Kom10, Theorem 3.1] and [Kom10, Theorem 4.1] or [GKY17, Corollary 3.1], respectively. Further improved remainder terms have been also analysed in [Kom10].
3. There are other versions of Hardy inequalities that one can find in the literature, such as multi-particle inequalities (see, e.g., [Lun15] and references therein) or Besov space versions of Hardy inequalities, see [BCG06] and [BFKG12] for the settings of the Heisenberg group and on graded groups, respectively.

Proof of Theorem 7.1.1. Since for some constant C_Q we have that $C_Q d^{2-Q}$ is the fundamental solution of $\mathcal{L}$, for all $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$, it follows that

$$\int_{\mathbb{G}} \langle \nabla_H d^{2-Q}, \nabla_H u \rangle dx = -C_Q^{-1} u(0) = 0. \tag{7.7}$$

For $\epsilon > 0$, let us define

$$u_\epsilon := (|u|^2 + \epsilon^2)^{p/2} - \epsilon^p.$$

Then $u_\epsilon \geq 0$, $u_\epsilon \in C_0^\infty(\mathbb{G}\backslash\{0\})$, and it has the same support as u. Replacing u by $u_\epsilon d^{Q-p-\alpha}$ in inequality (7.7), we obtain

$$\int_{\mathbb{G}} \frac{\langle \nabla_H d, \nabla_H u_\epsilon \rangle}{d^{p+\alpha-1}} dx + (Q-p-\alpha) \int_{\mathbb{G}} \frac{u_\epsilon}{d^{p+\alpha}} |\nabla_H d|^2 dx = 0.$$

Then we can estimate

$$(Q-p-\alpha) \int_{\mathbb{G}} \frac{u_\epsilon}{d^{p+\alpha}} |\nabla_H d|^2 dx = -p \int_{\mathbb{G}} (|u|^2 + \epsilon^2)^{(p-2)/2} u \langle \nabla_H u, \nabla_H d \rangle \frac{1}{d^{p+\alpha-1}} dx$$

$$\leq p\int_{\mathbb{G}}\frac{(|u|^2+\epsilon^2)^{(p-2)/2}|u||\nabla_H u||\nabla_H d|}{d^{p+\alpha-1}}dx$$
$$\leq p\int_{\mathbb{G}}\frac{(|u|^2+\epsilon^2)^{(p-1)/2}|\nabla_H u||\nabla_H d|}{d^{p+\alpha-1}}dx.$$

Letting $\epsilon\to 0$, by the dominated convergence theorem we obtain the estimate

$$(Q-p-\alpha)\int_{\mathbb{G}}\frac{|u|^p}{d^{p+\alpha}}|\nabla_H d|^2 dx\leq p\int_{\mathbb{G}}\frac{|u|^{p-1}|\nabla_H u||\nabla_H d|}{d^{p+\alpha-1}}dx.$$

By Hölder's inequality, this implies

$$(Q-p-\alpha)\int_{\mathbb{G}}\frac{|u|^p}{d^{p+\alpha}}|\nabla_H d|^2dx\leq p\left(\int_{\mathbb{G}}\frac{|u|^p}{d^{p+\alpha}}|\nabla_H d|^2dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{G}}\frac{|\nabla_H u|^p}{|\nabla_H d|^{p-2}d^{\alpha}}dx\right)^{\frac{1}{p}},$$

which gives (7.6).

Let us now show that the constant $\left(\frac{Q-p-\alpha}{p}\right)^p$ in inequality (7.6) is sharp. Let $f\in C_0^\infty(0,+\infty)$. Since $f(d)\in C_0^\infty(\mathbb{G}\backslash\{0\})$, using the polar decomposition in Proposition 1.2.10 with respect to d, we have

$$\begin{aligned}
&\inf_{u\in C_0^\infty(\mathbb{G}\backslash\{0\})\backslash\{0\}}\frac{\int_{\mathbb{G}}\frac{|\nabla_H u|^p}{|\nabla_H d|^{p-2}d^{\alpha}}dx}{\int_{\mathbb{G}}\frac{|u|^p}{d^{p+\alpha}}|\nabla_H d|^2dx}\\
&\leq\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_{\mathbb{G}}\frac{|\nabla_H f|^p}{|\nabla_H d|^{p-2}d^{\alpha}}dx}{\int_{\mathbb{G}}\frac{|f|^p}{d^{p+\alpha}}|\nabla_H d|^2dx}\\
&=\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_0^\infty|f'(d)|^p d^{Q-\alpha-1}dd\cdot\int_{\wp}|\nabla d|^2d\sigma}{\int_0^\infty|f(d)|^pd^{Q-p-\alpha-1}dd\cdot\int_{\wp}|\nabla d|^2d\sigma}\\
&=\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_0^\infty|f'(d)|^pd^{Q-\alpha-1}dd}{\int_0^\infty|f(d)|^pd^{Q-p-\alpha-1}dd}\\
&=\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_{\mathbb{R}^Q}\frac{|\nabla f(|x|)|^p}{|x|^{\alpha}}dx}{\int_{\mathbb{R}^Q}\frac{|f(|x|)|^p}{|x|^{p+\alpha}}dx}=\left(\frac{Q-p-\alpha}{p}\right)^p,
\end{aligned}$$

where we abuse the notation by writing dd for the integration with respect to the radial variable determined by d. The last equality follows from the fact that the Euclidean weighted Hardy inequalities

$$\int_{\mathbb{R}^Q}\frac{|\nabla f(|x|)|^p}{|x|^{\alpha}}dx\geq\left(\frac{Q-p-\alpha}{p}\right)^p\int_{\mathbb{R}^Q}\frac{|f(|x|)|^p}{|x|^{p+\alpha}}dx$$

hold for all $f\in C_0^\infty(\mathbb{R}^Q\backslash\{0\})$ and the constant here sharp and is attained as a limit of radial functions, as it was shown by Davies and Hinz [DH98]. This completes the proof. $\square$

Another type of Hardy inequality is also known on polarizable Carnot groups. We give it next following [Kom10, Theorem 3.1] and its proof.

Theorem 7.1.3 (Another type of weighted Hardy inequalities with $\mathcal{L}$-gauge)**.** *Let $\mathbb{G}$ be a polarizable Carnot group of homogeneous dimension $Q \geq 3$. Let $1 < p < Q$ and let $\alpha \in \mathbb{R}$ be such that $\alpha > -Q$. Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the inequality*

$$\int_{\mathbb{G}} d^{\alpha+p} \frac{|\nabla_H d \cdot \nabla_H f|^p}{|\nabla_H d|^{2p}} dx \geq \left(\frac{Q+\alpha}{p}\right)^p \int_{\mathbb{G}} d^\alpha |f|^p dx, \tag{7.8}$$

where the constant $\left(\frac{Q+\alpha}{p}\right)^p$ is sharp.

Proof of Theorem 7.1.3. Let us first recall the formula (1.105), that is,

$$\nabla_H \left(\frac{d}{|\nabla_H d|^2} \nabla_H d\right) = Q$$

in $\mathbb{G}\backslash\mathcal{Z}$, where $\mathcal{Z} := \{0\} \bigcup \{x \in \mathbb{G}\backslash\{0\} : \nabla_H d = 0\}$ has Haar measure zero, and $\nabla_H d \neq 0$ for a.e. $x \in \mathbb{G}$. By using this formula as well as Green's formula (see Theorem 1.4.6) we obtain

$$(Q+\alpha) \int_{\mathbb{G}} d^\alpha |f|^p dx = -p \int_{\mathbb{G}} \frac{|f|^{p-2} f d^{\alpha+1}}{|\nabla_H d|^2} \nabla_H d \cdot \nabla_H f dx.$$

Moreover, by using Hölder's and Young's inequalities we can estimate

$$\begin{aligned}(Q+\alpha) \int_{\mathbb{G}} d^\alpha |f|^p dx &\leq p \left(\int_{\mathbb{G}} d^\alpha |f|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{G}} \frac{d^{\alpha+p} |\nabla_H d \cdot \nabla_H f|^p}{|\nabla_H d|^{2p}} dx\right)^{1/p} \\ &\leq (p-1)\epsilon^{-p/(p-1)} \int_{\mathbb{G}} d^\alpha |f|^p dx + \epsilon^p \int_{\mathbb{G}} \frac{d^{\alpha+p} |\nabla_H d \cdot \nabla_H f|^p}{|\nabla_H d|^{2p}} dx\end{aligned}$$

for any $\epsilon > 0$, that is,

$$\epsilon^{-p}(Q+\alpha-(p-1)\epsilon^{-p/(p-1)}) \int_{\mathbb{G}} d^\alpha |f|^p dx \leq \int_{\mathbb{G}} \frac{d^{\alpha+p} |\nabla_H d \cdot \nabla_H f|^p}{|\nabla_H d|^{2p}} dx.$$

Since the function $\epsilon \to \epsilon^{-p}(Q+\alpha-(p-1)\epsilon^{-p/(p-1)})$ attains its maximum $\left(\frac{Q+\alpha}{p}\right)^p$ at $\epsilon^{p/(p-1)} = \frac{p}{Q+\alpha}$, we obtain the inequality

$$\left(\frac{Q+\alpha}{p}\right)^p \int_{\mathbb{G}} d^\alpha |f|^p dx \leq \int_{\mathbb{G}} \frac{d^{\alpha+p} |\nabla_H d \cdot \nabla_H f|^p}{|\nabla_H d|^{2p}} dx.$$

Now let us show that $\left(\frac{Q+\alpha}{p}\right)^p$ is the best constant, that is, we show that we have

$$C_H := \inf_{0 \neq f \in C_0^\infty(\mathbb{G})} \frac{\int_{\mathbb{G}} \frac{d^{\alpha+p} |\nabla_H d \cdot \nabla_H f|^p}{|\nabla_H d|^{2p}} dx}{\int_{\mathbb{G}} d^\alpha |f|^p dx} = \left(\frac{Q+\alpha}{p}\right)^p.$$

Obviously, one has

$$\left(\frac{Q+\alpha}{p}\right)^p \leq \frac{\int_{\mathbb{G}} \frac{d^{\alpha+p}|\nabla_H d\cdot\nabla_H f|^p}{|\nabla_H d|^{2p}}dx}{\int_{\mathbb{G}} d^\alpha |f|^p dx}$$

for all $f\in C_0^\infty(\mathbb{G}\setminus\{0\})$, that is, $\left(\frac{Q+\alpha}{p}\right)^p \leq C_H$. So, we need to show the converse, namely, that $C_H \geq \left(\frac{Q+\alpha}{p}\right)^p$. For this, consider the following family of d-radial functions

$$f_\epsilon(d) := \begin{cases} d^{\frac{Q+\alpha}{p}+\epsilon} & \text{if} \quad d\in[0,1], \\ d^{-\left(\frac{Q+\alpha}{p}+\epsilon\right)} & \text{if} \quad d>1, \end{cases}$$

with $\epsilon>0$. Note that $f_\epsilon(d)$ can be also approximated by smooth functions with compact support in $\mathbb{G}$. We can also readily calculate that

$$\frac{d^{\alpha+p}|\nabla_H d\cdot\nabla_H f_\epsilon|^p}{|\nabla_H d|^{2p}} = \begin{cases} \left(\frac{Q+\alpha}{p}+\epsilon\right)^p d^{Q+2\alpha-p\epsilon} & \text{if} \quad d\in[0,1], \\ \left(\frac{Q+\alpha}{p}+\epsilon\right)^p d^{-Q-\epsilon} & \text{if} \quad d>1. \end{cases}$$

Denoting by $\mathbb{B}_1=\{x\in\mathbb{G}: d(x)\leq 1\}$ the unit d-ball, we have

$$\int_{\mathbb{G}} d^\alpha |f_\epsilon|^p dx = \int_{\mathbb{B}_1} d^{Q+2\alpha-p\epsilon}dx + \int_{\mathbb{G}\setminus\mathbb{B}_1} d^{-Q-\epsilon}dx.$$

For every $\epsilon>0$, the weights $d^{Q+2\alpha+p\epsilon}$ and $d^{-Q-p\epsilon}$ are integrable at 0 and ∞, respectively. Thus, the integral $\int_{\mathbb{G}} d^\alpha|f_\epsilon|^p dx$ is finite. Therefore, we get

$$\begin{aligned}\left(\frac{Q+\alpha}{p}+\epsilon\right)^p \int_{\mathbb{G}} d^\alpha|f_\epsilon|^p dx &= \left(\frac{Q+\alpha}{p}+\epsilon\right)^p \left[\int_{\mathbb{B}_1} d^{Q+2\alpha-p\epsilon}dx + \int_{\mathbb{G}\setminus\mathbb{B}_1} d^{-Q-\epsilon}dx\right] \\ &= \int_{\mathbb{G}} d^{\alpha+p}\frac{|\nabla_H d\cdot\nabla_H f|^p}{|\nabla_H d|^{2p}}dx.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\left(\frac{Q+\alpha}{p}+\epsilon\right)^p \int_{\mathbb{G}} d^{\alpha+p}\frac{|\nabla_H d\cdot\nabla_H f|^p}{|\nabla_H d|^{2p}}dx &\geq \left(\frac{Q+\alpha}{p}+\epsilon\right)^p \int_{\mathbb{G}} d^\alpha|f_\epsilon|^p dx \\ &= \int_{\mathbb{G}} d^{\alpha+p}\frac{|\nabla_H d\cdot\nabla_H f|^p}{|\nabla_H d|^{2p}}dx.\end{aligned}$$

That is, $C_H \leq \left(\frac{Q+\alpha}{p}+\epsilon\right)^p$ and letting $\epsilon\to 0$ we obtain $\left(\frac{Q+\alpha}{p}\right)^p \leq C_H$. This yields $C_H = \left(\frac{Q+\alpha}{p}\right)^p$, showing the sharpness of the constant. $\square$

7.2 Weighted L^p-Rellich inequalities

In this section we discuss the Rellich inequality with weights given in terms of the $\mathcal{L}$-gauge.

Theorem 7.2.1 (Weighted L^p-Rellich inequalities with $\mathcal{L}$-gauge)**.** *Let $\mathbb{G}$ be a stratified group of homogeneous dimension $Q \geq 3$. Let $\alpha \in \mathbb{R}$ and let $1 < p < (Q-\alpha)/2$. Then for all $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|\nabla_H d|^{2(p-1)} d^\alpha} dx \geq \left(\frac{[(p-1)Q+\alpha](Q-2p-\alpha)}{p^2}\right)^p \int_{\mathbb{G}} \frac{|\nabla_H d|^2}{d^{2p+\alpha}} |u|^p dx, \quad (7.9)$$

where the constant $\left(\frac{[(p-1)Q+\alpha](Q-2p-\alpha)}{p^2}\right)^p$ in inequality (7.9) is sharp.

