Chapter 6

Horizontal Inequalities on Stratified Groups

In this chapter we discuss versions of some of the inequalities from the previous chapters in the setting of stratified groups. Because of the stratified structure here we can use the horizontal gradient in the estimates.

As already outlined in the introduction there are three versions of estimates on stratified groups available in the literature:

(A) Using the homogeneous semi-norm, sometimes called the $\mathcal{L}$-gauge, given by the appropriate power of the fundamental solution of the sub-Laplacian $\mathcal{L}$. Thus, if $d(x)$ is the $\mathcal{L}$-gauge, then $d(x)^{2-Q}$ is a constant multiple of Folland's [Fol75] fundamental solution of the sub-Laplacian $\mathcal{L}$, with Q being the homogeneous dimension of the stratified group $\mathbb{G}$; these will be discussed in Chapter 7.

(B) Using the Carnot–Carathéodory distance, i.e., the control distance associated to the sub-Laplacian.

(C) Using the Euclidean distance on the first stratum of the group.

The constants in the corresponding inequalities may depend on the quasi-norm that one is using. There is an extensive literature on Hardy type inequalities of stratified Lie groups, see, e.g., [DGP11], [GL90], [GK08], [Gri03], [JS11], [KS16], [KÖ13], [Lia13], [NZW01]). For example, in the case (A) the Hardy inequality takes the form

$$\left\| \frac{f}{d(x)} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \left\| \nabla_H f \right\|_{L^p(\mathbb{G})}, \quad Q \geq 3,\ 1 < p < Q, \tag{6.1}$$

where Q is the homogeneous dimension of the stratified group $\mathbb{G}$, ∇_H is the horizontal gradient, and $d(x)$ is the $\mathcal{L}$-gauge from (A). The analysis in the case (A) in terms of the fundamental solution of the sub-Laplacian will be the subject of Chapter 11. The results on Hardy and other inequalities for the case (B) are less extensive, mostly devoted to the case of the Heisenberg group.

In this chapter we concentrate on the case (C). Thus, here throughout we adopt the notations from Section 1.4.8 concerning the stratified groups. In this

M. Ruzhansky, D. Suragan, *Hardy Inequalities on Homogeneous Groups*,
Progress in Mathematics 327, https://doi.org/10.1007/978-3-030-02895-4_7

case there are several additional properties available assisting the analysis, such as formulae (1.72) and (1.73), making certain calculations possible. It is also worth noting that generally, for the same types of inequalities, the optimal constants in case (C) are different than the optimal constants in case (A).

Some analysis of inequalities of the case (C) is known in the literature, see, e.g., [BT02a] and [D'A04b]. However, in this chapter we aim at developing an independent point of view based on the divergence relations (1.72) and (1.73).

Notation. As already mentioned, throughout this chapter we adopt the notations from Section 1.4.8 concerning the stratified groups. In particular, $\mathbb{G}$ will always be a stratified group of homogeneous dimension Q, with N being the dimension of the first stratum. Also, x' will denote the variables in the first stratum of $\mathbb{G}$, and ∇_H will denote the horizontal gradient. To simplify the notation, we denote simply by

$$|x'| = \sqrt{x_1'^2 + \cdots + x_N'^2}$$

the Euclidean norm on the first stratum of $\mathbb{G}$, which can be identified with $\mathbb{R}^N$.

6.1 Horizontal L^p-Caffarelli–Kohn–Nirenberg type inequalities

In this section we establish the horizontal version on stratified groups of the L^p-Caffarelli–Kohn–Nirenberg type inequalities from Section 3.3.1. In particular, this would imply the horizontal version, as in the case (C) above, of the L^p-Hardy inequality with the sharp constant:

$$\left\| \frac{f}{|x'|} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{N-p} \left\| \nabla_H f \right\|_{L^p(\mathbb{G})}, \quad 1 < p < N. \tag{6.2}$$

We obtain it as a special case of the following more general inequality, see Remark 6.1.2, Part 3.

Theorem 6.1.1 (Horizontal L^p-Caffarelli–Kohn–Nirenberg inequalities)**.** *For any α, $\beta \in \mathbb{R}$ and every complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, we have*

$$\frac{|N-\gamma|}{p} \left\| \frac{f}{|x'|^{\frac{\gamma}{p}}} \right\|^p_{L^p(\mathbb{G})} \leq \left\| \frac{1}{|x'|^{\alpha}} \nabla_H f \right\|_{L^p(\mathbb{G})} \left\| \frac{f}{|x'|^{\frac{\beta}{p-1}}} \right\|^{p-1}_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{6.3}$$

where $\gamma = \alpha + \beta + 1$. If $\gamma \neq N$ then the constant $\frac{|N-\gamma|}{p}$ is sharp.

Before proving Theorem 6.1.1, let us point out some of its consequences.

Remark 6.1.2.

1. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$, so (6.3) gives the L^p-Caffarelli–Kohn–Nirenberg type inequality for $\mathbb{R}^n$ with the sharp constant:

$$\frac{|n-\gamma|}{p} \left\| \frac{f}{|x|_E^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{R}^n)}^p \leq \left\| \frac{1}{|x|_E^{\alpha}} \nabla f \right\|_{L^p(\mathbb{R}^n)} \left\| \frac{f}{|x|_E^{\frac{\beta}{p-1}}} \right\|_{L^p(\mathbb{R}^n)}^{p-1}, \tag{6.4}$$

for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, and $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$. In this case it becomes a special case of Theorem 3.3.3 because a particular (Euclidean) norm is used. In this case the inequality of this type has been analysed in, e.g., [Cos08] and [DJSJ13].

2. (Horizontal weighted L^p-Hardy inequality) In the case

$$\beta = \gamma \left(1 - \frac{1}{p}\right),$$

i.e., with $\beta = (\alpha+1)(p-1)$ and $\gamma = p(\alpha+1)$, inequality (6.3) implies the *horizontal weighted L^p-Hardy type inequality*

$$\frac{|N - p(\alpha+1)|}{p} \left\| \frac{f}{|x'|^{\alpha+1}} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{1}{|x'|^{\alpha}} \nabla_H f \right\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{6.5}$$

for any $f \in C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ and all $\alpha \in \mathbb{R}$, with sharp constant in (6.5) for $p(\alpha+1) \neq N$.

3. (Horizontal L^p-Hardy inequality) In particular, in the case of $\alpha = 0$, the inequality (6.5) implies the following stratified group version of *horizontal L^p-Hardy inequality* with the sharp constant:

$$\left\| \frac{f}{|x'|} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{N-p} \|\nabla_H f\|_{L^p(\mathbb{G})}, \quad 1 < p < N. \tag{6.6}$$

Such a type of inequalities was also considered in [D'A04b], and in [Yen16] in the case of the Heisenberg group.

In the case $p = 2$ this inequality can be in turn sharpened to the following inequality: If $N \geq 3$ and $\alpha \in \mathbb{R}$, then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ we have

$$\left\| \frac{f}{|x'|} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{N-2} \left\| \frac{x' \cdot \nabla_H f}{|x'|} \right\|_{L^2(\mathbb{G})}, \tag{6.7}$$

where the constant $\frac{2}{N-2}$ is sharp. This refinement will be shown in Theorem 6.4.4 by using the factorization method.

4. Clearly, when $\mathbb{G} = (\mathbb{R}^n, +)$, $n \geq 3$, (6.6) implies the classical Hardy inequality for $\mathbb{R}^n$:
$$\left\| \frac{f}{|x|_E} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)},$$
for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, and $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$.

Similar to Corollary 3.3.5, Theorem 6.1.1, and even the corresponding Hardy inequality, immediately implies a version of the Heisenberg–Pauli–Weyl uncertainty principle.

Corollary 6.1.3 (Horizontal Heisenberg–Pauli–Weyl uncertainty principle)**.** *For all* $f \in C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ *we have*
$$\|f\|^2_{L^2(\mathbb{G})} \leq \frac{p}{N-p} \|\nabla_H f\|_{L^p(\mathbb{G})} \||x'|f\|_{L^{\frac{p}{p-1}}(\mathbb{G})}, \quad 1 < p < N. \tag{6.8}$$

Proof of Corollary 6.1.3. Using (6.6) the Hölder inequality we immediately obtain
$$\begin{aligned} \|f\|^2_{L^2(\mathbb{G})} &\leq \left\| \frac{1}{|x'|} f \right\|_{L^p(\mathbb{G})} \||x'|f\|_{L^{\frac{p}{p-1}}(\mathbb{G})} \\ &\leq \frac{p}{N-p} \|\nabla_H f\|_{L^p(\mathbb{G})} \||x'|f\|_{L^{\frac{p}{p-1}}(\mathbb{G})}, \quad 1 < p < N, \end{aligned} \tag{6.9}$$
giving (6.8). □

Remark 6.1.4.

1. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, taking $N = n$, we get that (6.8) with $p = 2$ implies the classical uncertainty principle on $\mathbb{R}^n$, namely,
$$\left(\int_{\mathbb{R}^n} |f(x)|^2 dx \right)^2 \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} |\nabla f(x)|^2 dx \int_{\mathbb{R}^n} |x|_E^2 |f(x)|^2 dx, \tag{6.10}$$
for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. This is the Heisenberg–Pauli–Weyl uncertainty principle on $\mathbb{R}^n$. We note that we can also obtain (6.10) already as a consequence of the radial Heisenberg–Pauli–Weyl uncertainty principle on homogeneous groups, see Remark 3.3.6, Part 1. However, since the proofs of Theorem 3.3.3 and Theorem 6.1.1 are different, they give two different proofs of (6.10).
2. We can point out some inequalities with sharp constants as special cases of (6.3). For example, for $\alpha p = \alpha + \beta + 1$ we get
$$\frac{|N - \alpha p|}{p} \left\| \frac{f}{|x'|^\alpha} \right\|^p_{L^p(\mathbb{G})} \leq \left\| \frac{\nabla_H f}{|x'|^\alpha} \right\|_{L^p(\mathbb{G})} \left\| |x'|^{\frac{1}{p-1} - \alpha} f \right\|^{p-1}_{L^p(\mathbb{G})}. \tag{6.11}$$
Also, if $0 = \alpha + \beta + 1$ and $\alpha = -p$, then
$$\frac{N}{p} \|f\|^p_{L^p(\mathbb{G})} \leq \||x'|^p \nabla_H f\|_{L^p(\mathbb{G})} \left\| \frac{f}{|x'|} \right\|^{p-1}_{L^p(\mathbb{G})}, \tag{6.12}$$
with constants in both of these inequalities being sharp.