Proof of Theorem 7.2.1. For $\epsilon > 0$, let us set

$$u_\epsilon := (|u|^2+\epsilon^2)^{p/2} - \epsilon^p \qquad \text{and} \qquad \omega_\epsilon := (|u|^2+\epsilon^2)^{p/4} - \epsilon^{p/2}.$$

Then we can calculate

$$\begin{aligned}
\mathcal{L}u_\epsilon &= p(|u|^2+\epsilon^2)^{p/2-1}|\nabla_H u|^2 + p(p-2)(|u|^2+\epsilon^2)^{p/2-2}|u|^2|\nabla_H u|^2 \\
&\quad + p(|u|^2+\epsilon^2)^{p/2-1} u\mathcal{L}u \\
&\geq p(p-1)(|u|^2+\epsilon^2)^{p/2-2}|u|^2|\nabla_H u|^2 + p(|u|^2+\epsilon^2)^{p/2-1}u\mathcal{L}u \\
&= \frac{4(p-1)}{p}|\nabla_H \omega_\epsilon|^2 + p(|u|^2+\epsilon^2)^{p/2-1}u\mathcal{L}u.
\end{aligned}$$

Therefore, we have the estimate

$$-p\int_{\mathbb{G}} \frac{(|u|^2+\epsilon^2)^{p/2-1}u\mathcal{L}u}{d^{\alpha+2(p-1)}}dx \geq \frac{4(p-1)}{p}\int_{\mathbb{G}} \frac{|\nabla_H\omega_\epsilon|^2}{d^{\alpha+2(p-1)}}dx - \int_{\mathbb{G}} \frac{\mathcal{L}u_\epsilon}{d^{\alpha+2(p-1)}}dx.$$

The integration by parts in the last term, using (1.78), yields

$$-\int_{\mathbb{G}} \frac{\mathcal{L}u_\epsilon}{d^{\alpha+2(p-1)}}dx = (\alpha+2p-2)(Q-\alpha-2p)\int_{\mathbb{G}} \frac{u_\epsilon}{d^{\alpha+2p}}|\nabla_H d|^2 dx.$$

Using this and Theorem 7.1.1 we obtain

$$\begin{aligned}
-p\int_{\mathbb{G}} \frac{(|u|^2+\epsilon^2)^{p/2-1}u\mathcal{L}u}{d^{\alpha+2(p-1)}}dx &\geq \frac{4(p-1)}{p}\int_{\mathbb{G}} \frac{|\nabla_H\omega_\epsilon|^2}{d^{\alpha+2(p-1)}}dx - \int_{\mathbb{G}} \frac{\mathcal{L}u_\epsilon}{d^{\alpha+2(p-1)}}dx \\
&\geq \frac{(p-1)(Q-2p-\alpha)^2}{p}\int_{\mathbb{G}} \frac{\omega_\epsilon^2}{d^{\alpha+2p}}|\nabla_H d|^2 dx \\
&\quad + (\alpha+2p-2)(Q-\alpha-2p)\int_{\mathbb{G}} \frac{u_\epsilon}{d^{\alpha+2p}}|\nabla_H d|^2 dx.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{(p-1)(Q-2p-\alpha)^2}{p}\int_{\mathbb{G}}\frac{\omega_\epsilon^2}{d^{\alpha+2p}}|\nabla_H d|^2 dx\\
&\quad+(\alpha+2p-2)(Q-\alpha-2p)\int_{\mathbb{G}}\frac{u_\epsilon}{d^{\alpha+2p}}|\nabla_H d|^2 dx\\
&\qquad\le p\int_{\mathbb{G}}\frac{(|u|^2+\epsilon^2)^{p/2-1}|u||\mathcal{L}u|}{d^{\alpha+2(p-1)}}dx\le p\int_{\mathbb{G}}\frac{(|u|^2+\epsilon^2)^{(p-1)/2}|\mathcal{L}u|}{d^{\alpha+2(p-1)}}dx.
\end{aligned}$$

Letting $\epsilon\to 0+$, the dominated convergence theorem implies that

$$\frac{(Q-2p-\alpha)\left((p-1)Q+\alpha\right)}{p}\int_{\mathbb{G}}\frac{|u|^p}{d^{\alpha+2p}}|\nabla_H d|^2 dx\le p\int_{\mathbb{G}}\frac{|u|^{p-1}|\mathcal{L}u|}{d^{\alpha+2(p-1)}}dx.$$

By Hölder's inequality we can estimate

$$\begin{aligned}
&\frac{(Q-2p-\alpha)\left((p-1)Q+\alpha\right)}{p}\int_{\mathbb{G}}\frac{|u|^p}{d^{\alpha+2p}}|\nabla_H d|^2 dx\\
&\qquad\le p\left(\int_{\mathbb{G}}\frac{|u|^p}{d^{\alpha+2p}}|\nabla_H d|^2 dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}u|^p}{|\nabla_H d|^{2(p-1)}d^{\alpha}}dx\right)^{\frac{1}{p}},
\end{aligned}$$

which implies inequality (7.9).

The argument for the sharpness of the constant $\left(\frac{(Q-2p-\alpha)((p-1)Q+\alpha)}{p^2}\right)^p$ in (7.9) is similar to the sharpness argument in the proof of Theorem 7.1.1 (with the similar explanation for the notation dd). Namely, for functions $f\in C_0^\infty(0,+\infty)$, by using Proposition 1.2.10 we can estimate

$$\begin{aligned}
&\inf_{u\in C_0^\infty(\mathbb{G}\backslash\{0\})\backslash\{0\}}\frac{\int_{\mathbb{G}}\frac{|\mathcal{L}u|^p}{|\nabla_H d|^{2(p-1)}d^{\alpha}}dx}{\int_{\mathbb{G}}\frac{|u|^p}{d^{\alpha+2p}}|\nabla_H d|^2 dx}\\
&\quad\le\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_{\mathbb{G}}\frac{|\mathcal{L}f(d)|^p}{|\nabla_H d|^{2(p-1)}d^{\alpha}}dx}{\int_{\mathbb{G}}\frac{|f(d)|^p}{d^{\alpha+2p}}|\nabla_H d|^2 dx}\\
&\quad=\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_0^\infty|f''(d)+(Q-1)f'(d)/d|^p d^{Q-\alpha-1}dd\cdot\int_{\wp}|\nabla d|^2 d\sigma}{\int_0^\infty|f(d)|^p d^{Q-2p-\alpha-1}dd\cdot\int_{\wp}|\nabla d|^2 d\sigma}\\
&\quad=\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_0^\infty|f''(d)+(Q-1)f'(d)/d|^p d^{Q-\alpha-1}dd}{\int_0^\infty|f(d)|^p d^{Q-2p-\alpha-1}dd}\\
&\quad=\inf_{f\in C_0^\infty(0,+\infty)\backslash\{0\}}\frac{\int_{\mathbb{R}^Q}\frac{|\mathcal{L}f(|x|)|^p}{|x|^{\alpha}}dx}{\int_{\mathbb{R}^Q}\frac{|f(|x|)|^p}{|x|^{2p+\alpha}}dx}\\
&\quad=\left(\frac{((p-1)Q+\alpha)(Q-2p-\alpha)}{p^2}\right)^p,
\end{aligned}$$

where the last equality follows from the weighted Rellich inequalities

$$\int_{\mathbb{R}^Q} \frac{|\mathcal{L}f(|x|)|^p}{|x|^\alpha} dx \geq \left(\frac{((p-1)Q+\alpha)(Q-2p-\alpha)}{p^2} \right)^p \int_{\mathbb{R}^Q} \frac{|f(|x|)|^p}{|x|^{2p+\alpha}} dx$$

for all $f \in C_0^\infty(\mathbb{R}^Q \backslash \{0\})$, and the fact that the constant here is sharp and is attained in the limit of radial functions, [DH98]. This completes the proof. $\square$

Another Rellich inequality is also possible, see Remark 7.1.2, Part 2. We present it in the next statement following [Kom10, Theorem 4.1] and its proof.

Theorem 7.2.2 (Another type of weighted L^2-Rellich inequalities with $\mathcal{L}$-gauge)**.** *Let $\mathbb{G}$ be a stratified group of homogeneous dimension $Q \geq 3$, with the homogeneous $\mathcal{L}$-gauge norm d on $\mathbb{G}$. Let $\alpha \in \mathbb{R}$ be such that $Q + \alpha - 4 > 0$. Then for all $f \in C_0^\infty(\mathbb{G} \backslash \{0\})$ we have*

$$\int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_H d|^2} |\mathcal{L}f|^2 dx \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{G}} d^\alpha \frac{|\nabla_H d|^2}{d^4} |f|^2 dx, \tag{7.10}$$

where the constant $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is sharp.

Proof of Theorem 7.2.2. Recalling formula (7.5) for the $\mathcal{L}$-gauge, a direct calculation gives that

$$\mathcal{L}d^{\alpha-2} = (Q+\alpha-4)(\alpha-2)d^{\alpha-4}|\nabla_H d|^2 + \frac{\alpha-2}{2-Q} d^{Q+\alpha-4}\mathcal{L}\varepsilon. \tag{7.11}$$

As before, we can assume without loss of generality that f is real-valued. Then (7.11) implies

$$\int_{\mathbb{G}} f^2 \mathcal{L}d^{\alpha-2} dx = \int_{\mathbb{G}} d^{\alpha-2}(2f\mathcal{L}f + 2|\nabla_H f|^2) dx.$$

On the other hand, since ε is the fundamental solution of $\mathcal{L}$ we have

$$\int_{\mathbb{G}} f^2 \mathcal{L}d^{\alpha-2} dx = (Q+\alpha-4)(\alpha-2) \int_{\mathbb{G}} d^{\alpha-4}|\nabla_H d|^2 f^2 dx,$$

with $Q + \alpha - 4 > 0$. Thus, we have

$$\begin{aligned} (Q+\alpha-4)(\alpha-2) \int_{\mathbb{G}} d^{\alpha-4}|\nabla_H d|^2 f^2 dx - 2\int_{\mathbb{G}} d^{\alpha-2} f\mathcal{L}f dx \\ = 2\int_{\mathbb{G}} d^{\alpha-2}|\nabla_H f|^2 dx. \end{aligned} \tag{7.12}$$

Further, using the following weighted Hardy inequality (see Corollary 7.3.2, Part 1, related to Theorem 7.1.1 for $p = 2$)

$$\left(\frac{Q+\alpha-p}{p} \right)^p \int_{\mathbb{G}} d^\alpha \frac{|\nabla_H d|^p}{d^p} |f|^p dx \leq \int_{\mathbb{G}} d^\alpha |\nabla_H f|^p dx,$$

we arrive at

$$2\left(\frac{Q+\alpha-4}{2}\right)^2\int_{\mathbb{G}} d^{\alpha-4}|\nabla_H d|^2 f^2 dx$$
$$\leq (Q+\alpha-4)(\alpha-2)\int_{\mathbb{G}} d^{\alpha-4}|\nabla_H d|^2 f^2 dx - 2\int_{\mathbb{G}} d^{\alpha-2} f\mathcal{L}f dx.$$

It follows that

$$\left(\frac{Q+\alpha-4}{2}\right)\left(\frac{Q-\alpha}{2}\right)\int_{\mathbb{G}} d^{\alpha-4}|\nabla_H d|^2 f^2 dx \leq -\int_{\mathbb{G}} d^{\alpha-2} f\mathcal{L}f dx. \qquad (7.13)$$

By the Cauchy–Schwarz inequality we have

$$-\int_{\mathbb{G}} d^{\alpha-2} f\mathcal{L}f dx \leq \left(\int_{\mathbb{G}} d^{\alpha-4}|\nabla_H d|^2 f^2 dx\right)^{1/2}\left(\int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|\nabla_H d|^2} d^\alpha dx\right)^{1/2}. \qquad (7.14)$$

Now combination of the inequalities (7.14) and (7.13) yields (7.10).

Let us now show that the constant $C_R = \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is sharp, that is, we have the equality

$$C_R := \inf_{0\neq f\in C_0^\infty(\mathbb{G})} \frac{\int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_H d|^2}|\mathcal{L}f|^2 dx}{\int_{\mathbb{G}} d^\alpha \frac{|\nabla_H d|^2}{d^4} f^2 dx} = \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}.$$

Obviously, we have

$$\frac{\int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_H d|^2}|\mathcal{L}f|^2 dx}{\int_{\mathbb{G}} d^\alpha \frac{|\nabla_H d|^2}{d^4} f^2 dx} \leq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16},$$

that is, $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \leq C_R$. So, we need to show the converse, namely, that $C_R \leq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$. To do this we define a family of d-radial functions by

$$f_\epsilon(d) := \begin{cases} \left(\frac{Q+\alpha-4}{2}+\epsilon\right)^2 |\nabla_H d|^4 \frac{(Q-1)^2}{d^2} & \text{if} \quad d\leq 1, \\ \left(\frac{Q+\alpha-4}{2}+\epsilon\right)^2\left(\frac{Q+\alpha-4}{2}-\epsilon\right)^2 d^{-Q-\alpha-2\epsilon}|\nabla_H d|^4 & \text{if} \quad d>1, \end{cases}$$

for some $\epsilon > 0$. Denoting by $\mathbb{B}_1 = \{x\in\mathbb{G} : d(x)\leq 1\}$ the unit d-ball, we have

$$\int_{\mathbb{G}} d^\alpha \frac{|\mathcal{L}f_\epsilon|^2}{|\nabla_H d|^2} dx = A\int_{\mathbb{B}_1} d^{\alpha-2}|\nabla_H d|^2 dx + B\int_{\mathbb{G}\setminus\mathbb{B}_1} d^{-Q-2\epsilon}|\nabla_H d|^2 dx,$$

where

$$A = (Q-1)^2\left(\frac{Q+\alpha-4}{2}+\epsilon\right)^2$$

and

$$B = \left(\frac{Q+\alpha-4}{2}+\epsilon\right)^2\left(\frac{Q+\alpha-4}{2}-\epsilon\right)^2.$$

Since $|\nabla_H d|$ is uniformly bounded and $Q+\alpha-4>0$, the integral $\int_{\mathbb{B}_1} d^{\alpha-2}|\nabla_H d|^2 dx$ is finite. It implies that we have

$$\int_{\mathbb{G}} d^{\alpha}\frac{|\mathcal{L}f_\epsilon|^2}{|\nabla_H d|^2}dx = B\int_{\mathbb{G}\setminus\mathbb{B}_1} d^{-Q-2\epsilon}|\nabla_H d|^2 dx + O(1).$$

Moreover, we have

$$\int_{\mathbb{G}} d^{\alpha}\frac{|\nabla_H d|^2}{d^4}f_\epsilon^2 dx = \int_{\mathbb{B}_1} d^{\alpha}\frac{|\nabla_H d|^2}{d^4}f_\epsilon^2 dx + \int_{\mathbb{G}\setminus\mathbb{B}_1} d^{\alpha}\frac{|\nabla_H d|^2}{d^4}f_\epsilon^2 dx.$$

Since the first integral is finite we obtain

$$\int_{\mathbb{G}} d^{\alpha}\frac{|\nabla_H d|^2}{d^4}f_\epsilon^2 dx = \int_{\mathbb{G}\setminus\mathbb{B}_1} d^{-Q-2\epsilon}|\nabla_H d|^2 dx + O(1).$$

Taking $\epsilon \to 0$ and noting that

$$\int_{\mathbb{G}\setminus\mathbb{B}_1} d^{-Q-2\epsilon}|\nabla_H d|^2 dx \to \infty,$$

we arrive at

$$\frac{\int_{\mathbb{G}} \frac{d^{\alpha}}{|\nabla_H d|^2}|\mathcal{L}f_\epsilon|^2 dx}{\int_{\mathbb{G}} d^{\alpha}\frac{|\nabla_H d|^2}{d^4}f_\epsilon^2 dx} \leq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}.$$

This means that $C_R = \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$, so that the constant is sharp. □

7.3 Two-weight Hardy inequalities and uncertainty principles

In this section we consider Hardy inequalities with more general weights, presenting the approach of Goldstein, Kombe and Yener [GKY17]. This can be also extended further to Rellich inequalities, see [GKY18]. Other types of two-weight inequalities are known in the classical Euclidean setting, see, e.g., [GM11], and a more extensive exposition in [GM13].

Another general two-weight inequality on general homogeneous groups was given in Theorem 2.1.14, without making any assumptions on the weights ϕ, ψ there. However, in the following result, the weights V and W will be assumed to satisfy relation (7.15).

Theorem 7.3.1 (Two-weight L^p-Hardy inequality)**.** *Let $\mathbb{G}$ be a stratified group. Let $V \in C^1(\mathbb{G})$ and $W \in L^1_{\rm loc}(\mathbb{G})$ be non-negative functions, and let $\Phi \in C^\infty(\mathbb{G})$ be a positive function such that*

$$-\nabla_H \cdot (V(x)|\nabla_H \Phi|^{p-2}\nabla_H \Phi) \geq W(x)\Phi^{p-1} \tag{7.15}$$

holds almost everywhere.

Then there exists a positive constant $c_p > 0$ depending only on p such that for all $\phi \in C_0^\infty(\mathbb{G})$ we have:
if $p \geq 2$, then

$$\int_{\mathbb{G}} V(x)|\nabla_H \phi|^p dx \geq \int_{\mathbb{G}} W(x)|\phi|^p dx + c_p \int_{\mathbb{G}} V(x) \left|\nabla_H \frac{\phi}{\Phi}\right|^p \Phi^p dx, \tag{7.16}$$

and if $1 < p < 2$, then

$$\int_{\mathbb{G}} V(x)|\nabla_H \phi|^p dx \geq \int_{\mathbb{G}} W(x)|\phi|^p dx + c_p \int_{\mathbb{G}} V(x) \frac{\left|\nabla_H \frac{\phi}{\Phi}\right|^2 \Phi^2}{\left(\left|\frac{\phi}{\Phi}\nabla_H \Phi\right| + \left|\nabla_H \frac{\phi}{\Phi}\right| \Phi\right)^{2-p}} dx. \tag{7.17}$$

For $p = 2$ we have the equality in (7.16) with $c_2 = 1$.