Proof of Theorem 6.1.1. We may assume that $\gamma \neq N$ since for $\gamma = N$ the inequality (6.3) is trivial. By using the identity (1.73), the divergence theorem, and Schwarz' inequality, one calculates

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^\gamma} dx &= \frac{1}{N-\gamma} \int_{\mathbb{G}} |f(x)|^p \operatorname{div}_H \left(\frac{x'}{|x'|^\gamma} \right) dx \\
&= -\frac{1}{N-\gamma} \operatorname{Re} \int_{\mathbb{G}} p f(x) |f(x)|^{p-2} \frac{\overline{x' \cdot \nabla_H f}}{|x'|^\gamma} dx \\
&\leq \left| \frac{p}{N-\gamma} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x'|^\gamma} \left| x' \cdot \nabla_H f \right| dx \\
&\leq \left| \frac{p}{N-\gamma} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x'|^{\alpha+\beta}} \left| \nabla_H f(x) \right| dx \\
&\leq \left| \frac{p}{N-\gamma} \right| \left(\int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^{\alpha p}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^{\frac{\beta p}{p-1}}} dx \right)^{\frac{p-1}{p}} .
\end{aligned}$$

Here in the last line we used Hölder's inequality. This gives

$$\left| \frac{N-\gamma}{p} \right| \int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^\gamma} dx \leq \left(\int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^{\alpha p}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^{\frac{\beta p}{p-1}}} dx \right)^{\frac{p-1}{p}},$$

proving (6.3). Let us now show the sharpness of the constant. For this, we look at the equality condition in Hölder's inequality. Let us consider the function

$$g(x) = \begin{cases} e^{-\frac{C}{\lambda}|x'|^\lambda}, & \lambda := \alpha - \frac{\beta}{p-1} + 1 \neq 0, \\ \frac{1}{|x'|^C}, & \alpha - \frac{\beta}{p-1} + 1 = 0, \end{cases}$$

where $C = \left| \frac{N-\gamma}{p} \right|$ and $\gamma \neq N$. Then it can be checked that

$$\left| \frac{p}{N-\gamma} \right|^p \frac{|\nabla_H g(x)|^p}{|x'|^{\alpha p}} = \frac{|g(x)|^p}{|x'|^{\frac{\beta p}{p-1}}}.$$

Finally, approximating this function by functions in $C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ completes the proof. $\square$

6.1.1 Badiale–Tarantello conjecture

The idea of the proof of Theorem 6.1.1 implies the following similar fact in $\mathbb{R}^n$ which we may split into two factors as $\mathbb{R}^n = \mathbb{R}^N \times \mathbb{R}^{n-N}$. The best constant in the Hardy inequality of the type of inequality (6.13) was conjectured by Badiale and Tarantello in [BT02a, Remark 2.3]. Although it was subsequently established in [SSW03], here we follow [RS17e] to present an independent proof of a more general result.

Proposition 6.1.5 (L^p-Caffarelli–Kohn–Nirenberg inequality for Euclidean decomposition)**.** *Let $x = (x', x'') \in \mathbb{R}^N \times \mathbb{R}^{n-N}$, $1 \leq N \leq n$, and α, $\beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, and all $1 < p < \infty$, we have*

$$\frac{|N-\gamma|}{p} \left\| \frac{f}{|x'|^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{R}^n)}^p \leq \left\| \frac{1}{|x'|^\alpha} \nabla f \right\|_{L^p(\mathbb{R}^n)} \left\| \frac{f}{|x'|^{\frac{\beta}{p-1}}} \right\|_{L^p(\mathbb{R}^n)}^{p-1}, \tag{6.13}$$

where $\gamma = \alpha + \beta + 1$ and $|x'|$ is the Euclidean norm on $\mathbb{R}^N$. If $\gamma \neq N$ then the constant $\frac{|N-\gamma|}{p}$ is sharp.

Proof of Proposition 6.1.5*.* The proof is a modification of the proof of Theorem 6.1.1. For $\gamma = N$ the inequality (6.13) is trivial, so let us assume $\gamma \neq N$. Thus, by using the identity

$$\mathrm{div}_N \frac{x'}{|x'|^\gamma} = \frac{N-\gamma}{|x'|^\gamma},$$

for all $\gamma \in \mathbb{R}$ and $x' \in \mathbb{R}^N$ with $|x'| \neq 0$, where div_N is the standard divergence on $\mathbb{R}^N$, and applying the divergence theorem and Schwarz' inequality one can calculate

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^\gamma} dx &= \frac{1}{N-\gamma} \int_{\mathbb{R}^n} |f(x)|^p \mathrm{div}_N \left(\frac{x'}{|x'|^\gamma} \right) dx \\
&= -\frac{1}{N-\gamma} \mathrm{Re} \int_{\mathbb{R}^n} p f(x) |f(x)|^{p-2} \frac{\overline{x' \cdot \nabla_N f}}{|x'|^\gamma} dx \\
&\leq \left| \frac{p}{N-\gamma} \right| \int_{\mathbb{R}^n} \frac{|f(x)|^{p-1}}{|x'|^\gamma} |x' \cdot \nabla_N f| \, dx \\
&= \left| \frac{p}{N-\gamma} \right| \int_{\mathbb{R}^n} \frac{|f(x)|^{p-1}}{|x'|^\gamma} |x'_0 \cdot \nabla f| \, dx \\
&\leq \left| \frac{p}{N-\gamma} \right| \int_{\mathbb{R}^n} \frac{|f(x)|^{p-1}}{|x'|^{\alpha+\beta}} |\nabla f(x)| \, dx \\
&\leq \left| \frac{p}{N-\gamma} \right| \left(\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p}{|x'|^{\alpha p}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^{\frac{\beta p}{p-1}}} dx \right)^{\frac{p-1}{p}},
\end{aligned}$$

where $x'_0 = (x', 0) \in \mathbb{R}^n$, that is $|x'_0| = |x'|$, ∇_N is the standard gradient on $\mathbb{R}^N$, and ∇ is the gradient on $\mathbb{R}^n$. Here we have used Hölder's inequality in the last line. This gives

$$\left| \frac{N-\gamma}{p} \right| \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^\gamma} dx \leq \left(\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p}{|x'|^{\alpha p}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^{\frac{\beta p}{p-1}}} dx \right)^{\frac{p-1}{p}},$$

which proves (6.13). Again as in the proof of Theorem 6.1.1 let us examine the equality condition in the above Hölder inequality. Thus, we consider

$$g(x) = \begin{cases} e^{-\frac{C}{\lambda}|x'|^{\lambda}}, & \lambda := \alpha - \frac{\beta}{p-1} + 1 \neq 0, \\ \frac{1}{|x'|^{C}}, & \alpha - \frac{\beta}{p-1} + 1 = 0, \end{cases}$$

where $C = \left|\frac{N-\gamma}{p}\right|$ and $\gamma \neq N$. Then it can be directly checked that

$$\left|\frac{p}{N-\gamma}\right|^{p} \frac{|\nabla g(x)|^{p}}{|x'|^{\alpha p}} = \left|\frac{p}{N-\gamma}\right|^{p} \frac{|\nabla_N g(x)|^{p}}{|x'|^{\alpha p}} = \frac{|g(x)|^{p}}{|x'|^{\frac{\beta p}{p-1}}},$$

which satisfies the equality condition in Hölder's inequality. Approximating this function by functions in $C_0^{\infty}(\mathbb{R}^n\backslash\{x'=0\})$ shows that the constant $\left|\frac{N-\gamma}{p}\right|$ is sharp. $\square$

Remark 6.1.6. For $\beta = (\alpha+1)(p-1)$ and $\gamma = p(\alpha+1)$ the inequality (6.13) gives that

$$\frac{|N-p(\alpha+1)|}{p}\left\|\frac{f}{|x'|^{\alpha+1}}\right\|_{L^p(\mathbb{R}^n)} \leq \left\|\frac{1}{|x'|^{\alpha}}\nabla f\right\|_{L^p(\mathbb{R}^n)}, \ 1<p<\infty, \tag{6.14}$$

for all $f \in C_0^{\infty}(\mathbb{R}^n\backslash\{x'=0\})$ and for all $\alpha \in \mathbb{R}$, with the sharp constant. For $\alpha = 0$ and $1<p<N,\ 2 \leq N \leq n$, the inequality (6.14) implies that

$$\left\|\frac{f}{|x'|}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{N-p}\left\|\nabla f\right\|_{L^p(\mathbb{R}^n)}, \tag{6.15}$$

again with $\frac{p}{N-p}$ being the best constant.

6.1.2 Horizontal higher-order versions

We can iterate the L^p-Caffarelli–Kohn–Nirenberg type inequalities from Theorem 6.1.1 to obtain higher-order inequalities. Let us denote inductively

$$\nabla_H^2 f := \nabla_H|\nabla_H f| \ \text{ and } \ \nabla_H^m f := \nabla_H|\nabla_H^{m-1} f|, \quad m \in \mathbb{N}.$$

Then as a consequence of Theorem 6.1.1 we obtain

Corollary 6.1.7 (Higher-order horizontal L^p-Caffarelli–Kohn–Nirenberg type inequalities)**.** *For any $k, m \in \mathbb{N}$ and $1<p<\infty$ we have*

$$\frac{|N-\gamma|}{p}\left\|\frac{f}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{G})}^{p} \leq \widetilde{A}_{\alpha,m}\widetilde{A}_{\beta,k}\left\|\frac{1}{|x'|^{\alpha-m}}\nabla_H^{m+1} f\right\|_{L^p(\mathbb{G})}\left\|\frac{1}{|x'|^{\frac{\beta}{p-1}-k}}\nabla_H^{k} f\right\|_{L^p(\mathbb{G})}^{p-1}, \tag{6.16}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$*, where* $\gamma = \alpha+\beta+1$*, and all* $\alpha \in \mathbb{R}$ *such that* $\prod_{j=0}^{m-1}|N-p(\alpha-j)| \neq 0$*, and*

$$\widetilde{A}_{\alpha,m} := p^m \left[\prod_{j=0}^{m-1}|N-p(\alpha-j)|\right]^{-1},$$

as well as all $\beta \in \mathbb{R}$ *such that* $\prod_{j=0}^{k-1}\left|N-p\left(\frac{\beta}{p-1}-j\right)\right| \neq 0$*, and*

$$\widetilde{A}_{\beta,k} := p^{k(p-1)} \left[\prod_{j=0}^{k-1}\left|N-p\left(\frac{\beta}{p-1}-j\right)\right|\right]^{-(p-1)}.$$

Proof of Corollary 6.1.7. Taking $|\nabla_H f|$ instead of f and $\alpha-1$ instead of α in (6.5) we consequently get

$$\left\|\frac{\nabla_H f}{|x'|^\alpha}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{|N-p\alpha|}\left\|\frac{1}{|x'|^{\alpha-1}}\nabla_H^2 f\right\|_{L^p(\mathbb{G})},$$

for $\alpha \neq \frac{N}{p}$. Combining it with (6.5) we obtain

$$\left\|\frac{f}{|x'|^{\alpha+1}}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{|N-p(\alpha+1)|}\frac{p}{|N-p\alpha|}\left\|\frac{1}{|x'|^{\alpha-1}}\nabla_H^2 f\right\|_{L^p(\mathbb{G})},$$

for each $\alpha \in \mathbb{R}$ such that $\alpha \neq \frac{N}{p}-1$ and $\alpha \neq \frac{N}{p}$. This iteration process gives

$$\left\|\frac{f}{|x'|^{\theta+1}}\right\|_{L^p(\mathbb{G})} \leq A_{\theta,k}\left\|\frac{1}{|x'|^{\theta+1-k}}\nabla_H^k f\right\|_{L^p(\mathbb{G})}, \quad 1<p<\infty, \tag{6.17}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ and all $\theta \in \mathbb{R}$ such that $\prod_{j=0}^{k-1}|N-p(\theta+1-j)| \neq 0$, and

$$A_{\theta,k} := p^k \left[\prod_{j=0}^{k-1}|N-p(\theta+1-j)|\right]^{-1}.$$

Similarly, we have

$$\left\|\frac{\nabla_H f}{|x'|^{\vartheta+1}}\right\|_{L^p(\mathbb{G})} \leq A_{\vartheta,m}\left\|\frac{1}{|x'|^{\vartheta+1-m}}\nabla_H^{m+1} f\right\|_{L^p(\mathbb{G})}, \quad 1<p<\infty, \tag{6.18}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ and all $\vartheta \in \mathbb{R}$ such that $\prod_{j=0}^{m-1}|N-p(\vartheta+1-j)| \neq 0$, and

$$A_{\vartheta,m} := p^m \left[\prod_{j=0}^{m-1}|N-p(\vartheta+1-j)|\right]^{-1}.$$

Now putting $\vartheta+1=\alpha$ and $\theta+1=\frac{\beta}{p-1}$ into (6.18) and (6.17), respectively, from (6.3) we obtain (6.16). $\square$

6.2 Horizontal Hardy and Rellich inequalities

First of all let us record the horizontal Hardy inequalities discussed in Remark 6.1.2:

Corollary 6.2.1 (Horizontal L^p Hardy inequalities)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. Then for any $1 < p < \infty$, $\alpha \in \mathbb{R}$, and for all $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have*

$$\left\| \frac{1}{|x'|^\alpha} \nabla_H f \right\|_{L^p(\mathbb{G})} \geq \frac{|N - p(\alpha+1)|}{p} \left\| \frac{f}{|x'|^{\alpha+1}} \right\|_{L^p(\mathbb{G})}. \tag{6.19}$$

If $p(\alpha+1) \neq N$ then the constant $\frac{|N-p(\alpha+1)|}{p}$ is sharp.