Proof of Theorem 7.3.1. For the proof we follow [GKY17], relying on the following inequalities (see, for example, [Lin90, Appendix]): For any $1 < p < \infty$ there exists a positive constant $c_p > 0$ depending only on p such that for all $a, b \in \mathbb{R}^n$ we have

$$|a+b|^p \geq |a|^p + p|a|^{p-2} a \cdot b + c_p |b|^p, \quad \text{for} \quad p \geq 2, \tag{7.18}$$

and

$$|a+b|^p \geq |a|^p + p|a|^{p-2} a \cdot b + c_p \frac{|b|^2}{(|a| + |b|)^{2-p}}, \quad \text{for} \quad 1 < p < 2. \tag{7.19}$$

Let $\varphi := \frac{\phi}{\Phi}$, where $0 < \Phi \in C^\infty(\mathbb{G})$ and $\phi \in C_0^\infty(\mathbb{G})$. Applying the inequality (7.18) with $a = \varphi \nabla_H \Phi$ and $b = \Phi \nabla_H \varphi$, for $p \geq 2$ we get

$$\begin{aligned} |\nabla_H \phi|^p &= |\varphi \nabla_H \Phi + \Phi \nabla_H \varphi|^p \\ &\geq |\nabla_H \Phi|^p |\varphi|^p + \Phi |\nabla_H \Phi|^{p-2} \nabla_H \Phi \cdot \nabla_H (|\varphi|^p) + c_p |\nabla_H \varphi|^p \Phi^p. \end{aligned} \tag{7.20}$$

Multiplying this by $V(x)$ on both sides and integrating by parts yields

$$\begin{aligned} \int_{\mathbb{G}} V(x)|\nabla_H \phi|^p dx &\geq \int_{\mathbb{G}} V(x)|\nabla_H \Phi|^p |\varphi|^p dx + c_p \int_{\mathbb{G}} V(x)|\nabla_H \varphi|^p \Phi^p dx \\ &\quad - \int_{\mathbb{G}} \nabla_H \cdot \left(V(x)\Phi|\nabla_H \Phi|^{p-2}\nabla_H \Phi\right) |\varphi|^p dx \\ &= - \int_{\mathbb{G}} \nabla_H \cdot \left(V(x)\Phi|\nabla_H \Phi|^{p-2}\nabla_H \Phi\right) \Phi |\varphi|^p dx \\ &\quad + c_p \int_{\mathbb{G}} V(x)|\nabla_H \varphi|^p \Phi^p dx. \end{aligned}$$

Consequently, assumption (7.15) implies that

$$\int_{\mathbb{G}} V(x)|\nabla_H \phi|^p dx \geq \int_{\mathbb{G}} W(x)|\varphi|^p \Phi^p dx + c_p \int_{\mathbb{G}} V(x)|\nabla_H \varphi|^p \Phi^p dx.$$

Recalling that $\varphi = \frac{\phi}{\Phi}$ one gets (7.16).

For the case $1 < p < 2$ one can use inequality (7.19) with the same choice of a and b as above, and we leave the details to the reader. Also, the above arguments show that if $p = 2$, then (7.16) is an equality with $c_2 = 1$. $\square$

Let us now collect some consequences of Theorem 7.3.1 on polarizable Carnot groups, following [GKY17]. As before, we fix d to be the $\mathcal{L}$-gauge on a stratified group $\mathbb{G}$. Consequently, we denote by

$$B_R := \{x \in \mathbb{G} : d(x) < R\} \tag{7.21}$$

the ball of radius R with respect to the quasi-norm d.

Corollary 7.3.2 (Special cases of two-weight inequalities)**.** *Let $\mathbb{G}$ be a polarizable Carnot group. Then we have the following inequalities:*

1. *Let $\alpha \in \mathbb{R}$, $1 < p < Q + \alpha$, $\gamma > -1$. Then we have*
$$\int_{\mathbb{G}} d^{\alpha} |\nabla_H d|^{\gamma} |\nabla_H \phi|^p dx \geq \left(\frac{Q + \alpha - p}{p}\right)^p \int_{\mathbb{G}} d^{\alpha} \frac{|\nabla_H d|^{p+\gamma}}{d^p} |\phi|^p dx$$
for all $\phi \in C_0^{\infty}(\mathbb{G}\backslash\{0\})$.
2. *Let $Q = p > 1$ and $\alpha < -1$. Then we have*
$$\int_{B_R} \left(\log \frac{R}{d}\right)^{\alpha+p} |\nabla_H \phi|^p dx \geq \left(\frac{|\alpha + 1|}{p}\right)^p \int_{B_R} \left(\log \frac{R}{d}\right)^{\alpha} \frac{|\nabla_H d|}{d^p} |\phi|^p dx$$
for all $\phi \in C_0^{\infty}(B_R)$.
3. *Let $\alpha \in \mathbb{R}$ and $Q + \alpha > p > 1$. Then we have*
$$\int_{\mathbb{G}} d^{\alpha} |\nabla_H \phi|^p dx \geq \left(\frac{Q + \alpha - p}{p - 1}\right)^{p-1} (Q + \alpha) \int_{\mathbb{G}} d^{\alpha} \frac{|\nabla_H d|^p}{(1 + d^{\frac{p}{p-1}})^p} |\phi|^p dx$$
for all $\phi \in C_0^{\infty}(\mathbb{G})$.
4. *Let $1 < p < Q$ and $\alpha > 1$. Then we have*
$$\begin{aligned}&\int_{\mathbb{G}} (1 + d^{\frac{p}{p-1}})^{\alpha(p-1)} |\nabla_H \phi|^p dx \\ &\qquad \geq Q \left(\frac{p(\alpha - 1)}{p - 1}\right)^{p-1} \int_{\mathbb{G}} \frac{|\nabla_H d|^p}{(1 + d^{\frac{p}{p-1}})^{(1-\alpha)(p-1)}} |\phi|^p dx\end{aligned}$$
for all $\phi \in C_0^{\infty}(\mathbb{G})$.
5. *Let $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$. If $\alpha\beta > 0$ and $m \leq \frac{Q-2}{2}$, then we have*
$$\begin{aligned}\int_{\mathbb{G}} \frac{(a + bd^{\alpha})^{\beta}}{d^{2m}} |\nabla_H \phi|^2 dx &\geq C(Q, m)^2 \int_{\mathbb{G}} \frac{(a + bd^{\alpha})^{\beta}}{d^{2m+2}} |\nabla_H d|^2 \phi^2 dx \\ &\quad + C(Q, m)\alpha\beta b \int_{\mathbb{G}} \frac{(a + bd^{\alpha})^{\beta-1}}{d^{2m-\alpha+2}} |\nabla_H d|^2 \phi^2 dx,\end{aligned}$$
for all $\phi \in C_0^{\infty}(\mathbb{G})$, where $C(Q, m) = \frac{Q-2m-2}{2}$.

6. *Let $Q = p > 1$. Then we have*

$$\int_{B_R} |\nabla_H \phi|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{B_R} \frac{|\nabla_H d|^p}{(R-d)^p} |\phi|^p dx$$

for all $\phi \in C_0^\infty(B_R)$.

Proof of Corollary 7.3.2. To make the application of Theorem 7.3.1 rigorous one can replace the function d with its regularization $d_\epsilon := (\Gamma + \epsilon)^{\frac{1}{2-Q}}$ for $\epsilon > 0$, where Γ is the fundamental solution for $\mathcal{L}$, and after the application of Theorem 7.3.1 take the limit as $\epsilon \to 0$.

1. The inequality follows from Theorem 7.3.1 with the choice

$$V = d^\alpha |\nabla_H d|^\gamma \quad \text{with} \quad \Phi = d^{-\left(\frac{Q+\alpha-p}{p}\right)}.$$

2. This part follows by taking

$$V = \left(\log \frac{R}{d}\right)^{\alpha+p} \quad \text{and} \quad \Phi = \left(\log \frac{R}{d}\right)^{\frac{|\alpha+1|}{p}}.$$

3. This part follows by taking

$$V = d^\alpha \quad \text{and} \quad \Phi = \left(1 + d^{\frac{p}{p-1}}\right)^{-\left(\frac{Q+\alpha-p}{p},\right)}.$$

4. This part follows by taking

$$V = \left(1 + d^{\frac{p}{p-1}}\right)^{\alpha(p-1)} \quad \text{and} \quad \Phi = \left(1 + d^{\frac{p}{p-1}}\right)^{1-\alpha}.$$

5. This part follows by taking

$$V = \frac{(a + bd^\alpha)^\beta}{d^{2m}} \quad \text{and} \quad \Phi = d^{-\left(\frac{Q-2m-2}{2}\right)}.$$

6. This part follows by taking

$$V \equiv 1 \quad \text{and} \quad \Phi = (R-d)^{\frac{p-1}{p}}.$$

The proof is complete. □

Remark 7.3.3. The statement of Corollary 7.3.2, Part 1, was first shown by Wang and Niu [WN08]. Part 2 was shown in [D'A05, (3.40)]. Part 4 is a version of the Euclidean estimate [Skr13, (5.1)]. Part 5 is a version of the Euclidean estimate [GM11, (42)]. Part 6 was shown on the Heisenberg group in [HN03] and then for polarizable Carnot groups in [D'A05].

Theorem 7.3.1 also yields several versions of the uncertainty principles.

Corollary 7.3.4 (Special cases of two-weight uncertainty principles)**.** *Let $\mathbb{G}$ be a polarizable Carnot group. Then we have the following inequalities:*

1. *We have*
$$\left(\int_{\mathbb{G}} \frac{|\nabla_H \phi|^2}{|\nabla_H d|^2} dx\right)\left(\int_{\mathbb{G}} d^2|\phi|^2 dx\right) \geq \frac{Q^2}{4}\left(\int_{\mathbb{G}} |\phi|^2 dx\right)^2$$
for all $\phi \in C_0^\infty(\mathbb{G})$.

2. *We have*
$$\left(\int_{\mathbb{G}} |\nabla_H \phi|^2 dx\right)\left(\int_{\mathbb{G}} d^2|\nabla_H d|^2|\phi|^2 dx\right) \geq \frac{Q^2}{4}\left(\int_{\mathbb{G}} |\nabla_H d|^2|\phi|^2 dx\right)^2$$
for all $\phi \in C_0^\infty(\mathbb{G})$.

3. *We have*
$$\left(\int_{\mathbb{G}} |\nabla_H \phi|^2 dx\right)\left(\int_{\mathbb{G}} |\nabla_H d|^2|\phi|^2 dx\right) \geq \frac{(Q-1)^2}{4}\left(\int_{\mathbb{G}} \frac{|\nabla_H d|^2}{d}|\phi|^2 dx\right)^2$$
for all $\phi \in C_0^\infty(\mathbb{G})$.

Proof of Corollary 7.3.4. 1. This inequality was first shown by Kombe in [Kom10], extending the Euclidean uncertainty principle (2). Considering
$$V = \frac{1}{|\nabla_H d|^2} \quad \text{and} \quad \Phi = e^{-\alpha d^2},$$
for $\alpha > 0$, Theorem 7.3.1 implies
$$\int_{\mathbb{G}} \frac{1}{|\nabla_H d|^2}|\nabla_H \phi|^2 dx \geq 2\alpha Q \int_{\mathbb{G}} |\phi|^2 dx - 4\alpha^2 \int_{\mathbb{G}} d^2|\phi|^2 dx.$$
Let now $A := -4\int_{\mathbb{G}} d^2\phi^2 dx$, $B := 2Q\int_{\mathbb{G}} \phi^2 dx$ and $C := -\int_{\mathbb{G}} \frac{|\nabla_H \phi|^2}{|\nabla_H d|^2} dx$. Then the above inequality can be expressed as $A\alpha^2 + B\alpha + C \leq 0$ for all $\alpha \in \mathbb{R}$. But this implies that $B^2 - 4AC \leq 0$, which proves the statement.

2. Let us take
$$V \equiv 1 \qquad \text{and} \qquad \Phi = e^{-\alpha d},$$
where $\alpha > 0$. Then by Theorem 7.3.1 we have
$$\int_{\mathbb{G}} |\nabla_H \phi|^2 dx \geq 2\alpha Q \int_{\mathbb{G}} |\nabla_H d|^2|\phi|^2 dx - 4\alpha^2 \int_{\mathbb{G}} d^2|\nabla_H d|^2|\phi|^2 dx.$$
The same argument as in Part 1 implies the statement.

3. The statement follows from Theorem 7.3.1 with
$$V \equiv 1 \qquad \text{and} \qquad \Phi = e^{-\alpha d},$$
for $\alpha > 0$, and the same argument as in Part 1. $\square$

In [GKY17] the authors showed that on polarizable Carnot groups the statement of Theorem 7.3.1 can be refined to give also the remainder estimates. Following [GKY17] we recapture this statement, its proof, and its consequences. In the following theorem we consider the case $p \geq 2$ noting that a similar result can be shown also for $1 < p < 2$, with a different reminder term, if one uses in the proof (7.19) instead of (7.18).

Theorem 7.3.5 (Two-weight L^p-Hardy inequalities with remainder estimates)**.** *Let $\mathbb{G}$ be a polarizable Carnot group and let Ω be a bounded domain in $\mathbb{G}$ with smooth boundary $\partial\Omega$. Assume that V is a non-negative C^1-function and that δ is a positive C^∞-function such that*

$$- \nabla_H \cdot \left(V(x) d^{p-Q} \frac{|\nabla_H \delta|^{p-2}}{\delta^{p-2}} \nabla_H \delta \right) \geq 0 \tag{7.22}$$

holds almost everywhere in Ω. Then for any $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \int_\Omega V(x) d^\alpha |\nabla_H \phi|^p dx \geq & \left(\frac{Q+\alpha-p}{p} \right)^p \int_\Omega V(x) d^\alpha \frac{|\nabla_H d|^p}{d^p} |\phi|^p dx \\ & + \left(\frac{Q+\alpha-p}{p} \right)^{p-1} \int_\Omega V(x) d^\alpha \frac{|\nabla_H d|^{p-2}}{d^{p-1}} \nabla_H d \cdot \nabla_H V |\phi|^p dx \\ & + \frac{c_p}{p^p} \int_\Omega V(x) d^\alpha \frac{|\nabla_H \delta|^p}{\delta^p} |\phi|^p dx, \end{aligned} \tag{7.23}$$

where $Q + \alpha > p \geq 2$, $\alpha \in \mathbb{R}$ and $c_p = c(p) > 0$.