From this we move on to Rellich inequalities.

Theorem 6.2.2 (Horizontal Rellich inequalities)**.** *Let $\mathbb{G}$ be a stratified group with $N \geq 3$ being the dimension of the first stratum. Let $\delta \in \mathbb{R}$ with $-N/2 \leq \delta \leq -1$. Then for all functions $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have*

$$\left\| \frac{\mathcal{L} f}{|x'|^\delta} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{(N - 2\delta - 4)(N + 2\delta)}{4} \right| \left\| \frac{f}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G})}. \tag{6.20}$$

If $N + 2\delta \neq 0$, then the constant in (6.20) is sharp.

We note that another version of the Rellich inequality will be given in Corollary 6.5.2.

Proof of Theorem 6.2.2. In the case $p = 2$ and $\alpha = \delta + 1$, Corollary 6.2.1 implies

$$\left\| \frac{\nabla_H f}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{N - 2\delta - 4}{2} \right| \left\| \frac{f}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G})}. \tag{6.21}$$

It also follows that the constant $\left|\frac{N-2\delta-4}{2}\right|$ is sharp when $N - 2\delta - 4 \neq 0$. On the other hand, Corollary 6.5.2 gives (see also Theorem 6.8.1 with $p = 2$, $\gamma = 2\beta$ and $\alpha = \beta - 1$)

$$\left\| \frac{\mathcal{L} f}{|x'|^{\beta-1}} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{N + 2\beta - 2}{2} \right| \left\| \frac{\nabla_H f}{|x'|^\beta} \right\|_{L^2(\mathbb{G})} \tag{6.22}$$

for $2 - N \leq 2\beta \leq 0$ and $N \geq 3$.

Putting $\delta + 1$ instead of β this gives

$$\left\| \frac{\mathcal{L} f}{|x'|^\delta} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{N + 2\delta}{2} \right| \left\| \frac{\nabla_H f}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G})} \tag{6.23}$$

for $2 - N \leq 2\delta + 2 \leq 0$ and $N \geq 3$. Combining (6.21) and (6.23), we obtain (6.20).

Now, to show the sharpness of the constant in (6.20), we observe first that in Theorem 6.2.2 the sharpness of the constant is reduced to that in Theorem

6.1.1 which in turn is obtained by checking the equality condition in Hölder's inequality. Namely, the function $|x'|^{C_1}$ satisfies this equality condition for any real number $C_1 \neq 0$. Similarly, in Theorem 6.8.1, the sharpness of the constant will be obtained again from the equality condition in Hölder's inequality, so that we see that the same function $|x'|^{C_1}$ satisfies the equality condition. Therefore, the constant in (6.23) is sharp when $N + 2\delta \neq 0$, so, the constant in (6.20) is sharp for $N + 2\delta \neq 0$. $\square$

6.3 Critical horizontal Hardy type inequality

For $p = N$ the inequality (6.2) fails, and in this section we consider its critical versions.

Theorem 6.3.1 (Critical horizontal Hardy inequality)**.** *For a bounded domain $\Omega \subset \mathbb{G}$ with $0 \in \Omega$ and for all $f \in C_0^\infty(\Omega \backslash \{x' = 0\})$ we have*

$$\left\| \frac{f}{|x'| \log \frac{R}{|x'|}} \right\|_{L^N(\Omega)} \leq \frac{N}{N-1} \left\| \frac{x'}{|x'|} \cdot \nabla_H f \right\|_{L^N(\Omega)}, \quad 1 < N < \infty, \tag{6.24}$$

where $R = \sup\limits_{x \in \Omega} |x'|$.

To show Theorem 6.3.1 we will first prove the following more abstract theorem, and then the proof of Theorem 6.3.1 will follows directly from this. Moreover, it will imply a number of other estimates, for example the critical L^N-Poincaré inequality.

Theorem 6.3.2. *Let $0 \in \Omega \subset \mathbb{G}$ be a bounded domain. Let $g : (1, \infty) \to \mathbb{R}$ be a C^2-function such that*

$$g'(t) < 0, \quad g''(t) > 0, \tag{6.25}$$

for all $t > 1$, and such that

$$\frac{(-g'(t))^{2(N-1)}}{(g''(t))^{N-1}} \leq C < \infty, \quad \text{for all } \; t > 1. \tag{6.26}$$

Then we have

$$\begin{aligned} &\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|f(x)|^N}{|x'|^N} \left(-g'\left(\log \frac{Re}{|x'|}\right)\right)^{N-2} g''\left(\log \frac{Re}{|x'|}\right) dx \\ &\leq \int_\Omega \frac{\left(-g'\left(\log \frac{Re}{|x'|}\right)\right)^{2(N-1)}}{\left(g''\left(\log \frac{Re}{|x'|}\right)\right)^{N-1}} \left| \frac{x'}{|x'|} \cdot \nabla_H f(x) \right|^N dx, \end{aligned} \tag{6.27}$$

for all $f \in C_0^\infty(\Omega \backslash \{x' = 0\})$, with $R = \sup\limits_{x \in \Omega} |x'|$.

Proof of Theorem 6.3.2. For $\epsilon > 0$ a direct calculation shows

$$|\nabla_H G_\epsilon(x)|^{N-2}\nabla_H G_\epsilon(x) = \left(-g'(F_\epsilon(x))\right)^{N-1}\left(\frac{|x'|^{N-2}x'}{(|x'|^2+\epsilon^2)^{N-1}}\right),$$

where

$$F_\epsilon(x) := \log\frac{R_\epsilon e}{\sqrt{|x'|^2+\epsilon^2}}, \quad R_\epsilon := \sup_{x\in\Omega}\sqrt{|x'|^2+2\epsilon^2},$$

and

$$G_\epsilon(x) = g(F_\epsilon(x)).$$

Since $g'(t) < 0$, with $\mathcal{L}_N$ as in (1.71), we have

$$\begin{aligned}\mathcal{L}_N G_\epsilon(x) &= \operatorname{div}_H(|\nabla_H G_\epsilon(x)|^{N-2}\nabla_H G_\epsilon(x)) \\ &= (N-1)\left(-g'(F_\epsilon(x))\right)^{N-2} g''(F_\epsilon(x))\frac{|x'|^N}{(|x'|^2+\epsilon^2)^N} \\ &\quad + (N-1)\left(-g'(F_\epsilon(x))\right)^{N-1}\frac{2\epsilon^2|x'|^{N-2}}{(|x'|^2+\epsilon^2)^N}.\end{aligned}$$

The divergence theorem gives

$$\begin{aligned}\int_\Omega |f|^N\mathcal{L}_N G_\epsilon(x)dx &= \int_\Omega |f|^N\operatorname{div}_H(|\nabla_H G_\epsilon(x)|^{N-2}\nabla_H G_\epsilon(x))dx \\ &= -\int_\Omega \nabla_H|f|^N\cdot(|\nabla_H G_\epsilon(x)|^{N-2}\nabla_H G_\epsilon(x))dx.\end{aligned} \tag{6.28}$$

We have

$$\begin{aligned}&\int_\Omega |f|^N\mathcal{L}_N G_\epsilon(x)dx \\ &\quad= (N-1)\int_\Omega |f|^N\left(-g'(F_\epsilon(x))\right)^{N-2} g''(F_\epsilon(x))\frac{|x'|^N}{(|x'|^2+\epsilon^2)^N}dx \\ &\qquad + (N-1)\int_\Omega |f|^N\left(-g'(F_\epsilon(x))\right)^{N-1}\frac{2\epsilon^2|x'|^{N-2}}{(|x'|^2+\epsilon^2)^N}dx \\ &\quad\geq (N-1)\int_\Omega |f|^N\left(-g'(F_\epsilon(x))\right)^{N-2} g''(F_\epsilon(x))\frac{|x'|^N}{(|x'|^2+\epsilon^2)^N}dx.\end{aligned} \tag{6.29}$$

Moreover,

$$\begin{aligned}&\left|-\int_\Omega \nabla_H|f(x)|^N\cdot(|\nabla_H G_\epsilon(x)|^{N-2}\nabla_H G_\epsilon(x))dx\right| \\ &\quad= \left|N\int_\Omega |f(x)|^{N-2}f(x)\left(-g'(F_\epsilon(x))\right)^{N-1}\left(\frac{|x'|^{N-2}x'\cdot\nabla_H f}{(|x'|^2+\epsilon^2)^{N-1}}\right)dx\right|\end{aligned}$$

$$
\begin{aligned}
&= N \int_{\Omega} |f(x)|^{N-1} \left(-g'(F_\epsilon(x))\right)^{N-1} \left(\frac{|x'|^{N-2} |x' \cdot \nabla_H f|}{(|x'|^2 + \epsilon^2)^{N-1}} \right) dx \\
&\leq N \left(\int_{\Omega} \left(-g'(F_\epsilon(x))\right)^{N-2} g''(F_\epsilon(x)) \frac{|x'|^N |f(x)|^N}{(|x'|^2 + \epsilon^2)^N} dx \right)^{\frac{N-1}{N}} \\
&\quad \times \left(\int_{\Omega} \left(-g'(F_\epsilon(x))\right)^{2(N-1)} \left(g''(F_\epsilon(x))\right)^{-(N-1)} \left| \frac{x'}{|x'|} \cdot \nabla_H f \right|^N dx \right)^{\frac{1}{N}}. \qquad (6.30)
\end{aligned}
$$

Combining (6.28), (6.29) and (6.30) we obtain

$$
\begin{aligned}
&(N-1) \int_{\Omega} |f|^N \left(-g'(F_\epsilon(x))\right)^{N-2} g''(F_\epsilon(x)) \frac{|x'|^N}{(|x'|^2 + \epsilon^2)^N} dx \\
&\quad \leq N \left(\int_{\Omega} \left(-g'(F_\epsilon(x))\right)^{N-2} g''(F_\epsilon(x)) \frac{|x'|^N |f(x)|^N}{(|x'|^2 + \epsilon^2)^N} dx \right)^{\frac{N-1}{N}} \\
&\quad\quad \times \left(\int_{\Omega} \left(-g'(F_\epsilon(x))\right)^{2(N-1)} \left(g''(F_\epsilon(x))\right)^{-(N-1)} \left| \frac{x'}{|x'|} \cdot \nabla_H f \right|^N dx \right)^{\frac{1}{N}},
\end{aligned}
$$

which means

$$
\begin{aligned}
&\left(\frac{N-1}{N} \right)^N \int_{\Omega} |f|^N \left(-g'(F_\epsilon(x))\right)^{N-2} g''(F_\epsilon(x)) \frac{|x'|^N}{(|x'|^2 + \epsilon^2)^N} dx \\
&\quad \leq \int_{\Omega} \left(-g'(F_\epsilon(x))\right)^{2(N-1)} \left(g''(F_\epsilon(x))\right)^{-(N-1)} \left| \frac{x'}{|x'|} \cdot \nabla_H f \right|^N dx.
\end{aligned}
$$

Now letting $\epsilon \to 0$ we obtain (6.27). $\square$

Proof of Theorem 6.3.1. If we take

$$g(t) = -\log(t-1),$$

for $t > 1$, then we see that this function satisfies all assumptions of Theorem 6.3.2. That is,

$$g'(t) = -\frac{1}{t-1} < 0, \quad g''(t) = \frac{1}{(t-1)^2} > 0,$$

and

$$\frac{(-g'(t))^{2(N-1)}}{(g''(t))^{N-1}} = 1, \quad \text{for all } \ t > 1.$$

Therefore, putting

$$g'\left(\log \frac{Re}{|x'|}\right) = -\frac{1}{\log \frac{R}{|x'|}} \qquad \text{and} \qquad g''\left(\log \frac{Re}{|x'|}\right) = \frac{1}{\left(\log \frac{R}{|x'|}\right)^2}$$

in (6.27) we obtain (6.24). $\square$

One can obtain a number of inequalities from Theorem 6.3.2 by choosing different functions $g(t)$. For example, we get the following analogue of the L^N-Poincaré inequality for the horizontal gradient.

Corollary 6.3.3 (Horizontal critical Poincaré inequality)**.** *Let* $R := \sup_{x\in\Omega}|x'|$. *Then for all* $f \in C_0^\infty(\Omega\backslash\{x'=0\})$ *we have*

$$\|f\|_{L^N(\Omega)} \le R\,\|\nabla_H f\|_{L^N(\Omega)}\,. \tag{6.31}$$

Proof. Let us take

$$g(t) = e^{\frac{Nt}{1-N}}, \quad t > 1,$$

in (6.27). Then we have

$$\|f\|_{L^N(\Omega)} \le \left\| |x'|\frac{x'}{|x'|}\cdot\nabla_H f\right\|_{L^N(\Omega)}.$$

For $R = \sup_{x\in\Omega}|x'|$, the Cauchy–Schwarz inequality implies (6.31). □

Remark 6.3.4. In the Euclidean case the idea of proving Theorem 6.3.1 using Theorem 6.3.2 was realized in [Tak15]. In this section our presentation followed [RS17e].

6.4 Two-parameter Hardy–Rellich inequalities by factorization

In this section we apply the factorization method, similar to the ideas explained in Section 2.1.5, but now in the setting of stratified groups. As a result we obtain two-parameter inequalities analogous to the Gesztesy–Littlejohn type inequalities described in Example 2.1.13. The presentation of this section follows [RY17].

Theorem 6.4.1 (Two-parameter Hardy–Rellich inequalities)**.** *Let* $\mathbb{G}$ *be a stratified group with* $N \ge 2$ *being the dimension of the first stratum, and let* $\alpha, \beta \in \mathbb{R}$. *Then for all complex-valued functions* $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ *we have*

$$\begin{aligned}\|\mathcal{L}f\|^2_{L^2(\mathbb{G})} \ge (\alpha(N-2)-2\beta)\left\|\frac{\nabla_H f}{|x'|}\right\|^2_{L^2(\mathbb{G})}\\ -\alpha^2\left\|\frac{x'\cdot\nabla_H f}{|x'|^2}\right\|^2_{L^2(\mathbb{G})} + C_{N,\alpha,\beta}\left\|\frac{f}{|x'|^2}\right\|^2_{L^2(\mathbb{G})},\end{aligned} \tag{6.32}$$

where

$$C_{N,\alpha,\beta} = \alpha(N-4)(N-2) - \alpha^2(N-2) + 2\beta(4-N) - \beta^2 + \alpha\beta(N-2).$$

Remark 6.4.2.

1. Using the Cauchy–Schwarz inequality
$$\int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^4} dx \leq \int_{\mathbb{G}} \frac{|(\nabla_H f)(x)|^2}{|x'|^2} dx,$$
inequality (6.32) implies the inequality
$$\|\mathcal{L}f\|^2_{L^2(\mathbb{G})} \geq (\alpha(N-2)-2\beta-\alpha^2) \left\| \frac{\nabla_H f}{|x'|} \right\|^2_{L^2(\mathbb{G})} + C_{N,\alpha,\beta} \left\| \frac{f}{|x'|^2} \right\|^2_{L^2(\mathbb{G})}. \quad (6.33)$$

2. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the usual (full) gradient, so (6.32) implies for $\alpha, \beta \in \mathbb{R}$ and for any $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ with $n \geq 2$ the inequality
$$\begin{aligned} \|\Delta f\|^2_{L^2(\mathbb{R}^n)} \geq (\alpha(n-2) - 2\beta) & \left\| \frac{\nabla f}{|x|} \right\|^2_{L^2(\mathbb{R}^n)} \\ & - \alpha^2 \left\| \frac{x \cdot \nabla f}{|x|^2} \right\|^2_{L^2(\mathbb{R}^n)} + C_{n,\alpha,\beta} \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \quad (6.34)$$

   This can be also compared with inequality (2.34).

Proof of Theorem 6.4.1. For two parameters $\alpha, \beta \in \mathbb{R}$, let us define
$$T_{\alpha,\beta} := -\mathcal{L} + \alpha \frac{x' \cdot \nabla_H}{|x'|^2} + \frac{\beta}{|x'|^2}.$$
One can readily check that its formal adjoint is given by
$$T^+_{\alpha,\beta} := -\mathcal{L} - \alpha \frac{x' \cdot \nabla_H}{|x'|^2} - \frac{\alpha(N-2) - \beta}{|x'|^2},$$
for $x' \neq 0$. Then, by a direct calculation for any function $f \in C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ we have
$$\begin{aligned} &(T^+_{\alpha,\beta} T_{\alpha,\beta} f)(x) \\ &= \left(-\mathcal{L} - \alpha \frac{x' \cdot \nabla_H}{|x'|^2} - \frac{\alpha(N-2) - \beta}{|x'|^2} \right) \left(-(\mathcal{L}f)(x) + \alpha \frac{x' \cdot (\nabla_H f)}{|x'|^2} + \frac{\beta f(x)}{|x'|^2} \right) \\ &= (\mathcal{L}^2 f)(x) + \alpha \left(-\mathcal{L} \left(\frac{x' \cdot (\nabla_H f)}{|x'|^2} \right)(x) + \frac{x' \cdot \nabla_H}{|x'|^2} (\mathcal{L}f)(x) + \frac{N-2}{|x'|^2} (\mathcal{L}f)(x) \right) \\ &\quad + \beta \left(-\mathcal{L} \left(\frac{f}{|x'|^2} \right)(x) - \frac{(\mathcal{L}f)(x)}{|x'|^2} \right) \\ &\quad + \alpha\beta \left(-\frac{x' \cdot \nabla_H}{|x'|^2} \left(\frac{f}{|x'|^2} \right)(x) + \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} - \frac{(N-2) f(x)}{|x'|^4} \right) \end{aligned}$$

$$+\alpha^2\left(-\left(\frac{x'\cdot\nabla_H}{|x'|^2}\right)\left(\frac{x'\cdot(\nabla_H f)}{|x'|^2}\right)(x)-(N-2)\frac{x'\cdot(\nabla_H f)(x)}{|x'|^4}\right)+\beta^2\frac{f(x)}{|x'|^4}.$$

Now we calculate in the other direction,

$$\begin{aligned}
&(T_{\alpha,\beta}T^{+}_{\alpha,\beta}f)(x)\\
&=\left(-\mathcal{L}+\alpha\frac{x'\cdot\nabla_H}{|x'|^2}+\frac{\beta}{|x'|^2}\right)\\
&\qquad\times\left(-(\mathcal{L}f)(x)-\alpha\frac{x'\cdot(\nabla_H f)(x)}{|x'|^2}-\frac{(\alpha(N-2)-\beta)f(x)}{|x'|^2}\right)\\
&=(\mathcal{L}^2 f)(x)+\alpha\left(\mathcal{L}\left(\frac{x'\cdot(\nabla_H f)}{|x'|^2}\right)(x)-\frac{x'\cdot\nabla_H}{|x'|^2}(\mathcal{L}f)(x)+(N-2)\mathcal{L}\left(\frac{f}{|x'|^2}\right)(x)\right)\\
&\quad+\beta\left(-\mathcal{L}\left(\frac{f}{|x'|^2}\right)(x)-\frac{(\mathcal{L}f)(x)}{|x'|^2}\right)\\
&\quad+\alpha\beta\left(\frac{x'\cdot\nabla_H}{|x'|^2}\left(\frac{f}{|x'|^2}\right)(x)-\frac{x'\cdot(\nabla_H f)(x)}{|x'|^4}-\frac{(N-2)f(x)}{|x'|^4}\right)\\
&\quad+\alpha^2\left(-\left(\frac{x'\cdot\nabla_H}{|x'|^2}\right)\left(\frac{x'\cdot(\nabla_H f)}{|x'|^2}\right)(x)-(N-2)\frac{x'\cdot\nabla_H}{|x'|^2}\left(\frac{f}{|x'|^2}\right)(x)\right)\\
&\quad+\beta^2\frac{f(x)}{|x'|^4}.
\end{aligned}\tag{6.35}$$

Using that

$$\begin{aligned}
\mathcal{L}\left(\frac{f}{|x'|^2}\right)(x)&=\sum_{j=1}^{N}X_j^2\left(\frac{f}{|x'|^2}\right)(x)\\
&=\sum_{j=1}^{N}X_j\left(\frac{X_jf}{|x'|^2}-\frac{2x'_jf}{|x'|^4}\right)(x)\\
&=\sum_{j=1}^{N}\left(\frac{(X_j^2f)(x)}{|x'|^2}-\frac{4x'_j(X_jf)(x)}{|x'|^4}+\frac{8(x'_j)^2f(x)}{|x'|^6}-\frac{2f(x)}{|x'|^4}\right)\\
&=\frac{(\mathcal{L}f)(x)}{|x'|^2}-\frac{4x'\cdot(\nabla_H f)(x)}{|x'|^4}-(2N-8)\frac{f(x)}{|x'|^4}
\end{aligned}$$

and

$$\begin{aligned}
\frac{x'\cdot\nabla_H}{|x'|^2}\left(\frac{f}{|x'|^2}\right)(x)&=\frac{f(x)}{|x'|^2}\sum_{j=1}^{N}x'_jX_j(|x'|^{-2})+\frac{x'\cdot(\nabla_H f)(x)}{|x'|^4}\\
&=-2\sum_{j=1}^{N}\frac{(x'_j)^2f(x)}{|x'|^6}+\frac{x'\cdot(\nabla_H f)(x)}{|x'|^4}\\
&=-2\frac{f(x)}{|x'|^4}+\frac{x'\cdot(\nabla_H f)(x)}{|x'|^4},
\end{aligned}$$

in (6.35), we can write the sum $(T^+_{\alpha,\beta}T_{\alpha,\beta}f)(x)+(T_{\alpha,\beta}T^+_{\alpha,\beta}f)(x)$ as

$$\begin{aligned}
&(T^+_{\alpha,\beta}T_{\alpha,\beta}f)(x)+(T_{\alpha,\beta}T^+_{\alpha,\beta}f)(x)\\
&\quad=2(\mathcal{L}^2f)(x)+2\alpha(N-2)\left(\frac{(\mathcal{L}f)(x)}{|x'|^2}-2\frac{x'\cdot(\nabla_Hf)(x)}{|x'|^4}+(4-N)\frac{f(x)}{|x'|^4}\right)\\
&\qquad+2\beta\left(-\frac{2(\mathcal{L}f)(x)}{|x'|^2}+\frac{4x'\cdot(\nabla_Hf)(x)}{|x'|^4}+(2N-8)\frac{f(x)}{|x'|^4}\right)-2\alpha\beta(N-2)\frac{f(x)}{|x'|^4}\\
&\qquad+2\alpha^2\Bigg(-\left(\frac{x'\cdot\nabla_H}{|x'|^2}\right)\left(\frac{x'\cdot(\nabla_Hf)}{|x'|^2}\right)(x)\\
&\qquad\qquad-(N-2)\frac{x'\cdot(\nabla_Hf)(x)}{|x'|^4}+(N-2)\frac{f(x)}{|x'|^4}\Bigg)\\
&\qquad+2\beta^2\frac{f(x)}{|x'|^4}.
\end{aligned}\tag{6.36}$$