Proof. For any $\phi \in C_0^\infty(\Omega)$ we set $\varphi := d^{-\gamma}\phi$ with $\gamma < 0$, a constant that will be chosen later. By a direct computation we have

$$\nabla_H (d^\gamma \phi) = \gamma d^{\gamma-1} \varphi \nabla_H d + d^\gamma \nabla_H \varphi.$$

Applying inequality (7.18) with $a = \gamma d^{\gamma-1}\varphi \nabla_H d$ and $b = d^\gamma \nabla_H \varphi$ we get

$$\begin{aligned} |\nabla_H \phi|^p \geq & |\gamma|^p d^{p(\gamma-1)} |\nabla_H d|^p |\phi|^p \\ & + \gamma |\gamma|^{p-2} d^{p(\gamma-1)+1} |\nabla_H d|^{p-2} \nabla_H d \cdot \nabla_H (|\varphi|^p) + c_p d^{p\gamma} |\nabla_H \varphi|^p. \end{aligned} \tag{7.24}$$

Multiplying both sides of (7.24) by $V(x)d^\alpha$ and integrating by parts we get

$$\begin{aligned} \int_\Omega V(x) d^\alpha |\nabla_H \phi|^p dx \geq & |\gamma|^p \int_\Omega V(x) d^{\alpha+p(\gamma-1)} |\nabla_H d|^p |\varphi|^p dx \\ & - \gamma |\gamma|^{p-2} \int_\Omega \nabla_H \cdot (V(x) d^{\alpha+p(\gamma-1)+1} |\nabla_H d|^{p-2} \nabla_H d) |\varphi|^p dx \\ & + c_p \int_\Omega V(x) d^{\alpha+p\gamma} |\nabla_H \varphi|^p dx. \end{aligned} \tag{7.25}$$

Using (1.77) and (1.102) we have

$$\begin{aligned}\nabla_H \cdot (V(x)d^{\alpha+p(\gamma-1)+1}|\nabla_H d|^{p-2}\nabla_H d) \\ = d^{\alpha+p(\gamma-1)+1}|\nabla_H d|^{p-2}\nabla_H d \cdot \nabla_H V \\ + [Q+\alpha+p(\gamma-1)]V(x)d^{\alpha+p(\gamma-1)}|\nabla_H d|^p.\end{aligned} \tag{7.26}$$

Using (7.26) we can rewrite (7.25) as

$$\begin{aligned}\int_\Omega V(x)d^\alpha|\nabla_H\phi|^p dx \geq \zeta(Q,\alpha,p;\gamma)\int_\Omega V(x)d^{\alpha+p(\gamma-1)}|\nabla_H d|^p|\varphi|^p dx \\ -\gamma|\gamma|^{p-2}\int_\Omega d^{\alpha+p(\gamma-1)+1}|\nabla_H d|^{p-2}\nabla_H d\cdot\nabla_H V|\varphi|^p dx \\ + c_p\int_\Omega V(x)d^{\alpha+p\gamma}|\nabla_H\varphi|^p dx,\end{aligned}$$

where $\zeta(Q,\alpha,p;\gamma) = |\gamma|^p - \gamma|\gamma|^{p-2}(Q+\alpha+\gamma p-p)$. Since $\gamma<0$ we can choose $\gamma = (p-\alpha-Q)/p$. Therefore, we have

$$\begin{aligned}\int_\Omega V(x)d^\alpha|\nabla_H\phi|^p dx \geq \left(\frac{Q+\alpha-p}{p}\right)^p\int_\Omega V(x)\frac{|\nabla_H d|^p}{d^Q}|\varphi|^p dx \\ + \left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_\Omega \frac{|\nabla_H d|^{p-2}}{d^{Q-1}}\nabla_H d\cdot\nabla_H V|\varphi|^p dx \\ + c_p\int_\Omega V(x)d^{p-Q}|\nabla_H\varphi|^p dx.\end{aligned} \tag{7.27}$$

Let us analyse the last term in (7.27). Let us define $\vartheta := \delta^{-1/p}\varphi$, where $0<\delta\in C^\infty(\Omega)$ and $\varphi\in C_0^\infty(\Omega)$. It follows from (7.18) that

$$\begin{aligned}|\nabla_H\phi|^p = |\frac{1}{p}\delta^{\frac{1-p}{p}}\vartheta\nabla_H\delta + \delta^{\frac{1}{p}}\nabla_H\vartheta|^p \\ \geq \frac{1}{p^p}\frac{|\nabla_H\delta|^p}{\delta^{p-1}}|\vartheta|^p + \frac{1}{p^{p-1}}\frac{|\nabla_H\delta|^{p-2}}{\delta^{p-2}}\nabla_H\delta\cdot\nabla_H(|\vartheta|^p) + c_p\delta^p|\nabla_H\vartheta|^p.\end{aligned} \tag{7.28}$$

Since $c_p\delta^p|\nabla_H\vartheta|^p\geq 0$, integrating by parts in (7.28) we get

$$\begin{aligned}c_p\int_\Omega V(x)d^{p-Q}|\nabla_H\varphi|^p dx \geq \frac{c_p}{p^p}\int_\Omega V(x)d^{p-Q}\frac{|\nabla_H\delta|^p}{\delta^{p-1}}|\vartheta|^p dx \\ -\frac{c_p}{p^{p-1}}\int_\Omega \nabla_H\cdot\left(V(x)d^{p-Q}\frac{|\nabla_H\delta|^{p-2}}{\delta^{p-2}}\nabla_H\delta\right)|\vartheta|^p dx.\end{aligned}$$

Using (7.22) and the substitution $\vartheta := \delta^{-1/p}d^{\frac{Q+\alpha-p}{p}}\phi$ we obtain

$$c_p\int_\Omega V(x)d^{p-Q}|\nabla_H\varphi|^p dx \geq \frac{c_p}{p^p}\frac{c_p}{p^p}\int_\Omega V(x)d^{p-Q}\frac{|\nabla_H\delta|^p}{\delta^p}|\phi|^p dx. \tag{7.29}$$

Combining (7.27) and (7.29), and using $\varphi = d^{\frac{Q+\alpha-p}{p}}\phi$ we obtain (7.23). □

Let us now list several consequences of Theorem 7.3.5 for some specific choices of V and δ. We recall the notation

$$B_R := \{x \in \mathbb{G} : d(x) < R\}$$

for the ball of radius R with respect to the quasi-norm d, already used in (7.21).

Corollary 7.3.6 (A collection of L^p-Hardy inequalities with remainders)**.** *Let $\mathbb{G}$ be a polarizable Carnot group and let Ω be a bounded domain in $\mathbb{G}$ with smooth boundary $\partial\Omega$. Then we have the following statements.*

1. *For all $\phi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega d^\alpha |\nabla_H \phi|^p dx \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega d^\alpha \frac{|\nabla_H d|^p}{d^p} |\phi|^p dx + \frac{c_p}{p^p} \int_\Omega d^\alpha \frac{|\nabla_H d|^p}{(d \log(\frac{R}{d}))} |\phi|^p dx,$$

where $Q+\alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p > 0$ and $R > \sup_{x\in\Omega} d(x)$.

2. *For all $\phi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega d^\alpha |\nabla_H \phi|^p dx \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega d^\alpha \frac{|\nabla_H d|^p}{d^p} |\phi|^p dx + \frac{c_p}{p^p} \int_\Omega d^\alpha \frac{|\nabla_H d|^p}{d^p (\log \frac{R}{d})^p (\log(\log \frac{R}{d}))^p} |\phi|^p dx,$$

where $Q+\alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p > 0$ and $R > e \sup_{x\in\Omega} d(x)$.

3. *For all $\phi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega e^d d^\alpha |\nabla_H \phi|^p dx \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega e^d d^\alpha \frac{|\nabla_H d|^p}{d^p} |\phi|^p dx + \left(\frac{Q+\alpha-p}{p}\right)^{p-1} \int_\Omega e^d d^\alpha \frac{|\nabla_H d|^p}{d^{p-1}} |\phi|^p dx + \frac{c_p}{p^p} \int_\Omega e^d d^\alpha |\nabla_H d|^p |\phi|^p dx,$$

where $Q+\alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p > 0$.

4. *For all $\phi \in C_0^\infty(B_R)$ we have*

$$\int_{B_R} d^\alpha |\nabla_H \phi|^p dx \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_{B_R} d^\alpha \frac{|\nabla_H d|^p}{d^p} |\phi|^p dx + \frac{c_p}{p^p} \int_{B_R} d^\alpha \frac{|\nabla_H d|^p}{(R-d)^p} |\phi|^p dx,$$

where $Q+\alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p > 0$.

Proof of Corollary 7.3.6. 1. The statement follows from Theorem 7.3.5 with

$$V \equiv 1 \quad \text{and} \quad \delta = \log\left(\frac{R}{d}\right).$$

This inequality is the stratified group version of the Euclidean inequality in [ACR02, (1.4)].

2. The statement follows from Theorem 7.3.5 with

$$V \equiv 1 \quad \text{and} \quad \delta = \log\left(\log\frac{R}{d}\right), \quad R > e \sup_{x\in\Omega} d(x).$$

3. The statement follows from Theorem 7.3.5 with

$$V = e^{d} \quad \text{and} \quad \delta = e^{-d}.$$

4. The statement follows from Theorem 7.3.5 with

$$V \equiv 1 \quad \text{and} \quad \delta = R - d.$$

As in the proof of Corollary 7.3.2, the above applications of Theorem 7.3.5 can be justified by considering the regularization $d_\epsilon := (\Gamma + \epsilon)^{\frac{1}{2-Q}}$ for $\epsilon > 0$, where Γ is the fundamental solution for $\mathcal{L}$, and after the application of Theorem 7.3.5 taking the limit as $\epsilon \to 0$. □

7.4 Rellich inequalities for sub-Laplacians with drift

In this section, we show the weighted Rellich inequality for sub-Laplacians with drift on polarizable Carnot groups expressing the weights in terms of the fundamental solution of the sub-Laplacian.

We recall that in Section 6.9 we already showed Rellich inequalities for sub-Laplacians with drift with weights expressed in terms of the variable x' from the first stratum.

In this section, we assume all the notation of Section 1.4.6 where sub-Laplacians with drift have been discussed.

Theorem 7.4.1 (Rellich inequality for sub-Laplacian with drift with $\mathcal{L}$-gauge weights on polarizable Carnot groups)**.** *Let $\mathbb{G}$ be a polarizable Carnot group of homogeneous dimension $Q \geq 3$ and let $\theta \in \mathbb{R}$ with $Q + 2\theta - 4 > 0$. Then for all functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left\|\frac{d^\theta}{|\nabla_H d|}\mathcal{L}_X f\right\|^2_{L^2(\mathbb{G},\mu_X)} \geq \frac{(Q+2\theta-4)^2(Q-2\theta)^2}{16}\left\|d^{\theta-2}|\nabla_H d| f\right\|^2_{L^2(\mathbb{G},\mu_X)}$$
$$+\gamma^4 b_X^4 \left\|\frac{d^\theta}{|\nabla_H d|} f\right\|^2_{L^2(\mathbb{G},\mu_X)} + \gamma^2 b_X^2\left(\frac{(Q+2\theta-2)(Q-2\theta-2)}{2}\right)\left\|d^{\theta-1} f\right\|^2_{L^2(\mathbb{G},\mu_X)}$$

$$+ 2\gamma^2 b_X^2 (Q-1)(3Q-4) \left\|d^{\theta-1} f\right\|^2_{L^2(\mathbb{G},\mu_X)}$$
$$+ 2\gamma^2 b_X^2 \int_{\mathbb{G}} \frac{d^{2\theta+2Q-2}}{|\nabla_H d|^4} \left(d^{1-Q}|\nabla_H d|\mathcal{L}(d^{1-Q}|\nabla_H d|) - 3|\nabla_H (d^{1-Q}|\nabla_H d|)|^2\right)$$
$$\times |f(x)|^2 d\mu_X(x), \tag{7.30}$$

where $\mathcal{L}_X$ and b_X are defined in (1.93) *and* (1.95)*, respectively.*

Remark 7.4.2. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the usual full gradient, $d = |x|_E$ is the Euclidean distance, hence $|\nabla_H d| = 1$, and setting $X = \sum_{i=1}^n a_i \partial_{x_i}$, the last two terms in (7.30) cancel each other, so that we obtain the same estimate as in (6.83).

Proof of Theorem 7.4.1. The proof follows [RY18b]. Let $g = g(x) \in C_0^\infty(\mathbb{G}\backslash\{0\})$ be such that $f = \chi^{-1/2} g$. By (1.101) we know that $L^2(\mathbb{G},\mu) \in g \mapsto \chi^{-1/2} g \in L^2(\mathbb{G},\mu_X)$. Then, we have

$$\left\|\frac{d^\theta}{|\nabla_H d|}\mathcal{L}_X f\right\|_{L^2(\mathbb{G},\mu_X)} = \left\|\frac{d^\theta}{|\nabla_H d|}\chi^{1/2}\mathcal{L}_X f\right\|_{L^2(\mathbb{G},\mu)}$$
$$= \left\|\frac{d^\theta}{|\nabla_H d|}\chi^{1/2}\mathcal{L}_X (\chi^{-1/2} g)\right\|_{L^2(\mathbb{G},\mu)},$$

where μ is the Haar (i.e., Lebesgue) measure on $\mathbb{G}$. By (1.100) and integration by parts, we calculate

$$\left\|\frac{d^\theta}{|\nabla_H d|}\mathcal{L}_X f\right\|^2_{L^2(\mathbb{G},\mu_X)} = \left\|\frac{d^\theta}{|\nabla_H d|}(\mathcal{L}_0 + \gamma^2 b_X^2) g\right\|^2_{L^2(\mathbb{G},\mu)}$$
$$= \left\|\frac{d^\theta}{|\nabla_H d|}\mathcal{L}_0 g\right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \mathrm{Re}\int_{\mathbb{G}} \frac{d^{2\theta}}{|\nabla_H d|^2}\mathcal{L}_0 g(x)\overline{g(x)}dx + \gamma^4 b_X^4 \left\|\frac{d^\theta}{|\nabla_H d|} g\right\|^2_{L^2(\mathbb{G},\mu)}$$
$$= \left\|\frac{d^\theta}{|\nabla_H d|}\mathcal{L}_0 g\right\|^2_{L^2(\mathbb{G},\mu)} - 2\gamma^2 b_X^2 \mathrm{Re}\sum_{j=1}^N \int_{\mathbb{G}} \frac{d^{2\theta}}{|\nabla_H d|^2} X_j^2 g(x)\overline{g(x)}dx$$
$$+ \gamma^4 b_X^4 \left\|\frac{d^\theta}{|\nabla_H d|} g\right\|^2_{L^2(\mathbb{G},\mu)}$$
$$= \left\|\frac{d^\theta}{|\nabla_H d|}\mathcal{L}_0 g\right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \left\|\frac{d^\theta}{|\nabla_H d|}\nabla_H g\right\|^2_{L^2(\mathbb{G},\mu)} + \gamma^4 b_X^4 \left\|\frac{d^\theta}{|\nabla_H d|} g\right\|^2_{L^2(\mathbb{G},\mu)}$$
$$+ 2\gamma^2 b_X^2 \mathrm{Re}\sum_{j=1}^N \int_{\mathbb{G}} X_j g(x)\overline{g(x)} X_j \left(\frac{d^{2\theta}}{|\nabla_H d|^2}\right) dx$$
$$= \left\|\frac{d^\theta}{|\nabla_H d|}\mathcal{L}_0 g\right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \left\|\frac{d^\theta}{|\nabla_H d|}\nabla_H g\right\|^2_{L^2(\mathbb{G},\mu)} + \gamma^4 b_X^4 \left\|\frac{d^\theta}{|\nabla_H d|} g\right\|^2_{L^2(\mathbb{G},\mu)}$$

$$
\begin{aligned}
&+ 4\gamma^2 b_X^2 \theta \mathrm{Re} \sum_{j=1}^{N} \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
&- 4\gamma^2 b_X^2 \mathrm{Re} \sum_{j=1}^{N} \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} dx \\
=: &\left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_0 g \right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \left\| \frac{d^\theta}{|\nabla_H d|} \nabla_H g \right\|^2_{L^2(\mathbb{G},\mu)} + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} g \right\|^2_{L^2(\mathbb{G},\mu)} \\
&+ I_1 + I_2. \qquad (7.31)
\end{aligned}
$$

Then, by Theorem 7.1.3 one has for $Q + 2\theta - 2 > 0$ that

$$
2\gamma^2 b_X^2 \left\| \frac{d^\theta}{|\nabla_H d|} \nabla_H g \right\|^2_{L^2(\mathbb{G},\mu)} \geq 2\gamma^2 b_X^2 \left(\frac{Q+2\theta-2}{2} \right)^2 \left\| d^{\theta-1} g \right\|^2_{L^2(\mathbb{G},\mu)}. \qquad (7.32)
$$

On the other hand by Theorem 7.2.2 we get for $Q + 2\theta - 4 > 0$ that

$$
\left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_0 g \right\|^2_{L^2(\mathbb{G},\mu)} \geq \frac{(Q+2\theta-4)^2 (Q-2\theta)^2}{16} \left\| d^{\theta-2} |\nabla_H d| g \right\|^2_{L^2(\mathbb{G},\mu)}. \qquad (7.33)
$$

Putting (7.32) and (7.33) into (7.31) we obtain for $Q + 2\theta - 4 > 0$ that

$$
\begin{aligned}
&\left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_X f \right\|^2_{L^2(\mathbb{G},\mu_X)} \\
&\quad \geq \frac{(Q+2\theta-4)^2 (Q-2\theta)^2}{16} \left\| d^{\theta-2} |\nabla_H d| g \right\|^2_{L^2(\mathbb{G},\mu)} \\
&\quad\quad + 2\gamma^2 b_X^2 \left(\frac{Q+2\theta-2}{2} \right)^2 \left\| d^{\theta-1} g \right\|^2_{L^2(\mathbb{G},\mu)} \\
&\quad\quad + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} g \right\|^2_{L^2(\mathbb{G},\mu)} + I_1 + I_2. \qquad (7.34)
\end{aligned}
$$

Let us calculate I_1 from (7.31):

$$
\begin{aligned}
I_1 &= 4\gamma^2 b_X^2 \theta \mathrm{Re} \sum_{j=1}^{N} \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
&= -4\gamma^2 b_X^2 \theta \mathrm{Re} \sum_{j=1}^{N} \int_{\mathbb{G}} g(x) \overline{X_j g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
&\quad - 4\gamma^2 b_X^2 \theta \mathrm{Re} \sum_{j=1}^{N} \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I_1 &= 4\gamma^2 b_X^2 \theta \mathrm{Re} \sum_{j=1}^{N} \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
&= -2\gamma^2 b_X^2 \theta \sum_{j=1}^{N} \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) dx.
\end{aligned}
$$

Putting $d = u^{\frac{1}{2-Q}}$ we calculate

$$
\begin{aligned}
\sum_{j=1}^{N} X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) &= (2-Q) \sum_{j=1}^{N} X_j \left(\frac{u^{\frac{2\theta-Q}{2-Q}}}{|\nabla_H u|^2} X_j u \right) \\
&= (2-Q) \sum_{j=1}^{N} \frac{2\theta - Q}{2-Q} u^{\frac{2\theta-2}{2-Q}} \frac{(X_j u)^2}{|\nabla_H u|^2} + (2-Q) \sum_{j=1}^{N} \frac{u^{\frac{2\theta-Q}{2-Q}} X_j^2 u}{|\nabla_H u|^2} \\
&\quad - 2(2-Q) \sum_{j=1}^{N} \frac{u^{\frac{2\theta-Q}{2-Q}} X_j u}{|\nabla_H u|^3} X_j |\nabla_H u|,
\end{aligned}
$$

which implies, using (1.104), that

$$
\begin{aligned}
I_1 &= -2\gamma^2 b_X^2 \theta \int_{\mathbb{G}} \left(\sum_{j=1}^{N} X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) \right) |g(x)|^2 dx \\
&= -2\gamma^2 b_X^2 \theta (2\theta + Q - 2) \int_{\mathbb{G}} u^{\frac{2\theta-2}{2-Q}} |g(x)|^2 dx \\
&= -2\gamma^2 b_X^2 \theta (2\theta + Q - 2) \int_{\mathbb{G}} |g(x)|^2 d^{2\theta-2} dx. \qquad (7.35)
\end{aligned}
$$

Now for I_2 by integration by parts we get

$$
\begin{aligned}
I_2 &= -4\gamma^2 b_X^2 \mathrm{Re} \sum_{j=1}^{N} \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} dx \\
&= 2\gamma^2 b_X^2 \sum_{j=1}^{N} \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right) dx.
\end{aligned}
$$

Using $d = u^{\frac{1}{2-Q}}$ one has

$$
\begin{aligned}
\sum_{j=1}^{N} X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right) &= (2-Q)^2 \sum_{j=1}^{N} X_j \left(\frac{u^{\frac{2\theta-3Q+3}{2-Q}}}{|\nabla_H u|^3} X_j \left(u^{\frac{Q-1}{2-Q}} |\nabla_H u| \right) \right) \\
&= (2-Q)^2 \sum_{j=1}^{N} X_j \left(\frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \frac{X_j u}{|\nabla_H u|^2} + u^{\frac{2\theta-2Q+2}{2-Q}} \frac{X_j |\nabla_H u|}{|\nabla_H u|^3} \right) \\
&=: (2-Q)^2 J_1 + (2-Q)^2 J_2. \qquad (7.36)
\end{aligned}
$$

Then, taking into account (1.104) we have for J_1 that

$$J_1 = \sum_{j=1}^{N} X_j \left(\frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \frac{X_j u}{|\nabla_H u|^2} \right) = \frac{(Q-1)(2\theta-Q)}{(Q-2)^2} u^{\frac{2\theta-2}{2-Q}} \sum_{j=1}^{N} \frac{(X_j u)^2}{|\nabla_H u|^2}$$
$$+ \frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \sum_{j=1}^{N} \frac{X_j^2 u}{|\nabla_H u|^2} - \frac{2(Q-1)}{(2-Q)} u^{\frac{2\theta-Q}{2-Q}} \sum_{j=1}^{N} \frac{X_j u X_j |\nabla_H u|}{|\nabla_H u|^3}$$
$$= \frac{(Q-1)(2\theta-Q)}{(Q-2)^2} u^{\frac{2\theta-2}{2-Q}} + \frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \frac{\mathcal{L}u}{|\nabla_H u|^2} + \frac{2(Q-1)^2}{(Q-2)^2} u^{\frac{2\theta-2}{2-Q}}. \quad (7.37)$$

Now we calculate for J_2 that

$$J_2 = \sum_{j=1}^{N} X_j \left(u^{\frac{2\theta-2Q+2}{2-Q}} \frac{X_j |\nabla_H u|}{|\nabla_H u|^3} \right)$$
$$= \frac{2\theta-2Q+2}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \sum_{j=1}^{N} \frac{X_j u X_j |\nabla_H u|}{|\nabla_H u|^3}$$
$$+ u^{\frac{2\theta-2Q+2}{2-Q}} \sum_{j=1}^{N} \frac{X_j^2 |\nabla_H u|}{|\nabla_H u|^3} - 3u^{\frac{2\theta-2Q+2}{2-Q}} \sum_{j=1}^{N} \frac{(X_j |\nabla_H u|)^2}{|\nabla_H u|^4}$$
$$= \frac{2\theta-2Q+2}{2-Q} \left(\frac{Q-1}{Q-2} \right) u^{\frac{2\theta-2}{2-Q}} + u^{\frac{2\theta-2Q+2}{2-Q}} \frac{\mathcal{L}|\nabla_H u|}{|\nabla_H u|^3}$$
$$- 3u^{\frac{2\theta-2Q+2}{2-Q}} \frac{|\nabla_H |\nabla_H u||^2}{|\nabla_H u|^4}, \quad (7.38)$$

where we have used (1.104) in the last equality. Plugging (7.37) and (7.38) into (7.36), we obtain

$$\sum_{j=1}^{N} X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right)$$
$$= (Q-1)(3Q-4) u^{\frac{2\theta-2}{2-Q}} + (2-Q)(Q-1) \frac{\mathcal{L}u}{|\nabla_H u|^2}$$
$$+ (Q-2)^2 u^{\frac{2\theta-2Q+2}{2-Q}} |\nabla_H u|^{-4} (|\nabla_H u| \mathcal{L} |\nabla_H u| - 3|\nabla_H |\nabla_H u||^2).$$

Then we get for I_2 the expression

$$I_2 = 2\gamma^2 b_X^2 \sum_{j=1}^{N} \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right) dx$$
$$= 2\gamma^2 b_X^2 (Q-1)(3Q-4) \int_{\mathbb{G}} u^{\frac{2\theta-2}{2-Q}} |g(x)|^2 dx$$
$$+ 2\gamma^2 b_X^2 (Q-2)^2 \int_{\mathbb{G}} u^{\frac{2\theta-2Q+2}{2-Q}} |\nabla_H u|^{-4} (|\nabla_H u| \mathcal{L} |\nabla_H u| - 3|\nabla_H |\nabla_H u||^2) |g(x)|^2 dx.$$

Setting here $u = d^{2-Q}$, we get for I_2 that

$$I_2 = 2\gamma^2 b_X^2 (Q-1)(3Q-4) \int_{\mathbb{G}} d^{2\theta-2} |g(x)|^2 dx$$
$$+ 2\gamma^2 b_X^2 \int_{\mathbb{G}} \frac{d^{2\theta+2Q-2}}{|\nabla_H d|^4} \left(d^{1-Q} |\nabla_H d| \mathcal{L}(d^{1-Q}|\nabla_H d|) - 3|\nabla_H (d^{1-Q}|\nabla_H d|)|^2 \right) |g(x)|^2 dx.$$

Thus, by this and (7.35) we have

$$I_1 + I_2 = 2\gamma^2 b_X^2 ((Q-1)(3Q-4) - \theta(2\theta+Q-2)) \left\| d^{\theta-1} g \right\|^2_{L^2(\mathbb{G},\mu)}$$
$$+ 2\gamma^2 b_X^2 \int_{\mathbb{G}} \frac{d^{2\theta+2Q-2}}{|\nabla_H d|^4} \left(d^{1-Q} |\nabla_H d| \mathcal{L}(d^{1-Q}|\nabla_H d|) - 3|\nabla_H (d^{1-Q}|\nabla_H d|)|^2 \right) |g(x)|^2 dx.$$

Combining this with (7.34) and taking into account (1.101) we obtain Theorem 7.4.1. □

7.5 Hardy inequalities on the complex affine group

The aim of this section is to show that some of the above techniques are also applicable for non-unimodular Lie groups. For example, consider the complex affine groups:

Definition 7.5.1 (Complex affine group)**.** The complex affine group is the semi-direct product

$$\mathbb{G} = \mathbb{C} \rtimes \mathbb{C}^*,$$

where $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers. This means that $\mathbb{G}$ is equal to $\mathbb{C} \times \mathbb{C}^*$ as a set, with the group composition law of the complex affine group $\mathbb{G}$ given by

$$(x, y) \circ (x', y') = (x + yx', yy')$$

for all $x, x' \in \mathbb{C}$ and $y, y' \in \mathbb{C}^*$. We will be also using the notation $x := t + is$ and $y := \tau + i\varsigma$. The complex affine group is a Lie group, with its Lie algebra denoted by $\mathfrak{g}$.

We now fix a basis $\{X_1, X_2, X_3, X_4\}$ of $\mathfrak{g}$ given by

$$X_1 = \frac{\partial}{\partial t}, \quad X_3 = t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial \tau} + \varsigma\frac{\partial}{\partial \varsigma},$$
$$X_2 = \frac{\partial}{\partial s}, \quad X_4 = -s\frac{\partial}{\partial t} + t\frac{\partial}{\partial s} - \varsigma\frac{\partial}{\partial \tau} + \tau\frac{\partial}{\partial \varsigma}.$$

These right invariant vector fields correspond to the canonical basis elements of $\mathfrak{g}$, and it will be convenient to work with right invariant vector fields here. Therefore, the positive (sub-)Laplacian

$$\Delta_X = -\sum_{j=1}^{4} X_j^2 \tag{7.39}$$

is called a *right invariant canonical Laplacian of the complex affine group* $\mathbb{G}$. The fundamental solution of the Laplacian Δ_X was computed explicitly by Gaudry and Sjögren [GS98] in the following form

$$\varepsilon = \frac{1}{4\pi^2} \frac{|y|^2}{|x|^2 + |1-y|^2}.$$

We will also use the notation

$$\nabla_X = (X_1, X_2, X_3, X_4)$$

for the right invariant (canonical) gradient on $\mathbb{G}$. The right invariant and the left invariant Haar measures on $\mathbb{G}$ are defined by

$$d\mu_r = dx \frac{dy}{|y|^2}, \quad d\mu_l = dx \frac{dy}{|y|^4},$$

with the modular function $m(x, y) = |y|^2$, respectively. In addition, one has the following integration rules with respect to the modular function

$$\int_{\mathbb{G}} f(\eta\zeta) d\mu_l(\eta) = m^{-1}(\zeta) \int_{\mathbb{G}} f(\eta) d\mu_l(\eta),$$
$$\int_{\mathbb{G}} f(\eta^{-1}) m^{-1}(\eta) d\mu_l(\eta) = \int_{\mathbb{G}} f(\eta) d\mu_l(\eta).$$

We now present a Hardy type inequality on $\mathbb{G}$ with the proof relying on properties of the fundamental solution of the right invariant canonical Laplacian Δ_X on the complex affine group $\mathbb{G}$ given in (7.39).

Theorem 7.5.2 (Hardy inequalities on the complex affine group)**.** *Let $\mathbb{G}$ be the complex affine group. Let $\alpha \in \mathbb{R}$, $\alpha > 2 - \beta$, $\beta > 2$. Then we have*

$$\int_{\mathbb{G}} \varepsilon^{\frac{\alpha}{2-\beta}} |\nabla_X u|^2 \, d\mu_l \geq \left(\frac{\beta + \alpha - 2}{2} \right)^2 \int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}} |\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\mu_l, \quad (7.40)$$

for all $u \in C_0^\infty(\mathbb{G})$, where $\nabla_X = (X_1, X_2, X_3, X_4)$.

Proof of Theorem 7.5.2. By using formula (2.8) we can assume without loss of generality that u is real-valued. Then let us set $u = d^\gamma q$ for some real-valued functions $d > 0$, q, and a constant $\gamma \neq 0$ to be chosen later. We use our usual notation for the potential theory considerations:

$$\widetilde{\nabla} u := \sum_{k=1}^{4} (X_k u) \, X_k.$$

Then we can calculate

$$
\begin{aligned}
(\widetilde{\nabla} u)u &= (\widetilde{\nabla} d^\gamma q) d^\gamma q \\
&= \sum_{k=1}^{4} X_k(d^\gamma q) X_k(d^\gamma q) \\
&= \gamma^2 d^{2\gamma-2} \sum_{k=1}^{4} (X_k d)^2 q^2 + 2\gamma d^{2\gamma-1} q \sum_{k=1}^{4} X_k d\, X_k q + d^{2\gamma} \sum_{k=1}^{4} (X_k q)^2 \\
&= \gamma^2 d^{2\gamma-2} ((\widetilde{\nabla} d)d) q^2 + 2\gamma d^{2\gamma-1} q (\widetilde{\nabla} d) q + d^{2\gamma} (\widetilde{\nabla} q) q.
\end{aligned}
$$

Integrating by parts we observe that

$$
\begin{aligned}
2\gamma \int_{\mathbb{G}} d^{\alpha+2\gamma-1} q (\widetilde{\nabla} d) q d\mu_l &= \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} (\widetilde{\nabla} d^{\alpha+2\gamma}) q^2 d\mu_l \\
&= \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} (\widetilde{\nabla} q^2) d^{\alpha+2\gamma} d\mu_l \\
&= -\frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} q^2 \Delta_X d^{\alpha+2\gamma} d\mu_l.
\end{aligned}
$$

In particular, because of this, we will later choose γ so that $d^{\alpha+2\gamma} = \varepsilon$. Consequently, we have

$$
\begin{aligned}
\int_{\mathbb{G}} d^\alpha (\widetilde{\nabla} u) u d\mu_l &= \gamma^2 \int_{\mathbb{G}} d^{\alpha+2\gamma-2} ((\widetilde{\nabla} d)d)\, q^2 d\mu_l + \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} (\widetilde{\nabla} d^{\alpha+2\gamma}) q^2 d\mu_l \\
&\quad + \int_{\mathbb{G}} d^{\alpha+2\gamma} (\widetilde{\nabla} q) q d\mu_l \\
&= \gamma^2 \int_{\mathbb{G}} d^{\alpha+2\gamma-2} ((\widetilde{\nabla} d)d)\, q^2 d\mu_l \\
&\quad - \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} q^2 \Delta_X d^{\alpha+2\gamma} d\mu_l + \int_{\mathbb{G}} d^{\alpha+2\gamma} (\widetilde{\nabla} q) q d\mu_l \\
&\geq \gamma^2 \int_{\mathbb{G}} d^{\alpha+2\gamma-2} ((\widetilde{\nabla} d)d)\, q^2 d\mu_l - \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} q^2 \Delta_X d^{\alpha+2\gamma} d\mu_l,
\end{aligned}
\tag{7.41}
$$

since $d > 0$ and $(\widetilde{\nabla} q) q = |\nabla_X q|^2 \geq 0$. On the other hand, it can be readily checked that for a vector field X we have

$$
\begin{aligned}
&\frac{\gamma}{\alpha+2\gamma} X^2(d^{\alpha+2\gamma}) = \gamma X(d^{\alpha+2\gamma-1} X d) = \frac{\gamma}{2-\beta} X(d^{\alpha+2\gamma+\beta-2} X(d^{2-\beta})) \\
&\quad = \frac{\gamma}{2-\beta} (\alpha+2\gamma+\beta-2) d^{\alpha+2\gamma+\beta-3} (Xd) X(d^{2-\beta}) + \frac{\gamma}{2-\beta} d^{\alpha+2\gamma+\beta-2} X^2(d^{2-\beta}) \\
&\quad = \gamma(\alpha+2\gamma+\beta-2) d^{\alpha+2\gamma-2} (Xd)^2 + \frac{\gamma}{2-\beta} d^{\alpha+2\gamma+\beta-2} X^2(d^{2-\beta}).
\end{aligned}
$$

Consequently, we get the equality

$$-\frac{\gamma}{\alpha+2\gamma}\Delta_X d^{\alpha+2\gamma} = -\gamma(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma-2}(\widetilde{\nabla}d)d - \frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}\Delta_X d^{2-\beta}. \tag{7.42}$$