In order to simplify this, let us rewrite the following expression:

$$\begin{aligned}
&-2\left(\frac{x'\cdot\nabla_H}{|x'|^2}\right)\left(\frac{x'\cdot(\nabla_Hf)}{|x'|^2}\right)(x)-2(N-2)\frac{x'\cdot(\nabla_Hf)(x)}{|x'|^4}+2(N-2)\frac{f(x)}{|x'|^4}\\
&\quad=-\frac{2\sum_{j,k=1}^N(x'_jX_j)(x'_k(X_kf))(x)}{|x'|^4}-2\sum_{j,k=1}^N x'_j(-2)|x'|^{-3}X_j|x'|\frac{x'_k(X_kf)(x)}{|x'|^2}\\
&\qquad-2(N-2)\frac{x'\cdot(\nabla_Hf)(x)}{|x'|^4}+2(N-2)\frac{f(x)}{|x'|^4}\\
&\quad=-\frac{2\sum_{k=1}^N x'_k(X_kf)(x)}{|x'|^4}-\frac{2\sum_{j,k=1}^N x'_jx'_kX_j(X_kf)(x)}{|x'|^4}+\frac{4\sum_{k=1}^N x'_k(X_kf)(x)}{|x'|^4}\\
&\qquad-2(N-2)\frac{x'\cdot(\nabla_Hf)(x)}{|x'|^4}+2(N-2)\frac{f(x)}{|x'|^4}\\
&\quad=-2(N-3)\frac{x'\cdot(\nabla_Hf)(x)}{|x'|^4}-\frac{2\sum_{j,k=1}^N x'_jx'_k(X_jX_kf)(x)}{|x'|^4}+2(N-2)\frac{f(x)}{|x'|^4}.
\end{aligned}$$

Now putting this in (6.36), we obtain

$$\begin{aligned}
&(T^+_{\alpha,\beta}T_{\alpha,\beta}f)(x)+(T_{\alpha,\beta}T^+_{\alpha,\beta}f)(x)\\
&\quad=2(\mathcal{L}^2f)(x)+(2\alpha(N-2)-4\beta)\frac{(\mathcal{L}f)(x)}{|x'|^2}\\
&\qquad+(-4\alpha(N-2)-2\alpha^2(N-3)+8\beta)\frac{x'\cdot(\nabla_Hf)(x)}{|x'|^4}\\
&\qquad+(2\alpha(N-2)(4-N)+2\alpha^2(N-2)-2\alpha\beta(N-2)\\
&\qquad+(4N-16)\beta+2\beta^2)\frac{f(x)}{|x'|^4}-2\alpha^2\frac{\sum_{j,k=1}^N x'_jx'_k(X_jX_kf)(x)}{|x'|^4}.
\end{aligned}\tag{6.37}$$

In general, the non-negativity of $T^+_{\alpha,\beta}T_{\alpha,\beta} + T_{\alpha,\beta}T^+_{\alpha,\beta}$ and integration by parts imply

$$\int_{\mathbb{G}} |(T_{\alpha,\beta}f)(x)|^2 dx + \int_{\mathbb{G}} |(T^+_{\alpha,\beta}f)(x)|^2 dx = \int_{\mathbb{G}} \overline{f(x)}((T^+_{\alpha,\beta}T_{\alpha,\beta} + T_{\alpha,\beta}T^+_{\alpha,\beta})f)(x)dx \geq 0.$$

Putting (6.37) into this inequality, one calculates

$$\begin{aligned} 2\int_{\mathbb{G}} |(\mathcal{L}f)(x)|^2 dx &+ (2\alpha(N-2) - 4\beta)\int_{\mathbb{G}} \frac{\overline{f(x)}(\mathcal{L}f)(x)}{|x'|^2}dx \\ &+ (-4\alpha(N-2) - 2\alpha^2(N-3) + 8\beta)\int_{\mathbb{G}} \frac{\overline{f(x)}(x'\cdot(\nabla_H f)(x))}{|x'|^4}dx \\ &+ (2\alpha(N-2)(4-N) + 2\alpha^2(N-2) - 2\alpha\beta(N-2) \\ &+ (4N-16)\beta + 2\beta^2)\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^4}dx \\ &- 2\alpha^2 \sum_{j,k=1}^{N} \int_{\mathbb{G}} \frac{\overline{f(x)}x'_j x'_k (X_j X_k f)(x)}{|x'|^4}dx \geq 0. \end{aligned} \tag{6.38}$$

Using the identities

$$\begin{aligned} \int_{\mathbb{G}} \frac{\overline{f(x)}(\mathcal{L}f)(x)}{|x'|^2}dx &= -\sum_{j=1}^{N}\int_{\mathbb{G}} \frac{\overline{(X_j f)(x)}(X_j f)(x)}{|x'|^2} \\ &\quad - \sum_{j=1}^{N}\int_{\mathbb{G}} \overline{f(x)}(-2)|x'|^{-3}X_j|x'|(X_j f)(x)dx \\ &= 2\int_{\mathbb{G}} \frac{\overline{f(x)}(x'\cdot(\nabla_H f)(x))}{|x'|^4}dx - \int_{\mathbb{G}} \frac{|(\nabla_H f)(x)|^2}{|x'|^2}dx \end{aligned}$$

and

$$\begin{aligned} &\sum_{j,k=1}^{N}\int_{\mathbb{G}} \frac{\overline{f(x)}x'_j x'_k (X_j X_k f)(x)}{|x'|^4}dx \\ &= -(N-1)\sum_{k=1}^{N}\int_{\mathbb{G}} \frac{\overline{f(x)}x'_k (X_k f)(x)}{|x'|^4}dx - 2\sum_{k=1}^{N}\int_{\mathbb{G}} \frac{\overline{f(x)}x'_k (X_k f)(x)}{|x'|^4}dx \\ &\quad - \sum_{j,k=1}^{N}\int_{\mathbb{G}} \frac{x'_j x'_k \overline{(X_j f)(x)}(X_k f)(x)}{|x'|^4}dx \\ &\quad - \sum_{j,k=1}^{N}\int_{\mathbb{G}} \overline{f(x)}x'_j x'_k (X_k f)(x)(-4)|x'|^{-5}X_j|x'|dx \end{aligned}$$

$$
\begin{aligned}
&= -(N+1)\sum_{k=1}^{N}\int_{\mathbb{G}}\frac{\overline{f(x)}x'_k(X_kf)(x)}{|x'|^4}dx + 4\sum_{j,k=1}^{N}\int_{\mathbb{G}}\frac{\overline{f(x)}(x'_j)^2x'_k(X_kf)(x)}{|x'|^6}dx \\
&\quad - \int_{\mathbb{G}}\frac{|x'\cdot(\nabla_Hf)(x)|^2}{|x'|^4}dx \\
&= -(N-3)\int_{\mathbb{G}}\frac{\overline{f(x)}(x'\cdot(\nabla_Hf)(x))}{|x'|^4}dx - \int_{\mathbb{G}}\frac{|x'\cdot(\nabla_Hf)(x)|^2}{|x'|^4}dx
\end{aligned}
$$

in (6.38), we obtain

$$
\begin{aligned}
&2\int_{\mathbb{G}}|(\mathcal{L}f)(x)|^2dx + (4\alpha(N-2)-8\beta-4\alpha(N-2)+8\beta-2\alpha^2(N-3)+2\alpha^2(N-3)) \\
&\quad\times\int_{\mathbb{G}}\frac{\overline{f(x)}(x'\cdot(\nabla_Hf)(x))}{|x'|^4}dx \\
&\quad + (2\alpha(N-2)(4-N)+2\alpha^2(N-2)-2\alpha\beta(N-2)+(4N-16)\beta+2\beta^2) \\
&\quad\times\int_{\mathbb{G}}\frac{|f(x)|^2}{|x'|^4}dx - (2\alpha(N-2)-4\beta)\int_{\mathbb{G}}\frac{|(\nabla_Hf)(x)|^2}{|x'|^2}dx \\
&\quad + 2\alpha^2\int_{\mathbb{G}}\frac{|x'\cdot(\nabla_Hf)(x)|^2}{|x'|^4}dx \geq 0,
\end{aligned}
$$

which implies (6.32). □

The factorization method can be used to give an elementary proof of the horizontal L^2-weighted inequality given in Remark 6.1.2, Part 3.

Proposition 6.4.3 (Horizontal L^2-Hardy inequality)**.** *Let $\mathbb{G}$ be a stratified group with $N \geq 3$ being the dimension of the first stratum. Let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$
\|\nabla_H f\|_{L^2(\mathbb{G})} \geq \frac{N-2}{2}\left\|\frac{f}{|x'|}\right\|_{L^2(\mathbb{G})}, \tag{6.39}
$$

where the constant $\frac{N-2}{2}$ is sharp.

Proof of Proposition 6.4.3. Let

$$
\widetilde{T}_\alpha := \nabla_H + \alpha\frac{x'}{|x'|^2}.
$$

One can readily check that its formal adjoint is given by

$$
\widetilde{T}_\alpha^+ = -\mathrm{div}_H + \alpha\frac{x'}{|x'|^2},
$$

where $x' \neq 0$.

Using (1.73) we have

$$\begin{aligned}\widetilde{T}_\alpha^+\widetilde{T}_\alpha f &= -(\mathcal{L}f)(x) - \alpha \mathrm{div}_H\left(\frac{x'}{|x'|^2}f\right)(x) + \alpha\frac{x'\cdot(\nabla_H f)(x)}{|x'|^2} + \alpha^2\frac{f(x)}{|x'|^2}\\ &= -(\mathcal{L}f)(x) - \alpha \mathrm{div}_H\left(\frac{x'}{|x'|^2}\right)f(x) + \alpha^2\frac{f(x)}{|x'|^2}\\ &= -(\mathcal{L}f)(x) + \frac{\alpha(\alpha+2-N)}{|x'|^2}f(x).\end{aligned}$$

By integrating by parts and using the non-negativity of $\widetilde{T}_\alpha^+\widetilde{T}_\alpha$ we have

$$\begin{aligned}0 &\le \int_{\mathbb{G}} |\widetilde{T}_\alpha f|^2 dx\\ &= \int_{\mathbb{G}} \overline{f(x)}(\widetilde{T}_\alpha^+\widetilde{T}_\alpha f)(x)dx\\ &= \int_{\mathbb{G}} |\nabla_H f|^2 dx + \alpha(\alpha+2-N)\int_{\mathbb{G}}\frac{|f(x)|^2}{|x'|^2}dx.\end{aligned}$$

It follows from this that

$$\int_{\mathbb{G}} |\nabla_H f(x)|^2 dx \ge \alpha(N-2-\alpha)\int_{\mathbb{G}}\frac{|f(x)|^2}{|x'|^2}dx.$$

By maximizing the constant with respect to α we obtain (6.39). The sharpness of the constant follows from Remark 6.1.2, Part 3. □

By modifying the differential expression $\widetilde{T}_\alpha$ in the proof of Proposition 6.4.3 we can also show the following refinement of the L^2-Hardy inequality (6.39). The fact that it is indeed a refinement, that is, that (6.40) implies (6.39) follows by the Cauchy–Schwarz inequality. Consequently, the sharpness of the constant in (6.40) also follows from the sharpness of the constant in (6.39).