We now substitute (7.42) into (7.41) and use that $q^2 = d^{-2\gamma}u^2$, so that

$$\int_{\mathbb{G}} d^{\alpha}(\widetilde{\nabla}u)u d\mu_l \geq \big(-\gamma^2-\gamma(\alpha+\beta-2)\big)\int_{\mathbb{G}} d^{\alpha-2}((\widetilde{\nabla}d)d)u^2 d\mu_l$$
$$-\frac{\gamma}{2-\beta}\int_{\mathbb{G}}(\Delta_X d^{2-\beta})d^{\alpha+\beta-2}u^2 dx.$$

We now take $d := \varepsilon^{\frac{1}{2-\beta}}$, and since $\beta > 2$ and ε is the fundamental solution to Δ_X we have

$$\int_{\mathbb{G}}(\Delta_X \varepsilon)\varepsilon^{\frac{\alpha+\beta-2}{2-\beta}}u^2 dx = 0, \ \alpha > 2-\beta, \ \beta > 2.$$

Thus, we obtain

$$\int_{\mathbb{G}} \varepsilon^{\frac{\alpha}{2-\beta}}(\widetilde{\nabla}u)u\, d\mu_l \geq (-\gamma^2-\gamma(\alpha+\beta-2))\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}}(\widetilde{\nabla}\varepsilon^{\frac{1}{2-\beta}})\varepsilon^{\frac{1}{2-\beta}}u^2\, d\mu_l.$$

Taking $\gamma = \frac{2-\beta-\alpha}{2}$, we obtain (7.40). □

As usual, a Hardy inequality, such as the one in Theorem 7.5.2, implies uncertainty principles:

Corollary 7.5.3 (Uncertainty principles on the complex affine group)**.** *Let $\mathbb{G}$ be the complex affine group and let $\beta > 2$. Then for all $u \in C_0^\infty(\mathbb{G})$ we have*

$$\int_{\mathbb{G}} \varepsilon^{\frac{2}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2|u|^2 d\mu_l \int_{\mathbb{G}}|\nabla_X u|^2 d\mu_l \geq \left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{G}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2|u|^2 d\mu_l\right)^2, \tag{7.43}$$

as well as

$$\int_{\mathbb{G}} \frac{\varepsilon^{\frac{2}{2-\beta}}}{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2}|u|^2 d\mu_l \int_{\mathbb{G}}|\nabla_X u|^2 d\mu_l \geq \left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{G}}|u|^2 d\mu_l\right)^2. \tag{7.44}$$

Proof of Corollary 7.5.3. Taking $\alpha = 0$ in the inequality (7.40) and using Hardy inequality in Theorem 7.5.2, we get

$$\int_{\mathbb{G}} \varepsilon^{\frac{2}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2|u|^2 d\mu_l \int_{\mathbb{G}}|\nabla_X u|^2 d\mu_l$$
$$\geq \left(\frac{\beta-2}{2}\right)^2 \int_{\mathbb{G}} \varepsilon^{\frac{2}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2|u|^2 d\mu_l \int_{\mathbb{G}} \frac{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2}{\varepsilon^{\frac{2}{2-\beta}}}|u|^2\, d\mu_l$$
$$\geq \left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{G}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2|u|^2 d\mu_l\right)^2,$$

which shows (7.43). The proof of (7.44) is similar. □

7.6 Hardy inequalities for Baouendi–Grushin operators

In this section, we describe a special case of the Hardy inequalities on homogeneous groups, namely, inequalities associated to the Baouendi–Grushin vector fields on $\mathbb{R}^n$. However, the described methods also work for non-smooth vector fields allowing a singularity at the origin, so we include such cases in our exposition as well.

Definition 7.6.1 (Baouendi–Grushin operator and vector fields)**.** Let

$$z = (x_1, \ldots, x_m, y_1, \ldots, y_k) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$$

with $k, m \geq 1$, $k + m = n$. Let $\gamma \geq 0$. Let us consider the (Baouendi–Grushin) vector fields

$$X_i = \frac{\partial}{\partial x_i},\ i = 1, \ldots, m, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j},\ j = 1, \ldots, k.$$

The corresponding subelliptic gradient, which is the n-dimensional vector field, is then defined as

$$\nabla_\gamma := (X_1, \ldots, X_m, Y_1, \ldots, Y_k) = (\nabla_x, |x|^\gamma \nabla_y). \tag{7.45}$$

The *Baouendi–Grushin operator* on $\mathbb{R}^{m+k}$ is defined by

$$\Delta_\gamma = \sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{k} Y_j^2 = \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma, \tag{7.46}$$

where Δ_x and Δ_y are the Laplace operators in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively.

If γ is an even positive integer then the vector fields X_i, Y_j are smooth, and Δ_γ is hypoelliptic as a sum of squares of C^∞ vector fields satisfying Hörmander's condition

$$\operatorname{rank} \operatorname{Lie}[X_1, \ldots, X_m, Y_1, \ldots, Y_k] = n.$$

For any $\gamma \geq 0$ the dilation structure on $\mathbb{R}^{m+k}$ associated to Δ_γ is

$$\delta_\lambda(x, y) := (\lambda x, \lambda^{1+\gamma} y)$$

for $\lambda > 0$. Indeed, it is easy to check that this dilation structure makes the vector fields homogeneous,

$$X_i(\delta_\lambda) = \lambda \delta_\lambda(X_i), \quad Y_i(\delta_\lambda) = \lambda \delta_\lambda(Y_i),$$

and hence also

$$\nabla_\gamma \circ \delta_\lambda = \lambda \delta_\lambda \nabla_\gamma.$$

The homogeneous dimension of $\mathbb{R}^m \times \mathbb{R}^k$ with respect to this dilation is

$$Q = m + (1 + \gamma)k. \tag{7.47}$$

Definition 7.6.2 (Baouendi–Grushin distance)**.** Let $\rho(z)$ be the distance function, for $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ defined by

$$\rho = \rho(z) := (|x|^{2(1+\gamma)} + (1+\gamma)^2|y|^2)^{\frac{1}{2(1+\gamma)}}. \tag{7.48}$$

It is easy to check that it satisfies

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}. \tag{7.49}$$

We will formulate a refined version of the Hardy inequality for Baouendi–Grushin vector fields, and then in Remark 7.6.4 we will put it in the context of the existing rich literature on this subject.

Theorem 7.6.3 (Refined Hardy inequality for Baouendi–Grushin vector fields)**.** *Let $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ with $k, m \geq 1$, $k + m = n$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that*

$$Q + \alpha_1 - 2 > 0 \text{ and } m + \gamma\alpha_2 > 0.$$

Then for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \left(\left| \frac{d}{d|x|} f \right|^2 + |x|^{2\gamma} |\nabla_y f|^2 \right) dxdy \\ \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 dxdy,\end{aligned} \tag{7.50}$$

with sharp constant $\left(\frac{Q+\alpha_1-2}{2}\right)^2$.

Remark 7.6.4.

1. First, a Hardy inequality for Grushin operators was obtained by Garofalo [Gar93], who has shown the inequality

$$\begin{aligned}\int_{\mathbb{R}^n} (|\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2) dxdy \\ \geq \left(\frac{Q-2}{2} \right)^2 \int_{\mathbb{R}^n} \left(\frac{|x|^{2\gamma}}{|x|^{2+2\gamma} + (1+\gamma)^2 |y|^2} \right) |f|^2 dxdy,\end{aligned} \tag{7.51}$$

   where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$ with $n = m+k$, $m, k \geq 1$, $\gamma \geq 0$, $Q = m+(1+\gamma)k$ and $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^k \backslash \{(0,0)\})$. Theorem 7.6.3 gives (7.51) when $\alpha_1 = \alpha_2 = 0$ in view of the inequality

$$\left| \frac{d}{d|x|} f \right| \leq |\nabla_x f|. \tag{7.52}$$

2. Weighted L^p-versions of (7.51) were investigated by D'Ambrosio in [D'A04a] who has obtained the following estimate: Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $p > 1$, $k, m \geq 1$, $\alpha, \beta \in \mathbb{R}$ be such that $m + (1+\gamma)k > \alpha - \beta$ and $m > \gamma p - \beta$. Then for every $f \in D^{1,p}_{\gamma}(\Omega, |x|^{\beta-\gamma p}\rho^{(1+\gamma)p-\alpha})$ we have
$$\int_{\Omega} |\nabla_{\gamma} f|^p |x|^{\beta-\gamma p} \rho^{(1+\gamma)p-\alpha} dxdy \geq \left(\frac{Q+\beta-\alpha}{p}\right)^p \int_{\Omega} |f|^p \frac{|x|^{\beta}}{\rho^{\alpha}} dxdy, \quad (7.53)$$
where $D^{1,p}_{\gamma}(\Omega, \omega)$ stands for the closure of $C_0^{\infty}(\Omega)$ in the norm $\left(\int_{\Omega} |\nabla_{\gamma} f|^p \omega dzdy\right)^{1/p}$ for a weight $\omega \in L^1_{\mathrm{loc}}(\Omega)$ with $\omega > 0$ a.e. on Ω.

   If $0 \in \Omega$, then the constant $\left(\frac{Q+\beta-\alpha}{p}\right)^p$ in (7.53) is sharp. The inequality (7.53) has also been obtained in [Kom15], and in [SJ12] for $\Omega = \mathbb{R}^n$ with sharp constant.

   In view of (7.52), Theorem 7.6.3 refines (7.53) when $p = 2$ and $\Omega = \mathbb{R}^n$. We also mention that in the case $p = 2$ inequality (7.53) has been also shown in [Kom15] and [SJ12] by different methods.
3. In [SJ12], a Hardy–Rellich type inequality for the Baouendi–Grushin operator was obtained in L^2 with sharp constant:
$$\left(\frac{Q-\alpha-2}{2}\right)^2 \int_{\mathbb{R}^n} |\nabla_{\gamma} f|^2 \rho^{\alpha} dxdy \leq \int_{\mathbb{R}^n} |\Delta_{\gamma} f|^2 \rho^{\alpha+2} |\nabla_{\gamma}\rho|^{-2} dxdy,$$
where $p > 1$, $\frac{2-Q}{3} \leq \alpha \leq Q-2$, $f \in C_0^{\infty}(\mathbb{R}^n \backslash \{0\})$.
4. Inequalities of the above types have been also studied for subelliptic operators of different types, see, e.g., [Gar93], [GL90], [D'A04b], [D'A04a] and [DGN06], and also with remainder estimates, see, e.g., [DGN10] and references therein.
5. Magnetic Hardy inequalities for the Baouendi–Grushin operators have been obtained in [LRY17]. There, the authors also obtained Hardy inequalities for the magnetic Landau Hamiltonian.

Proof of Theorem 7.6.3. For the proof we follow [LRY17]. We denote
$$r := |x| \text{ and } F(r,y) := \rho^{\alpha_1} |\nabla_{\gamma}\rho|^{\alpha_2}.$$
Then, using (7.48) and (7.49) we can write
$$F(r,y) = \rho^{\alpha_1}|\nabla_{\gamma}\rho|^{\alpha_2} = r^{\alpha_2\gamma}\rho^{\alpha_1-\alpha_2\gamma} = r^{\alpha_2\gamma}\left(r^{2(1+\gamma)} + (1+\gamma)^2|y|^2\right)^{\frac{\alpha_1-\alpha_2\gamma}{2(1+\gamma)}}. \quad (7.54)$$
Let us first calculate the following expression
$$\int_{\mathbb{R}^k}\int_0^{\infty} \left(\left|\left(\partial_r + \alpha\frac{\partial_r\rho}{\rho}\right)f\right|^2 + r^{2\gamma}\left|\left(\nabla_y + \alpha\frac{\nabla_y\rho}{\rho}\right)f\right|^2\right) r^{m-1}F(r,y)drdy$$
$$= \int_{\mathbb{R}^k}\int_0^{\infty} \left(|\partial_r f|^2 + r^{2\gamma}|\nabla_y f|^2\right) r^{m-1}F(r,y)drdy$$

$$
\begin{aligned}
&+\alpha^2\int_{\mathbb{R}^k}\int_0^\infty\left(\left|\frac{\partial_r\rho}{\rho}\right|^2+r^{2\gamma}\left|\frac{\nabla_y\rho}{\rho}\right|^2\right)|f|^2r^{m-1}F(r,y)drdy\\
&+2\alpha\mathrm{Re}\int_{\mathbb{R}^k}\int_0^\infty\frac{\partial_r\rho}{\rho}r^{m-1}F(r,y)\overline{\partial_r f}\cdot fdrdy\\
&+2\alpha\mathrm{Re}\int_{\mathbb{R}^k}\int_0^\infty\frac{\nabla_y\rho}{\rho}\cdot\overline{\nabla_y f}r^{2\gamma+m-1}F(r,y)fdrdy\\
=:{}&I_1+I_2+I_3+I_4.
\end{aligned}\tag{7.55}
$$

We now calculate the terms I_2, I_3, I_4. Using the expressions

$$
\frac{\partial_r\rho}{\rho}=\frac{r^{2\gamma+1}}{\rho^{2\gamma+2}}\qquad\text{and}\qquad\frac{\nabla_y\rho}{\rho}=\frac{(\gamma+1)y}{\rho^{2\gamma+2}},
$$

we calculate

$$
\left|\frac{\partial_r\rho}{\rho}\right|^2+r^{2\gamma}\left|\frac{\nabla_y\rho}{\rho}\right|^2=\frac{r^{4\gamma+2}+r^{2\gamma}(\gamma+1)^2|y|^2}{\rho^{4\gamma+4}}=\frac{r^{2\gamma}}{\rho^{2\gamma+2}}=\frac{|\nabla_\gamma\rho|^2}{\rho^2}.\tag{7.56}
$$

Thus, we obtain

$$
I_2=\alpha^2\int_{-\infty}^\infty\int_0^\infty\frac{|\nabla_\gamma\rho|^2}{\rho^2}|f|^2r^{m-1}F(r,y)drdy.\tag{7.57}
$$

For I_3, we integrate by parts to get

$$
\begin{aligned}
I_3=-\alpha&\int_{\mathbb{R}^k}\int_0^\infty(2\gamma+m+\gamma\alpha_2)\rho^{\alpha_1-\alpha_2\gamma-2\gamma-2}r^{2\gamma+m-1+\gamma\alpha_2}|f|^2drdy\\
&-\alpha\int_{\mathbb{R}^k}\int_0^\infty(\alpha_1-\alpha_2\gamma-2\gamma-2)\rho^{\alpha_1-\alpha_2\gamma-4\gamma-4}r^{4\gamma+m+\gamma\alpha_2+1}|f|^2drdy.
\end{aligned}
$$

Since $F(r,y)=r^{\alpha_2\gamma}\rho^{\alpha_1-\alpha_2\gamma}$ by (7.54), we obtain

$$
\begin{aligned}
I_3={}&-\alpha\int_{\mathbb{R}^k}\int_0^\infty\left((2\gamma+m+\gamma\alpha_2)\frac{r^{2\gamma}}{\rho^{2\gamma+2}}+(\alpha_1-\alpha_2\gamma-2\gamma-2)\frac{r^{4\gamma+2}}{\rho^{4\gamma+4}}\right)\\
&\qquad\times r^{m-1}F(r,y)|f|^2drdy\\
={}&-\alpha\int_{\mathbb{R}^k}\int_0^\infty\left(2\gamma+m+\gamma\alpha_2+(\alpha_1-\alpha_2\gamma-2\gamma-2)\frac{r^{2\gamma+2}}{\rho^{2\gamma+2}}\right)\\
&\qquad\times\frac{|\nabla_\gamma\rho|^2}{\rho^2}|f|^2r^{m-1}F(r,y)drdy.
\end{aligned}
$$

Similarly, we have for I_4 that

$$
\begin{aligned}
I_4={}&-\alpha\int_{\mathbb{R}^k}\int_0^\infty\mathrm{div}_y\left(F(r,y)\frac{\nabla_y\rho}{\rho}\right)r^{2\gamma+m-1}|f|^2drdy\\
={}&-\alpha(\gamma+1)\int_{\mathbb{R}^k}\int_0^\infty\mathrm{div}_y\left(\rho^{\alpha_1-\alpha_2\gamma-2\gamma-2}y\right)r^{\alpha_2\gamma+2\gamma+m-1}|f|^2drdy
\end{aligned}
$$

$$= -\alpha \int_{\mathbb{R}^k}\int_0^\infty \left((\alpha_1-\alpha_2\gamma-2\gamma-2)\rho^{\alpha_1-\alpha_2\gamma-2\gamma-3}\frac{(\gamma+1)^2|y|^2}{\rho^{2\gamma+1}}\right) \times r^{\alpha_2\gamma+2\gamma+m-1}|f|^2 drdy$$
$$-\alpha \int_{\mathbb{R}^k}\int_0^\infty k(\gamma+1)\rho^{\alpha_1-\alpha_2\gamma-2\gamma-2} r^{\alpha_2\gamma+2\gamma+m-1}|f|^2 drdy.$$