Theorem 6.4.4 (Refined horizontal L^2-Hardy inequality)**.** *Let $\mathbb{G}$ be a stratified group with $N \ge 3$ being the dimension of the first stratum. Let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$\left\|\frac{x'\cdot\nabla_H f}{|x'|}\right\|_{L^2(\mathbb{G})} \ge \frac{N-2}{2}\left\|\frac{f}{|x'|}\right\|_{L^2(\mathbb{G})}, \tag{6.40}$$

where the constant $\frac{N-2}{2}$ is sharp.

Proof of Theorem 6.4.4. Let us define

$$\widehat{T}_\alpha := \frac{x'\cdot\nabla_H}{|x'|} + \frac{\alpha}{|x'|}.$$

One can readily check that its formal adjoint is given by

$$\widehat{T}_\alpha^+ := -\frac{x'\cdot\nabla_H}{|x'|} + \frac{\alpha - N + 1}{|x'|},$$

where $x' \neq 0$. Using (1.73) we get

$$\begin{aligned}
&\left(\frac{x'\cdot\nabla_H}{|x'|}\right)\left(\frac{x'\cdot(\nabla_H f)(x)}{|x'|}\right)\\
&\quad= \frac{\sum_{j,k=1}^N (x'_j X_j)(x'_k (X_k f)(x))}{|x'|^2} + \sum_{j,k=1}^N x'_j(-1)|x'|^{-2}X_j|x'|\frac{x'_k(X_k f)(x)}{|x'|}\\
&\quad= \frac{\sum_{k=1}^N x'_k (X_k f)(x)}{|x'|^2} + \frac{\sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^2} - \frac{\sum_{k=1}^N x'_k (X_k f)(x)}{|x'|^2}\\
&\quad= \frac{\sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^2}
\end{aligned}$$

and

$$\begin{aligned}
(x'\cdot\nabla_H)\left(\frac{f}{|x'|}\right) &= \frac{\sum_{k=1}^N x'_k (X_k f)(x)}{|x'|} + \sum_{k=1}^N x'_k(-1)|x'|^{-2}X_k|x'|f(x)\\
&= \frac{x'\cdot(\nabla_H f)(x)}{|x'|} - \frac{f(x)}{|x'|}.
\end{aligned}$$

From these identities we get

$$\begin{aligned}
\widehat{T}_\alpha^+\widehat{T}_\alpha f(x) = &-\frac{\sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^2}\\
&-(N-1)\frac{x'\cdot(\nabla_H f)(x)}{|x'|^2} + \frac{\alpha(\alpha+2-N)}{|x'|^2}f(x).
\end{aligned}$$

Using (1.73) again we have

$$\begin{aligned}
&\sum_{j,k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}x'_j x'_k (X_j X_k f)(x)}{|x'|^2}dx\\
&= -(N-1)\sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}x'_k (X_k f)(x)}{|x'|^2}dx - 2\sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}x'_k (X_k f)(x)}{|x'|^2}dx\\
&\quad - \sum_{j,k=1}^N \int_{\mathbb{G}} \frac{x'_j x'_k \overline{(X_j f)(x)}(X_k f)(x)}{|x'|^2}dx\\
&\quad - \sum_{j,k=1}^N \int_{\mathbb{G}} \overline{f(x)}x'_j x'_k (X_k f)(x)(-2)|x'|^{-3}X_j|x'|dx
\end{aligned}$$

$$
\begin{aligned}
&= -(N+1)\sum_{k=1}^{N}\int_{\mathbb{G}} \frac{\overline{f(x)}x_k'(X_k f)(x)}{|x'|^2}dx + 2\sum_{j,k=1}^{N}\int_{\mathbb{G}} \frac{\overline{f(x)}(x_j')^2 x_k'(X_k f)(x)}{|x'|^4}dx \\
&\quad - \int_{\mathbb{G}} \frac{|x'\cdot(\nabla_H f)(x)|^2}{|x'|^2}dx \\
&= -(N-1)\int_{\mathbb{G}} \frac{\overline{f(x)}(x'\cdot(\nabla_H f)(x))}{|x'|^2}dx - \int_{\mathbb{G}} \frac{|x'\cdot(\nabla_H f)(x)|^2}{|x'|^2}dx. \qquad (6.41)
\end{aligned}
$$

Taking into account this, integrating by parts, and using the non-negativity of the operator $\widehat{T}_\alpha^+\widehat{T}_\alpha$, we get

$$
\begin{aligned}
0 \le &\int_{\mathbb{G}} |\widehat{T}_\alpha f|^2 dx = \int_{\mathbb{G}} \overline{f(x)}(\widehat{T}_\alpha^+\widehat{T}_\alpha f)(x)dx \\
= &-\int_{\mathbb{G}} \left(\frac{\sum_{j,k=1}^{N} x_j' x_k' \overline{f(x)}(X_j X_k f)(x)}{|x'|^2} + \frac{(N-1)\overline{f(x)}(x'\cdot(\nabla_H f)(x))}{|x'|^2} \right) dx \\
&+ \alpha(\alpha - N + 2)\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2}dx.
\end{aligned}
$$

Consequently, using (6.41) we obtain

$$
\int_{\mathbb{G}} \left(\frac{|x'\cdot(\nabla_H f)(x)|^2}{|x'|^2} + \alpha(\alpha - N + 2)\frac{|f(x)|^2}{|x'|^2} \right) dx \ge 0.
$$

It now follows that

$$
\int_{\mathbb{G}} \frac{|x'\cdot(\nabla_H f)(x)|^2}{|x'|^2}dx \ge \alpha((N-2)-\alpha)\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2}dx.
$$

By maximizing $\alpha((N-2)-\alpha)$ with respect to α we obtain (6.40). $\square$

Further two-parameter inequalities by factorization method are possible in the setting of the Heisenberg group, for which we can refer the reader to [RY17].

6.5 Hardy–Rellich type inequalities and embedding results

We now discuss several refinements of the Hardy–Rellich inequalities with respect to the variables in the first stratum. We recall that N stands for the dimension of the first stratum of a stratified Lie group $\mathbb{G}$ here. As a consequence, we formulate several corollaries for the embeddings of the appearing function spaces.

Theorem 6.5.1 (Horizontal L^2-Hardy–Rellich type inequalities)**.** *Let α, $\beta \in \mathbb{R}$. Let $N \geq 2$ be the dimension of the first stratum of a stratified Lie group $\mathbb{G}$, and let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^N$. Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)^2 \leq \int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx. \tag{6.42}$$

Moreover, if $\alpha+\beta+1 \leq 0$ then we have

$$\frac{N+\alpha+\beta-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx \leq \left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{1/2}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{1/2}. \tag{6.43}$$

The inequality (6.43) can be considered as a special case ($p=2$) of Theorem 6.8.1. In particular, taking $\alpha=\beta+1$, we obtain the following Rellich type inequality:

Corollary 6.5.2 (Horizontal L^2-Rellich type inequality)**.** *Let N be the dimension of the first stratum of a stratified Lie group $\mathbb{G}$ and let $\alpha \leq 0$. Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$\frac{(N+2\alpha-2)^2}{4}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx \leq \int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\alpha-2}}dx. \tag{6.44}$$

Furthermore, we have

$$\frac{(N+2\alpha-2)^2(N-2\alpha-2)^2}{16}\int_{\mathbb{G}}\frac{|f(x)|^2}{|x'|^{2\alpha+2}}dx \leq \int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\alpha-2}}dx. \tag{6.45}$$

We can compare it with another version given in Theorem 6.2.2.

Proof of Corollary 6.5.2. Inequality (6.44) follows from Theorem 6.5.1 by taking $\alpha=\beta+1$. Inequality (6.45) follows from (6.44) and Corollary 6.2.1 with $p=2$ which says that

$$\left\|\frac{1}{|x'|^\alpha}\nabla_H f\right\|_{L^2(\mathbb{G})} \geq \frac{|N-2(\alpha+1)|}{2}\left\|\frac{f}{|x'|^{\alpha+1}}\right\|_{L^2(\mathbb{G})}.$$

Note that the sharpness of the constant follows from the fact that in both inequalities the best constants are attained when there are equalities in the corresponding Hölder inequalities in their proofs, and these are attained on powers of $|x'|$. $\square$

We note that another version of the horizontal Rellich inequality is given in Theorem 6.2.2.

Note that when $\mathbb{G} = (\mathbb{R}^n, +)$, that is, $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$, then (6.42) implies the following Hardy–Rellich type inequality for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$:

$$\left(\frac{n-(\alpha+\beta+3)}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{|x|_E^{\alpha+\beta+1}} dx + (\alpha+\beta+1) \int_{\mathbb{R}^n} \frac{(x \cdot \nabla f)^2}{|x|_E^{\alpha+\beta+3}} dx \right)^2 \leq \int_{\mathbb{R}^n} \frac{|\Delta f|^2}{|x|_E^{2\beta}} dx \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{|x|_E^{2\alpha}} dx, \tag{6.46}$$

where $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$. This inequality was also discussed in [Cos08] and [DJSJ13].

Proof of Theorem 6.5.1. First we note that for all $s \in \mathbb{R}$ we have

$$\int_{\mathbb{G}} \left| \frac{\nabla_H f}{|x'|^\alpha} + s \frac{x'}{|x'|^{\beta+1}} \mathcal{L} f \right|^2 dx \geq 0,$$

that is,

$$\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}} dx + 2s \int_{\mathbb{G}} \frac{x' \cdot \nabla_H f}{|x'|^{\alpha+\beta+1}} \mathcal{L} f dx + s^2 \int_{\mathbb{G}} \frac{|\mathcal{L} f|^2}{|x'|^{2\beta}} dx \geq 0. \tag{6.47}$$

Since

$$\int_{\mathbb{G}} \frac{x' \cdot \nabla_H f}{|x'|^{\alpha+\beta+1}} \mathcal{L} f dx = \int_{\mathbb{G}} \operatorname{div}_H (\nabla_H f) \left(\frac{x' \cdot \nabla_H f}{|x'|^{\alpha+\beta+1}} \right) dx$$

by using the divergence theorem (Theorem 1.4.5) and (1.73) we obtain

$$\int_{\mathbb{G}} \operatorname{div}_H (\nabla_H f) \left(\frac{x' \cdot \nabla_H f}{|x'|^{\alpha+\beta+1}} \right) dx = -\frac{1}{2} \int_{\mathbb{G}} \frac{x'}{|x'|^{\alpha+\beta+1}} \cdot \nabla_H (|\nabla_H f|^2) dx - \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}} dx + (\alpha+\beta+1) \int_{\mathbb{G}} \frac{(x' \cdot \nabla_H f)^2}{|x'|^{\alpha+\beta+3}} dx.$$

Again by Theorem 1.4.5 and (1.73) we have the equality

$$-\frac{1}{2} \int_{\mathbb{G}} \frac{x'}{|x'|^{\alpha+\beta+1}} \cdot \nabla_H (|\nabla_H f|^2) dx = \frac{N-(\alpha+\beta+1)}{2} \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}} dx.$$

Thus,

$$\int_{\mathbb{G}} \frac{x' \cdot \nabla_H f}{|x'|^{\alpha+\beta+1}} \mathcal{L} f dx = \frac{N-(\alpha+\beta+3)}{2} \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}} dx + (\alpha+\beta+1) \int_{\mathbb{G}} \frac{(x' \cdot \nabla_H f)^2}{|x'|^{\alpha+\beta+3}} dx. \tag{6.48}$$