Since $\frac{r^{2\gamma}}{\rho^{2\gamma+2}} = \frac{|\nabla_\gamma \rho|^2}{\rho^2}$ and $F(r,y) = r^{\alpha_2\gamma}\rho^{\alpha_1-\alpha_2\gamma}$ by (7.56) and (7.54), respectively, we have

$$I_4 = -\alpha \int_{\mathbb{R}^k}\int_0^\infty \left((\alpha_1-\alpha_2\gamma-2\gamma-2)\frac{(\gamma+1)^2|y|^2}{\rho^{2\gamma+2}} + k(\gamma+1)\right) \times \frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1} F(r,y) drdy.$$

Then, taking into account (7.48) we get

$$I_3 + I_4 = -\alpha \int_{\mathbb{R}^k}\int_0^\infty (\alpha_1-\alpha_2\gamma-2\gamma-2+2\gamma+m+\gamma\alpha_2+k(\gamma+1)) \times \frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1} F(r,y) drdy.$$

Finally, using that $Q = m + (1+\gamma)k$ we obtain

$$I_3 + I_4 = -\alpha \int_{\mathbb{R}^k}\int_0^\infty (Q+\alpha_1-2)\frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1} F(r,y) drdy. \tag{7.58}$$

Putting (7.57) and (7.58) in (7.55) we get

$$\int_{\mathbb{R}^k}\int_0^\infty \left(\left|\left(\partial_r + \alpha\frac{\partial_r\rho}{\rho}\right)f\right|^2 + r^{2\gamma}\left|\left(\nabla_y + \alpha\frac{\nabla_y\rho}{\rho}\right)f\right|^2\right) r^{m-1}F(r,y)drdy$$
$$= \int_{\mathbb{R}^k}\int_0^\infty \left(|\partial_r f|^2 + r^{2\gamma}|\nabla_y f|^2\right) r^{m-1}F(r,y)drdy$$
$$- \left((Q+\alpha_1-2)\alpha - \alpha^2\right)\int_{\mathbb{R}^k}\int_0^\infty \frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1}F(r,y)drdy.$$

By substituting $\alpha = \frac{Q+\alpha_1-2}{2}$ and taking into account (7.54), we obtain (7.50).

The sharpness of the constant $\left(\frac{Q+\alpha_1-2}{2}\right)^2$ in (7.50) follows from the inequalities

$$\int_{\mathbb{R}^n} \rho^{\alpha_1}|\nabla_\gamma\rho|^{\alpha_2}\left(|\nabla_x f|^2 + |x|^{2\gamma}|\nabla_y f|^2\right)dxdy$$
$$\geq \int_{\mathbb{R}^n} \rho^{\alpha_1}|\nabla_\gamma\rho|^{\alpha_2}\left(\left|\frac{d}{d|x|}f\right|^2 + |x|^{2\gamma}|\nabla_y f|^2\right)dxdy$$
$$\geq \left(\frac{Q+\alpha_1-2}{2}\right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1}|\nabla_\gamma\rho|^{\alpha_2}\frac{|\nabla_\gamma\rho|^2}{\rho^2}|f|^2 dxdy,$$

since it is known that this inequality is sharp in (7.53), see Remark 7.6.4, Part 2.

$\square$

7.7 Weighted L^p-inequalities with boundary terms

In this section, we present a generalization of the weighted L^p-Hardy, L^p-Caffarelli–Kohn–Nirenberg, and L^p-Rellich inequalities with respect to the inclusion of boundary terms in the setting of stratified Lie groups. The appearing weights are controlled by a real-valued function V with the property that $\mathcal{L}V$ does not change sign. In addition to the inequalities themselves we will also give their refined versions involving expressions appearing due to the boundary of the domain in which these inequalities are derived. As a consequence, one can recover many of the Hardy type inequalities and Heisenberg–Pauli–Weyl type uncertainty principles on stratified groups by choosing special cases of the real-valued function V and working with functions vanishing at the boundary. The exposition of this section follows [RSS18d]. In Section 11.4 we will discuss boundary terms again but emphasizing the use of the $\mathcal{L}$-gauge in that discussion.

Setting of this section

Thus, throughout this section Ω is an admissible domain in the stratified group $\mathbb{G}$, and V is a real-valued function in $L^1_{\mathrm{loc}}(\Omega)$ with partial derivatives of order up to two in $L^1_{\mathrm{loc}}(\Omega)$, and such that $\mathcal{L}V$ is of one sign. Also, as usual, N denotes the dimension of the first stratum of the group $\mathbb{G}$ and ∇_H the horizontal gradient on $\mathbb{G}$. Then, as in (1.87), the vector field $\widetilde{\nabla} u$ is defined by

$$\widetilde{\nabla} u := \sum_{k=1}^{N} (X_k u)\, X_k. \tag{7.59}$$

7.7.1 Hardy and Caffarelli–Kohn–Nirenberg inequalities

We start with Hardy and Caffarelli–Kohn–Nirenberg inequalities with generalized weights.

Theorem 7.7.1 (L^p-Hardy inequality with generalized weight and boundary term)**.** *Let $1 < p < \infty$. Let V be a real-valued function such that $\mathcal{L}V < 0$ holds a.e. in Ω. Then for all complex-valued functions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ we have the inequality*

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} \leq p \left\| \frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_H u| \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{p-1}_{L^p(\Omega)} - \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle. \tag{7.60}$$

Proof of Theorem 7.7.1. Let us denote

$$\upsilon_\epsilon := (|u|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon.$$

Then $\upsilon_\epsilon^p \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and using Green's first formula in Theorem 1.4.6 and the fact that $\mathcal{L}V < 0$ we get

$$\int_\Omega |\mathcal{L}V| \upsilon_\epsilon^p dx = -\int_\Omega \mathcal{L}V \upsilon_\epsilon^p dx = \int_\Omega (\widetilde{\nabla} V) \upsilon_\epsilon^p dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle$$

$$
\begin{aligned}
&= \int_\Omega \nabla_H V \cdot \nabla_H \upsilon_\epsilon^p dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx\rangle \\
&\le \int_\Omega |\nabla_H V||\nabla_H \upsilon_\epsilon^p| dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx\rangle \\
&= p\int_\Omega \left(\frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}\right)|\mathcal{L}V|^{\frac{p-1}{p}} \upsilon_\epsilon^{p-1}|\nabla_H \upsilon_\epsilon| dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx\rangle,
\end{aligned}
$$

where, as usual, $(\widetilde{\nabla} u)v = \nabla_H u \cdot \nabla_H v$. We then have

$$
\nabla_H \upsilon_\epsilon = (|u|^2 + \epsilon^2)^{-\frac{1}{2}} |u| \nabla_H |u|,
$$

since $0 \le \upsilon_\epsilon \le |u|$. Thus, we also have

$$
\upsilon_\epsilon^{p-1} |\nabla_H \upsilon_\epsilon| \le |u|^{p-1} |\nabla_H |u||.
$$

On the other hand, let us write

$$
u(x) = R(x) + iI(x),
$$

where $R(x)$ and $I(x)$ denote the real and imaginary parts of u. We can restrict to the set where $u \neq 0$. Then we have

$$
(\nabla_H |u|)(x) = \frac{1}{|u|}(R(x)\nabla_H R(x) + I(x)\nabla_H I(x)) \quad \text{if} \quad u \neq 0.
$$

Since

$$
\left|\frac{1}{|u|}(R\nabla_H R + I\nabla_H I)\right|^2 \le |\nabla_H R|^2 + |\nabla_H I|^2,
$$

we get that $|\nabla_H |u|| \le |\nabla_H u|$ a.e. in Ω. Therefore,

$$
\begin{aligned}
\int_\Omega |\mathcal{L}V| \upsilon_\epsilon^p dx &\le p\int_\Omega \left(\frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}|\nabla_H u|\right)|\mathcal{L}V|^{\frac{p-1}{p}} |u|^{p-1} dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx\rangle \\
&\le p\left(\int_\Omega \left(\frac{|\nabla_H V|^p}{|\mathcal{L}V|^{(p-1)}}|\nabla_H u|^p\right) dx\right)^{\frac{1}{p}} \left(\int_\Omega |\mathcal{L}V||u|^p dx\right)^{\frac{p-1}{p}} - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx\rangle,
\end{aligned}
$$

where we have used Hölder's inequality in the last line. Thus, when $\epsilon \to 0$, we obtain (7.60). □

Remark 7.7.2.

1. If u vanishes on the boundary $\partial\Omega$, then (7.60) extends the Davies and Hinz result [DH98] to the following weighted L^p-Hardy type inequality on stratified groups:

$$
\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \le p \left\| \frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_H u| \right\|_{L^p(\Omega)}, \quad 1 < p < \infty. \tag{7.61}
$$

2. There are a number of other interesting consequences of Theorem 7.7.1 that we can record. We now discuss several such statements. First we present a horizontal L^p-Caffarelli–Kohn–Nirenberg type inequality with the boundary term on the stratified group $\mathbb{G}$. Incidentally, this also gives another proof of the horizontal L^p-Hardy type inequality (such as that in Theorem 6.2.1).

Corollary 7.7.3 (Horizontal L^p-Caffarelli–Kohn–Nirenberg inequality with boundary term)**.** *Let $1 < p < \infty$ and let $\alpha, \beta \in \mathbb{R}$. Let Ω be an admissible domain in a stratified group $\mathbb{G}$ with $N \geq 3$ being the dimension of the first stratum. Let $|\cdot|_E$ be the Euclidean norm on $\mathbb{R}^N$. Then for all $u \in C^2(\Omega\backslash\{x'=0\}) \cap C^1(\overline{\Omega}\backslash\{x'=0\})$ we have*

$$\frac{|N-\gamma|}{p}\left\|\frac{u}{|x'|_E^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^p \leq \left\|\frac{\nabla_H u}{|x'|_E^{\alpha}}\right\|_{L^p(\Omega)}\left\|\frac{u}{|x'|_E^{\frac{\beta}{p-1}}}\right\|_{L^p(\Omega)}^{p-1} - \frac{1}{p}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}|x'|_E^{2-\gamma}, dx\rangle, \tag{7.62}$$

for $2 < \gamma < N$ with $\gamma = \alpha+\beta+1$. In particular, if u vanishes on the boundary $\partial\Omega$, we have (6.3)*.*

Proof of Corollary 7.7.3*.* We will show that (7.62) follows as a special case of (7.60). Let us take

$$V(x) := |x'|_E^{2-\gamma}.$$

Then we have

$$|\nabla_H V| = |2-\gamma||x'|_E^{1-\gamma}, \qquad |\mathcal{L}V| = |(2-\gamma)(N-\gamma)||x'|_E^{-\gamma},$$

and observe that $\mathcal{L}V = (2-\gamma)(N-\gamma)|x'|_E^{-\gamma} < 0$. To use (7.60) we calculate the following expressions:

$$\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|_{L^p(\Omega)}^p = |(2-\gamma)(N-\gamma)|\left\|\frac{u}{|x'|_E^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^p,$$

$$\left\|\frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}\nabla_H u\right\|_{L^p(\Omega)} = \frac{|2-\gamma|}{|(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}}}\left\|\frac{|\nabla_H u|}{|x'|_E^{\frac{\gamma-p}{p}}}\right\|_{L^p(\Omega)},$$

$$\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|_{L^p(\Omega)}^{p-1} = |(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}}\left\|\frac{u}{|x'|_E^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^{p-1}.$$

Thus, (7.60) implies the inequality

$$\frac{|N-\gamma|}{p}\left\|\frac{u}{|x'|_E^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^p \leq \left\|\frac{\nabla_H u}{|x'|_E^{\frac{\gamma-p}{p}}}\right\|_{L^p(\Omega)}\left\|\frac{u}{|x'|_E^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^{p-1} - \frac{1}{p}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}|x'|_E^{2-\gamma}, dx\rangle.$$

If we denote $\alpha = \frac{\gamma-p}{p}$ and $\frac{\beta}{p-1} = \frac{\gamma}{p}$, we obtain (7.62). $\square$

Another interesting feature of Theorem 7.7.1 is that it also allows one to obtain inequalities with the $\mathcal{L}$-gauge d.

Let us give an example.

Corollary 7.7.4 (Hardy inequality with $\mathcal{L}$-gauge weights and boundary term)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$, and assume that $0 \notin \partial\Omega$. Let $2-Q < \alpha < 0$. Let $u \in C^1(\Omega\backslash\{0\}) \cap C(\overline{\Omega}\backslash\{0\})$. Then we have*

$$\frac{|Q+\alpha-2|}{p} \left\| d^{\frac{\alpha-2}{p}} |\nabla_H d|^{\frac{2}{p}} u \right\|_{L^p(\Omega)} \tag{7.63}$$
$$\leq \left\| d^{\frac{p+\alpha-2}{p}} |\nabla_H d|^{\frac{2-p}{p}} |\nabla_H u| \right\|_{L^p(\Omega)} - \frac{1}{p} \left\| d^{\frac{\alpha-2}{p}} |\nabla_H d|^{\frac{2}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} d^{\alpha-1} |u|^p \langle \widetilde{\nabla} d, dx \rangle.$$

Proof of Corollary 7.7.4. First, we can multiply both sides of the inequality (7.60) by $\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)}$, so that we have the inequality

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq p \left\| \frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_H u| \right\|_{L^p(\Omega)} - \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle. \tag{7.64}$$

Now, let us take $V := d^\alpha$. Since $d = \varepsilon^{\frac{1}{2-Q}}$ for the fundamental solution ε of $\mathcal{L}$, we have

$$\begin{aligned} \mathcal{L} d^\alpha &= \nabla_H (\nabla_H \varepsilon^{\frac{\alpha}{2-Q}}) = \nabla_H \left(\frac{\alpha}{2-Q} \varepsilon^{\frac{\alpha+Q-2}{2-Q}} \nabla_H \varepsilon \right) \\ &= \frac{\alpha(\alpha+Q-2)}{(2-Q)^2} \varepsilon^{\frac{\alpha-4+2Q}{2-Q}} |\nabla_H \varepsilon|^2 + \frac{\alpha}{2-Q} \varepsilon^{\frac{\alpha+Q-2}{2-Q}} \mathcal{L}\varepsilon. \end{aligned}$$

Since ε is the fundamental solution of $\mathcal{L}$, it follows that

$$\mathcal{L} d^\alpha = \frac{\alpha(\alpha+Q-2)}{(2-Q)^2} \varepsilon^{\frac{\alpha-4+2Q}{2-Q}} |\nabla_H \varepsilon|^2 = \alpha(\alpha+Q-2) d^{\alpha-2} |\nabla_H d|^2.$$

From this we can observe that $\mathcal{L} d^\alpha < 0$, and also a direct calculation yields the identities

$$\left\| |\mathcal{L} d^\alpha|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} = \alpha^{\frac{1}{p}} |Q+\alpha-2|^{\frac{1}{p}} \left\| d^{\frac{\alpha-2}{p}} |\nabla_H d|^{\frac{2}{p}} u \right\|_{L^p(\Omega)},$$

$$\left\| \frac{|\nabla_H d^\alpha|}{|\mathcal{L} d^\alpha|^{\frac{p-1}{p}}} |\nabla_H u| \right\|_{L^p(\Omega)} = \alpha^{\frac{1}{p}} |Q+\alpha-2|^{\frac{1-p}{p}} \left\| d^{\frac{\alpha-2+p}{p}} |\nabla_H d|^{\frac{2-p}{p}} |\nabla_H u| \right\|_{L^p(\Omega)},$$

$$\begin{aligned} &\left\| |\mathcal{L} d^\alpha|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} d^\alpha, dx \rangle \\ &\quad = \alpha^{\frac{1}{p}} |Q+\alpha-2|^{\frac{1-p}{p}} \left\| d^{\frac{\alpha-2}{p}} |\nabla_H d|^{\frac{2}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} d^{\alpha-1} |u|^p \langle \widetilde{\nabla} d, dx \rangle. \end{aligned}$$

Using (7.64) we arrive at

$$\frac{|Q+\alpha-2|}{p}\left\|d^{\frac{\alpha-2}{p}}|\nabla_H d|^{\frac{2}{p}}u\right\|_{L^p(\Omega)}$$
$$\leq \left\|d^{\frac{p+\alpha-2}{p}}|\nabla_H d|^{\frac{2-p}{p}}|\nabla_H u|\right\|_{L^p(\Omega)} - \frac{1}{p}\left\|d^{\frac{\alpha-2}{p}}|\nabla_H d|^{\frac{2}{p}}u\right\|^{1-p}_{L^p(\Omega)}\int_{\partial\Omega} d^{\alpha-1}|u|^p\langle\widetilde{\nabla}d, dx\rangle,$$

which implies (7.63). □

The inequality (7.64) implies the following generalized Heisenberg–Pauli–Weyl type uncertainty principle on stratified groups.

Corollary 7.7.5 (Weighted Heisenberg–Pauli–Weyl uncertainty principle with boundary term)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ and let $V \in C^2(\Omega)$ be real-valued. Then for all complex-valued functions $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ we have*

$$\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\|\frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}|\nabla_H u|\right\|_{L^p(\Omega)}$$
$$\geq \frac{1}{p}\|u\|^2_{L^p(\Omega)} + \frac{1}{p}\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^{1-p}_{L^p(\Omega)}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}V, dx\rangle. \tag{7.65}$$

In particular, if u vanishes on the boundary $\partial\Omega$, then we have

$$\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\|\frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}|\nabla_H u|\right\|_{L^p(\Omega)} \geq \frac{1}{p}\|u\|^2_{L^p(\Omega)}. \tag{7.66}$$

By setting $V = |x'|^\alpha$ in the inequality (7.66), we recover the Heisenberg–Pauli–Weyl type uncertainty principle on stratified groups.

Proof of Corollary 7.7.5. By using the extended Hölder inequality and (7.64) we have

$$\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\|\frac{|\nabla_H V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}|\nabla_H u|\right\|_{L^p(\Omega)}$$
$$\geq \frac{1}{p}\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|_{L^p(\Omega)}$$
$$+\frac{1}{p}\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^{1-p}_{L^p(\Omega)}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}V, dx\rangle,$$
$$\geq \frac{1}{p}\left\||u|^2\right\|_{L^{\frac{p}{2}}(\Omega)} + \frac{1}{p}\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^{1-p}_{L^p(\Omega)}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}V, dx\rangle.$$
$$= \frac{1}{p}\|u\|^2_{L^p(\Omega)} + \frac{1}{p}\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^{1-p}_{L^p(\Omega)}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}V, dx\rangle,$$

proving (7.65). □

7.7.2 Rellich inequalities

In this section, we describe weighted Rellich inequalities with boundary terms. We consider first the L^2 and then the L^p case.

Theorem 7.7.6 (L^2-Rellich inequality with generalized weight and boundary term)**.** *Let $V \in C^2(\Omega)$ be a real-valued function such that $\mathcal{L}V(x) < 0$ for all $x \in \Omega \subset \mathbb{G}$. Then for every $\epsilon > 0$ we have*

$$\begin{aligned}\left\| \frac{|V|}{|\mathcal{L}V|^{\frac{1}{2}}}\mathcal{L}u \right\|^2_{L^2(\Omega)} \geq 2\epsilon \left\| V^{\frac{1}{2}}|\nabla_H u| \right\|^2_{L^2(\Omega)} + \epsilon(1-\epsilon)\left\| |\mathcal{L}V|^{\frac{1}{2}} u \right\|^2_{L^2(\Omega)} \\ - \epsilon \int_{\partial\Omega} (|u|^2 \langle \widetilde{\nabla} V, dx\rangle - V\langle \widetilde{\nabla}|u|^2, dx\rangle),\end{aligned} \tag{7.67}$$

for all complex-valued functions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. In particular, if u vanishes on the boundary $\partial\Omega$, we have

$$\left\| \frac{|V|}{|\mathcal{L}V|^{\frac{1}{2}}}\mathcal{L}u \right\|^2_{L^2(\Omega)} \geq 2\epsilon \left\| V^{\frac{1}{2}}|\nabla_H u| \right\|^2_{L^2(\Omega)} + \epsilon(1-\epsilon)\left\| |\mathcal{L}V|^{\frac{1}{2}} u \right\|^2_{L^2(\Omega)}.$$

Remark 7.7.7. In the case of $\mathbb{R}^n$, an analogous L^2-Rellich inequality was proved by Schmincke [Sch72] and generalized further by Bennett [Ben89]. In the setting of stratified group this and other results of this section were obtained in [RSS18d].

Proof of Theorem 7.7.6. Using Green's second identity from Theorem 1.4.6 and the condition that $\mathcal{L}V(x) < 0$ in Ω, we obtain

$$\begin{aligned}\int_\Omega |\mathcal{L}V||u|^2 dx &= -\int_\Omega V\mathcal{L}|u|^2 dx - \int_{\partial\Omega} (|u|^2 \langle \widetilde{\nabla} V, dx\rangle - V\langle \widetilde{\nabla}|u|^2, dx\rangle) \\ &= -2\int_\Omega V\left(\mathrm{Re}(\overline{u}\mathcal{L}u) + |\nabla_H u|^2\right) dx - \int_{\partial\Omega} (|u|^2 \langle \widetilde{\nabla} V, dx\rangle - V\langle \widetilde{\nabla}|u|^2, dx\rangle).\end{aligned}$$

Using the Cauchy–Schwarz inequality this implies

$$\begin{aligned}\int_\Omega |\mathcal{L}V||u|^2 dx &\leq 2\left(\frac{1}{\epsilon}\int_\Omega \frac{|V|^2}{|\mathcal{L}V|}|\mathcal{L}u|^2 dx\right)^{1/2} \left(\epsilon \int_\Omega |\mathcal{L}V||u|^2 dx\right)^{1/2} \\ &\quad - 2\int_\Omega V|\nabla_H u|^2 dx - \int_{\partial\Omega} (|u|^2 \langle \widetilde{\nabla} V, dx\rangle - V\langle \widetilde{\nabla}|u|^2, dx\rangle) \\ &\leq \frac{1}{\epsilon}\int_\Omega \frac{|V|^2}{|\mathcal{L}V|}|\mathcal{L}u|^2 dx + \epsilon\int_\Omega |\mathcal{L}V||u|^2 dx \\ &\quad - 2\int_\Omega V|\nabla_H u|^2 dx - \int_{\partial\Omega} (|u|^2 \langle \widetilde{\nabla} V, dx\rangle - V\langle \widetilde{\nabla}|u|^2, dx\rangle),\end{aligned}$$

yielding (7.67). ∎

We now move to the case of L^p-estimates.

Theorem 7.7.8 (L^p-Rellich inequality with generalized weight and boundary term)**.** *Let $1 \leq p < \infty$. Let Ω be an admissible domain in a stratified group $\mathbb{G}$. If*

$$0 < V \in C(\Omega),\ \mathcal{L}V < 0 \ \textit{ and } \mathcal{L}(V^{\sigma}) \leq 0$$

on Ω for some $\sigma > 1$, then for all $u \in C_0^\infty(\Omega)$ we have

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq \frac{p^2}{(p-1)\sigma + 1} \left\| \frac{V}{|\mathcal{L}V|^{\frac{p-1}{p}}} \mathcal{L}u \right\|_{L^p(\Omega)}. \tag{7.68}$$

Before proving Theorem 7.7.8, let us make some remarks and establish some preliminary properties needed for its proof.

Remark 7.7.9.

1. Choosing $V = |x'|_E^{-(\alpha-2)}$ in Theorem 7.7.8, with the Euclidean distance $|\cdot|_E$ in the first stratum of $\mathbb{G}$, we obtain for any $2 < \alpha < N$ and all $u \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$, the inequality
$$\int_{\mathbb{G}} \frac{|u|^p}{|x'|_E^{\alpha}} dx \leq C^p_{(N,p,\alpha)} \int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|x'|_E^{\alpha-2p}} dx, \tag{7.69}$$
where
$$C_{(N,p,\alpha)} = \frac{p^2}{(N-\alpha)\left((p-1)N + \alpha - 2p\right)}. \tag{7.70}$$

2. Let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution of the sub-Laplacian $\mathcal{L}$. Assume that $Q \geq 3$, $\alpha < 2$, and $Q + \alpha - 4 > 0$. Choosing $V = d^{\alpha-2}$ in Theorem 7.7.8, with d being the $\mathcal{L}$-gauge as above, we obtain
$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{G}} d^{\alpha-4} |\nabla_H d|^2 |u|^2 dx \leq \int_{\mathbb{G}} \frac{d^{\alpha}}{|\nabla_H d|^2} |\mathcal{L}u|^2 dx. \tag{7.71}$$

Theorem 7.7.8 will be proved as follows: it is a consequence of Lemma 7.7.11, by putting $C = \frac{(p-1)(\sigma-1)}{p}$ in Lemma 7.7.10.

Lemma 7.7.10. *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. If $V \geq 0$, $\mathcal{L}V < 0$, and there exists a constant $C \geq 0$ such that*

$$C \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} \leq p(p-1) \left\| V^{\frac{1}{p}} |u|^{\frac{p-2}{p}} |\nabla_H u|^{\frac{2}{p}} \right\|^p_{L^p(\Omega)}, \quad 1 < p < \infty, \tag{7.72}$$

for all $u \in C_0^\infty(\Omega)$, then we have

$$(1 + C) \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq p \left\| \frac{V}{|\mathcal{L}V|^{\frac{p-1}{p}}} \mathcal{L}u \right\|_{L^p(\Omega)}, \tag{7.73}$$

for all $u \in C_0^\infty(\Omega)$. If $p = 1$ then the statement holds for $C = 0$.

Proof of Lemma 7.7.10. In view of (2.8) we can assume that u is real-valued. Let $\epsilon > 0$ and set

$$u_\epsilon := (|u|^2 + \epsilon^2)^{p/2} - \epsilon^p.$$

Then $0 \leq u_\epsilon \in C_0^\infty$ and

$$\int_\Omega |\mathcal{L}V| u_\epsilon dx = -\int_\Omega (\mathcal{L}V) u_\epsilon dx = -\int_\Omega V \mathcal{L} u_\epsilon dx,$$

where

$$\begin{aligned}
\mathcal{L}u_\epsilon = \mathcal{L}\left((|u|^2+\epsilon^2)^{\frac{p}{2}} - \epsilon^p\right) &= \nabla_H \cdot (\nabla_H((|u|^2+\epsilon^2)^{\frac{p}{2}} - \epsilon^p)) \\
&= \nabla_H (p(|u|^2+\epsilon^2)^{\frac{p-2}{2}} u \nabla_H u) \\
&= p(p-2)(|u|^2+\epsilon^2)^{\frac{p-4}{2}} u^2 |\nabla_H u|^2 \\
&\quad + p(|u|^2+\epsilon^2)^{\frac{p-2}{2}} |\nabla_H u|^2 + p(|u|^2+\epsilon^2)^{\frac{p-2}{2}} u \mathcal{L} u.
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_\Omega |\mathcal{L}V| u_\epsilon dx = & -\int_\Omega \left(p(p-2)u^2(u^2+\epsilon^2)^{\frac{p-4}{2}} + p(u^2+\epsilon^2)^{\frac{p-2}{2}}\right) V |\nabla_H u|^2 dx \\
& - p \int_\Omega V u (u^2+\epsilon^2)^{\frac{p-2}{2}} \mathcal{L} u dx.
\end{aligned}$$

Hence we have the inequality

$$\begin{aligned}
\int_\Omega |\mathcal{L}V| u_\epsilon + \left(p(p-2)u^2(u^2+\epsilon^2)^{\frac{p-4}{2}} + p(u^2+\epsilon^2)^{\frac{p-2}{2}}\right) V |\nabla_H u|^2 dx \\
\leq p \int_\Omega V |u| (u^2+\epsilon^2)^{\frac{p-2}{2}} |\mathcal{L} u| dx.
\end{aligned}$$

When $\epsilon \to 0$, the integrand on the left-hand side is non-negative and tends to

$$|\mathcal{L}V||u|^p + p(p-1)V|u|^{p-2}|\nabla_H u|^2$$

pointwise, only for $u \neq 0$ when $p < 2$, otherwise for any x. On the other hand, the integrand on the right-hand side is bounded by

$$V(\max |u|^2 + 1)^{(p-1)/2} \max |\mathcal{L}u|$$

and it is integrable because $u \in C_0^\infty(\Omega)$, and so the integral tends to

$$\int_\Omega V|u|^{p-1}|\mathcal{L}u| dx$$

by the dominated convergence theorem. It then follows by Fatou's lemma that

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} + p(p-1) \left\| V^{\frac{1}{p}} |u|^{\frac{p-2}{p}} |\nabla_H u|^{\frac{2}{p}} \right\|^p_{L^p(\Omega)} \leq p \left\| V^{\frac{1}{p}} |u|^{\frac{p-1}{p}} |\mathcal{L}u|^{\frac{1}{p}} \right\|^p_{L^p(\Omega)}.$$

By using (7.72), followed by Hölder's inequality, we obtain

$$(1+C)\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^p_{L^p(\Omega)} \le p\left\||\mathcal{L}V|^{(p-1)}V^{\frac{1}{p}}|u|^{\frac{p-1}{p}}|\mathcal{L}V|^{-(p-1)}|\mathcal{L}u|^{\frac{1}{p}}\right\|^p_{L^p(\Omega)}$$
$$\le p\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^{p-1}_{L^p(\Omega)}\left\|\frac{|V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}\mathcal{L}u\right\|_{L^p(\Omega)}.$$

This implies (7.73). $\square$

Lemma 7.7.11. *Let $1 < p < \infty$. Let Ω be an admissible domain in a stratified group $\mathbb{G}$. If*

$$0 < V \in C(\Omega),\ \mathcal{L}V < 0 \ \text{ and } \ \mathcal{L}V^\sigma \le 0$$

on Ω for some $\sigma > 1$, then we have

$$(\sigma-1)\int_\Omega |\mathcal{L}V||u|^p dx \le p^2\int_{\{x\in\Omega, u(x)\neq 0\}} V|u|^{p-2}|\nabla_H u|^2 dx < \infty, \tag{7.74}$$

for all $u \in C_0^\infty(\Omega)$.

Proof of Lemma 7.7.11. We shall use that

$$0 \ge \mathcal{L}(V^\sigma) = \sigma V^{\sigma-2}\left((\sigma-1)|\nabla_H V|^2 + V\mathcal{L}V\right), \tag{7.75}$$

and hence

$$(\sigma-1)|\nabla_H V|^2 \le V|\mathcal{L}V|.$$

First we consider the case $p=2$: we use the inequality (7.61) to get

$$(\sigma-1)\int_\Omega |\mathcal{L}V||u|^2 dx \le 4(\sigma-1)\int_\Omega \frac{|\nabla_H V|^2}{|\mathcal{L}V|}|\nabla_H u|^2 dx$$
$$\le 4\int_\Omega V|\nabla_H u|^2 dx$$
$$= 4\int_{\{x\in\Omega; u(x)\neq 0, |\nabla_H u|\neq 0\}} V|\nabla_H u|^2 dx, \tag{7.76}$$

the last equality valid since $|\{x\in\Omega; u(x)=0, |\nabla_H u|\neq 0\}| = 0$. This proves Lemma 7.7.11 for $p=2$.

For $p \neq 2$, denote

$$v_\epsilon := (u^2+\epsilon^2)^{p/4} - \epsilon^{p/2},$$

and let $\epsilon \to 0$. Since

$$0 \le v_\epsilon \le |u|^{\frac{p}{2}},$$

the left-hand side of (7.76), with u replaced by v_ϵ, tends to

$$(\sigma-1)\int_\Omega |\mathcal{L}V||u|^p dx$$

by the dominated convergence theorem. If $u \neq 0$, then

$$|\nabla_H v_\epsilon|^2 V = \left|\frac{p}{2} u (u^2+\epsilon^2)^{\frac{p-4}{4}} \nabla_H u\right|^2 V.$$

For $\epsilon \to 0$ we obtain

$$|\nabla_H u|^p V = \frac{p^2}{4} |u|^{p-2} |\nabla_H u|^2 V.$$

It follows as in the proof of Lemma 7.7.10, by using Fatou's lemma, that the right-hand side of (7.76) tends to

$$p^2 \int_{\{x\in\Omega; u(x)\neq 0, |\nabla_H u|\neq 0\}} V|u|^{p-2} |\nabla_H u|^2 dx,$$

and this completes the proof. □