Therefore, the inequality (6.47) can be rewritten as

$$s^2\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx+2s\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx\right.$$
$$\left.+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)+\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\geq 0.$$

Denoting

$$a:=\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx,$$

$$b:=\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx,$$

and

$$c:=\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx$$

we arrive at

$$as^2+2bs+c\geq 0,$$

which is equivalent to $b^2-ac\leq 0$. Thus, we have

$$\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)^2$$
$$\leq\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx.$$

This shows the inequality (6.42). Now let us show the inequality (6.43). By using Schwarz' and Hölder's inequality we obtain

$$\int_{\mathbb{G}}\frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\mathcal{L}f\,dx\leq\int_{\mathbb{G}}\frac{|\nabla_H f|}{|x'|^{\alpha+\beta}}\mathcal{L}f\,dx\leq\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{1/2}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{1/2}.$$

On the other hand, since $\alpha+\beta+1\leq 0$ by Schwarz' inequality we have

$$\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx$$
$$\geq\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx$$
$$=\frac{N+\alpha+\beta-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx.$$

Combining the above inequalities with (6.48) we obtain (6.42). $\square$

The special case of Theorem 6.1.1 with $p = 2$ can be also shown using the divergence formula techniques in the proof of Theorem 6.5.1:

Corollary 6.5.3 (Horizontal L^2-Caffarelli–Kohn–Nirenberg inequalities)**.** *Let $\mathbb{G}$ be a homogeneous stratified group with N being the dimension of the first stratum. Let $\alpha, \beta \in \mathbb{R}$. Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have*

$$\frac{|N-\gamma|}{2}\left\|\frac{f}{|x'|^{\frac{\gamma}{2}}}\right\|^2_{L^2(\mathbb{G})} \leq \left\|\frac{\nabla_H f}{|x'|^\alpha}\right\|_{L^2(\mathbb{G})}\left\|\frac{f}{|x'|^\beta}\right\|_{L^2(\mathbb{G})}, \tag{6.49}$$

where $\gamma = \alpha+\beta+1$, and the constant $\frac{|N-\gamma|}{2}$ is sharp.

Proof. For all $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$, $\alpha, \beta \in \mathbb{R}$ and $s \in \mathbb{R}$ we have

$$\int_{\mathbb{G}}\left|\frac{\nabla_H f}{|x'|^\beta} + s\frac{x'}{|x'|^{\alpha+1}}f\right|^2 dx \geqslant 0.$$

This can be written as

$$\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\beta}}dx + s^2\int_{\mathbb{G}}\frac{|f|^2}{|x'|^{2\alpha}}dx + 2s\int_{\mathbb{G}} f\frac{x'\cdot\nabla_H f}{|x'|^\gamma}dx \geqslant 0.$$

By the divergence theorem (Theorem 1.4.5) we have

$$\int_{\mathbb{G}} f\frac{x'\cdot\nabla_H f}{|x'|^\gamma}dx = -\frac{N-\gamma}{2}\int_{\mathbb{G}}\frac{|f|^2}{|x'|^\gamma}dx.$$

Denoting

$$a := \int_{\mathbb{G}}\frac{|f|^2}{|x'|^{2\alpha}}dx, \quad b := |N-\gamma|\int_{\mathbb{G}}\frac{|f|^2}{|x'|^\gamma}, \quad c := \int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\beta}}dx,$$

this means that

$$as^2 - bs + c \geqslant 0,$$

which is equivalent to $b^2 - 4ac \leqslant 0$, that is,

$$|N-\gamma|^2\left(\int_{\mathbb{G}}\frac{|f|^2}{|x'|^\gamma}\right)^2 \leqslant 4\left(\int_{\mathbb{G}}\frac{|f|^2}{|x'|^{2\alpha}}dx\right)\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\beta}}dx\right),$$

which gives (6.49). $\square$

The appearance of the horizontal weights in Theorem 6.5.1 prompts one to define the following weighted Sobolev type spaces on the stratified Lie group $\mathbb{G}$ (in Chapter 10 we will be discussing analogous spaces but there on general homogeneous groups).

Definition 6.5.4 (Sobolev types spaces with horizontal weights)**.** Let us define the following spaces:

(1) Let $L^2_\alpha(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{L^2_\alpha} := \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx\right)^{1/2}.$$

(2) Let $D^{1,2}_\gamma(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{D^{1,2}_\gamma(\mathbb{G})} := \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\gamma}} dx\right)^{1/2}.$$

(3) Let $D^{2,2}_\gamma(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{D^{2,2}_\gamma(\mathbb{G})} := \left(\int_{\mathbb{G}} \frac{|\mathcal{L} f|^2}{|x'|^{2\gamma}} dx\right)^{1/2}.$$

(4) Let $H^1_{\alpha,\beta}(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{H^1_{\alpha,\beta}} := \left(\int_{\mathbb{G}} \left[\frac{|f|^2}{|x'|^{2\alpha}} + \frac{|\nabla_H f|^2}{|x'|^{2\beta}}\right] dx\right)^{1/2}.$$

(5) Let $H^2_{\alpha,\beta}(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{H^2_{\alpha,\beta}(\mathbb{G})} := \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}} + \frac{|\mathcal{L} f|^2}{|x'|^{2\beta}} dx\right)^{1/2}.$$

Theorem 6.5.5 (Several horizontal embeddings)**.** *Let α, $\beta \in \mathbb{R}$. We have the following continuous embeddings*

(i) $H^2_{\alpha,\beta}(\mathbb{G}) \subset D^{2,2}_{\frac{\alpha+\beta+1}{2}}(\mathbb{G})$ *for* $\alpha+\beta-1 \neq N$.

(ii) $D^{2,2}_\alpha(\mathbb{G}) \subset D^{1,2}_{\alpha+1}(\mathbb{G})$ *for* $\alpha \leq \frac{N}{2}-2$.

(iii) $H^1_{\alpha,\beta}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G})$ *and* $H^1_{\beta,\alpha}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G})$ *for* $\gamma=\alpha+\beta+1$, *provided that* $\gamma \neq N$.

Proof of Theorem 6.5.5. Since $N \neq \alpha+\beta-1$, from (6.43) we obtain

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\frac{(\alpha+\beta+1)}{2}}} dx &\leq \frac{2}{|N+\alpha+\beta-1|}\left(\int_{\mathbb{G}} \frac{|\mathcal{L} f|^2}{|x'|^{2\beta}} dx\right)^{1/2}\left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}} dx\right)^{1/2} \\ &\leq \frac{2}{|N+\alpha+\beta-1|}\left(\int_{\mathbb{G}} \frac{|\mathcal{L} f|^2}{|x'|^{2\beta}} dx + \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}} dx\right),\end{aligned}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$. This proves Part (i).

Part (ii) follows from the inequality (6.43), namely assuming $\alpha+\beta+3 \leq N$ and letting $\beta=\alpha+1,\ \alpha \neq \frac{N}{2}$.

The first inequality in Part (iii) follows from inequality (6.49). Since the spaces are symmetric with respect to the parameters α, β we also have the second embedding. □

Using inequality (6.43) and choosing different values of α and β we can obtain a number of Heisenberg–Pauli–Weyl type uncertainty inequalities. Let us list some interesting cases.

Corollary 6.5.6 (Horizontal Heisenberg–Pauli–Weyl type uncertainty inequalities)**.** *We have the following inequalities:*

(1) *For $\alpha \leq \frac{N}{2} - 2$ and any $f \in H^2_{\alpha,\alpha+1}(\mathbb{G})$,*

$$\frac{|N+2\alpha|}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2(\alpha+1)}}dx \leq \left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2(\alpha+1)}}dx\right)^{1/2}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{1/2}.$$

(2) *For $N \geq 3$ and any $f \in D^{1,2}_0(\mathbb{G})$,*

$$\frac{|N-2|}{2}\int_{\mathbb{G}}|\nabla_H f|^2 dx \leq \left(\int_{\mathbb{G}}|x'|^{2(\alpha+1)}|\nabla_H f|^2 dx\right)^{1/2}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\alpha}}dx\right)^{1/2}.$$

(3) *For any $f \in D^{1,2}_1(\mathbb{G})$,*

$$\frac{N}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^2}dx \leq \left(\int_{\mathbb{G}}|\nabla_H f|^2 dx\right)^{1/2}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^2}dx\right)^{1/2}.$$

(4) *For $N \geq 2$ and any $f \in D^{1,2}_{1/2}(\mathbb{G})$,*

$$\frac{N-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|}dx \leq \left(\int_{\mathbb{G}}|x'|^2|\nabla_H f|^2 dx\right)^{1/2}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^2}dx\right)^{1/2}.$$

(5) *For $N \geq 2$ and any $f \in D^{1,2}_{1/2}(\mathbb{G})$,*

$$\frac{N-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|}dx \leq \left(\int_{\mathbb{G}}|\nabla_H f|^2 dx\right)^{1/2}\left(\int_{\mathbb{G}}|\mathcal{L}f|^2 dx\right)^{1/2}.$$

Moreover, the following inequalities hold true with sharp constants:

(6) *For any $f \in D^{1,2}(\mathbb{G})$, taking $\alpha = 1, \beta = 0$,*

$$\left(\frac{N-2}{2}\right)^2\int_{\mathbb{G}}\frac{|f|^2}{|x'|^2}dx \leq \int_{\mathbb{G}}|\nabla_H f|^2 dx.$$

(7) *For any $f \in H^1_{\beta+1,\beta}(\mathbb{G})$, taking $\alpha = \beta + 1$,*

$$\left(\frac{N-2(\beta+1)}{2}\right)^2\int_{\mathbb{G}}\frac{|f|^2}{|x'|^{2(\beta+1)}}dx \leq \int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\beta}}dx.$$

(8) *For any* $f \in H^1_{\alpha,\alpha+1}(\mathbb{G})$, *taking* $\beta = \alpha + 1$,

$$\left(\frac{N-2(\alpha+1)}{2}\right)^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2(\alpha+1)}} dx \leq \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2(\alpha+1)}} dx\right)^{1/2}.$$

(9) *For any* $f \in H^1_{-(\beta+1),\beta}(\mathbb{G})$, *taking* $\alpha = -(\beta+1)$, *then* $f \in L^2(\mathbb{G})$ *and*

$$\left(\frac{N}{2}\right) \int_{\mathbb{G}} |u|^2 dx \leq \left(\int_{\mathbb{G}} |x'|^{2(\beta+1)} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\beta}} dx\right)^{1/2}.$$

(10) *For any* $f \in H^1_{0,1}(\mathbb{G})$, *taking* $\alpha = 0, \beta = 1$, *then* $f \in L^2_1(\mathbb{G})$ *and*

$$\left|\frac{N-2}{2}\right| \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^2} dx\right)^{1/2}.$$

(11) *For any* $f \in H^1_{-1,1}(\mathbb{G})$, $N > 1$, *taking* $\alpha = -1, \beta = 1$, *then* $f \in L^2_{1/2}(\mathbb{G})$ *and*

$$\left(\frac{N-1}{2}\right) \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |x'|^2 |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^2} dx\right)^{1/2}.$$

(12) *For any* $f \in H^1(\mathbb{G}) = H^1_{0,0}(\mathbb{G})$, $N > 1$, *taking* $\alpha = 0, \beta = 0$, *then* $f \in L^2_{1/2}(\mathbb{G})$ *and*

$$\left(\frac{N-1}{2}\right) \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} |\nabla_H f|^2 dx\right)^{1/2}.$$

6.6 Horizontal Sobolev type inequalities

In this section, first, we are interested in Sobolev inequalities, so let us repeat them briefly again for the sake of the reader comparing to the full homogeneous group version discussed in Section 3.2.2. The (Euclidean) Sobolev inequality in its simplest form has the form

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C(p) \|\nabla g\|_{L^{p^*}(\mathbb{R}^n)},$$

for all $1 < p, p^* < \infty$ with

$$\frac{1}{p} = \frac{1}{p^*} - \frac{1}{n}.$$

Here ∇ is the usual gradient in $\mathbb{R}^n$. The following version of a Sobolev type inequality with respect to the operator $x \cdot \nabla$ instead of the standard gradient ∇ was considered in [BEHL08, OS09]:

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C'(p) \|x \cdot \nabla g\|_{L^q(\mathbb{R}^n)}. \tag{6.50}$$

By putting $g(x) = h(\lambda x)$, $\lambda > 0$, into this inequality, we see that $p = q$ is a necessary condition to have (6.50).

We can notice that in formula (6.50) the operator $x \cdot \nabla$ can be interpreted as the homogeneous Euler operator on general homogeneous groups (see Section 1.3.2) as well as an operator on stratified groups by substituting x and ∇ with the corresponding *horizontal* operations x' and ∇_H related to the first stratum of the group.

The homogeneous groups version of such inequalities was discussed in Section 3.2.2. Thus, we will now concentrate on the horizontal interpretation presenting a range of Caffarelli–Kohn–Nirenberg and weighted L^p-Sobolev type inequalities on stratified Lie groups. All the inequalities can be obtained with sharp constants.

The presentation of the following results follows [RSY17a].

We start with an L^p-weighted Sobolev type inequality.

Theorem 6.6.1 (Horizontal weighted L^p-Sobolev type inequality)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. For any $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, and all $\alpha \in \mathbb{R}$, we have*

$$\frac{|N-\alpha p|}{p}\left\|\frac{f}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{x'\cdot\nabla_H f}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{G})}, \quad 1<p<\infty, \tag{6.51}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. The constant $\frac{|N-\alpha p|}{p}$ is sharp when $N \neq \alpha p$.

Remark 6.6.2.

1. In the Abelian case $\mathbb{G}=(\mathbb{R}^n,+)$, that is, $N=n$ and $\nabla_H=\nabla=(\partial_{x_1},\ldots,\partial_{x_n})$, the inequality (6.51) yields the L^p-weighted Sobolev type inequality for $\mathbb{G}=\mathbb{R}^n$ with the sharp constant:

$$\frac{|n-\alpha p|}{p}\left\|\frac{f}{|x|_E^{\alpha}}\right\|_{L^p(\mathbb{R}^n)} \leq \left\|\frac{x\cdot\nabla f}{|x|_E^{\alpha}}\right\|_{L^p(\mathbb{R}^n)}, \tag{6.52}$$

   for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, and $|x|_E = \sqrt{x_1^2+\cdots+x_n^2}$. This Euclidean inequality was shown in [OS09].

2. Using Schwarz' inequality in the right-hand side of (6.51) we see that (6.51) is a refinement of the L^p-weighted Hardy inequality on stratified groups: For any $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, and all $\alpha \in \mathbb{R}$, we have

$$\frac{|N-\alpha p|}{p}\left\|\frac{f}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\nabla_H f}{|x'|^{\alpha-1}}\right\|_{L^p(\mathbb{G})}, \ 1<p<\infty, \tag{6.53}$$

   where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. If $N \neq \alpha p$ then the constant $\frac{|N-\alpha p|}{p}$ is sharp. Thus, (6.51) can be regarded as a refinement of (6.53). In the case of $p=2$ they are actually equivalent, see Theorem 6.6.3. These results have been obtained in [RSY17a].

3. For $\alpha = 0$, inequality (6.51) gives the weighted version of the homogeneous groups inequality in Proposition 3.2.1, Part (i). In particular, in the Euclidean case of $\mathbb{R}^n$ we have $N = n$, and this gives the weighted version of the inequality in Remark 3.2.2, Part 2.

Proof of Theorem 6.6.1. Let us assume $\alpha p \neq N$ since when $\alpha p = N$ there is nothing to prove. By using the identity (1.73) and the divergence theorem we obtain

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^{\alpha p}} &= \frac{1}{N-\alpha p}\int_{\mathbb{G}} |f(x)|^p \mathrm{div}_H\left(\frac{x'}{|x'|^{\alpha p}}\right) dx \\
&= -\frac{p}{N-\alpha p}\mathrm{Re}\int_{\mathbb{G}} p f(x)|f(x)|^{p-2}\frac{\overline{x'\cdot\nabla_H f}}{|x'|^{\alpha p}}dx \\
&\leq \left|\frac{p}{N-\alpha p}\right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x'|^{\alpha p}}|x'\cdot\nabla_H f| dx \\
&\leq \left|\frac{p}{N-\alpha p}\right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x'|^{\alpha(p-1)}}\frac{|x'\cdot\nabla_H f|}{|x'|^{\alpha}} dx \\
&\leq \left|\frac{p}{N-\alpha p}\right| \left(\frac{|f(x)|^p}{|x'|^{\alpha p}}dx\right)^{(p-1)/p}\left(\frac{|x'\cdot\nabla_H f|^p}{|x'|^{\alpha p}}dx\right)^{1/p},
\end{aligned}$$

which implies (6.51). Here in the last line the Hölder inequality has been used. Now it remains to show the sharpness of the constant. Observe that the function

$$h_1(x) = \frac{1}{|x'|^{\frac{|N-\alpha p|}{p}}}, \quad N \neq \alpha p,$$

satisfies the equality condition in the Hölder inequality

$$\left|\frac{p}{N-\alpha p}\right|^p \frac{|x'\cdot\nabla_H h_1(x)|^p}{|x'|^{\alpha p}} = \frac{|h_1(x)|^p}{|x'|^{\alpha p}}.$$

This means that the constant $\frac{|N-\alpha p|}{p}$ is sharp. $\square$

In the case of L^2 the horizontal Sobolev type inequality is actually equivalent to the Hardy inequality:

Theorem 6.6.3 (Equivalence of Sobolev type and Hardy inequalities in L^2)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum with $N \geq 3$. Then the following two statements are equivalent:*

(a) *For any $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, we have*

$$\|f\|_{L^2(\mathbb{G})} \leq \frac{2}{N}\|x'\cdot\nabla_H f\|_{L^2(\mathbb{G})}. \tag{6.54}$$

(b) *For any $g \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, we have*

$$\left\|\frac{g}{|x'|}\right\|_{L^2(\mathbb{G})} \leq \frac{2}{N-2}\left\|\frac{x'}{|x'|}\cdot\nabla_H g\right\|_{L^2(\mathbb{G})}. \tag{6.55}$$

Proof of Theorem 6.6.3. Setting $g=|x'|f$ we obtain that

$$\begin{aligned}\|x'\cdot\nabla_H f\|^2_{L^2(\mathbb{G})} &= \left\|-\frac{g}{|x'|}+\frac{x'}{|x'|}\cdot\nabla_H g\right\|^2_{L^2(\mathbb{G})} \qquad (6.56)\\ &= \left\|\frac{g}{|x'|}\right\|^2_{L^2(\mathbb{G})} - 2\mathrm{Re}\int_{\mathbb{G}}\frac{\overline{g(x)}}{|x'|}\frac{x'}{|x'|}\cdot\nabla_H g(x)dx + \left\|\frac{x'}{|x'|}\cdot\nabla_H g\right\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

By (1.73), one calculates

$$\begin{aligned}-2\mathrm{Re}\int_{\mathbb{G}}\frac{\overline{g(x)}}{|x'|}\frac{x'}{|x'|}\cdot\nabla_H g(x)dx &= -\int_{\mathbb{G}}\frac{x'}{|x'|^2}\nabla_H|g(x)|^2dx\\ &= \int_{\mathbb{G}}\mathrm{div}_H\left(\frac{x'}{|x'|^2}\right)|g(x)|^2dx\\ &= (N-2)\int_{\mathbb{G}}\frac{|g(x)|^2}{|x'|^2}dx.\end{aligned}$$

We obtain from the statement (a) and (6.56) that

$$\left\|\frac{g}{|x'|}\right\|^2_{L^2(\mathbb{G})} \leq \frac{4}{N^2}\left((N-1)\left\|\frac{g}{|x'|}\right\|^2_{L^2(\mathbb{G})} + \left\|\frac{x'}{|x'|}\cdot\nabla_H g\right\|^2_{L^2(\mathbb{G})}\right),$$

which implies (6.55). This shows that the statement (a) gives (b).

Conversely, assume that (b) holds. Put $f=g/|x'|$. Then we obtain

$$\begin{aligned}\left\|\frac{x'}{|x'|}\cdot\nabla_H(|x'|f)\right\|^2_{L^2(\mathbb{G})} &= \|f+x'\cdot\nabla_H f\|^2_{L^2(\mathbb{G})}\\ &= \|f\|^2_{L^2(\mathbb{G})} + 2\mathrm{Re}\int_{\mathbb{G}}x'f(x)\overline{\nabla_H f}dx + \|x'\cdot\nabla_H f\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

Using (1.73), we have

$$2\mathrm{Re}\int_{\mathbb{G}}x'f(x)\overline{\nabla_H f}dx = -N\|f\|^2_{L^2(\mathbb{G})}.$$

It follows from the statement (b) that

$$\|f\|^2_{L^2(\mathbb{G})} \leq \frac{4}{(N-2)^2}(\|x'\cdot\nabla_H f\|^2_{L^2(\mathbb{G})} - (N-1)\|f\|^2_{L^2(\mathbb{G})}),$$

which implies (6.54). □

6.7 Horizontal extended Caffarelli–Kohn–Nirenberg inequalities

We now present horizontal extended Caffarelli–Kohn–Nirenberg inequalities in the setting of stratified groups. We recall that another version of such inequalities on general homogeneous groups involving the radial derivative was discussed in Section 3.3.

Theorem 6.7.1 (Horizontal Caffarelli–Kohn–Nirenberg type inequalities)**.** *Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$, $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$ and a, b, $c \in \mathbb{R}$. In addition, assume that*

$$\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1 \quad \text{and} \quad c = \delta(a-1) + b(1-\delta).$$

Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum with $N \neq p(1-a)$. Then the following inequality holds:

$$\||x'|^c f\|_{L^r(\mathbb{G})} \leq \left|\frac{p}{N + p(a-1)}\right|^{\delta} \||x'|^a \nabla_H f\|^{\delta}_{L^p(\mathbb{G})} \left\||x'|^b f\right\|^{1-\delta}_{L^q(\mathbb{G})} \tag{6.57}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. The constant in the inequality (6.57) is sharp for $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1-a) + bq \neq 0$, or for $\delta = 0, 1$.

Remark 6.7.2.

1. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$ and $\nabla_H = \nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$, so (6.57) implies the following Caffarelli–Kohn–Nirenberg type inequality for $\mathbb{G} = \mathbb{R}^n$: Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$ and a, b, $c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a-1) + b(1-\delta)$. Then we have
$$\||x|^c f\|_{L^r(\mathbb{R}^n)} \leq \left|\frac{p}{n + p(a-1)}\right|^{\delta} \||x|^a \nabla f\|^{\delta}_{L^p(\mathbb{R}^n)} \left\||x|^b f\right\|^{1-\delta}_{L^q(\mathbb{R}^n)}, \tag{6.58}$$
for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, and $n \neq p(1-a)$. The constant in the inequality (6.58) is sharp for $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1-a) + bq \neq 0$, or for $\delta = 0, 1$.
2. The inequalities (6.58) give an extension of the Caffarelli–Kohn–Nirenberg in equalities Theorem 3.3.3 with respect to the range of indices. For example, let us take $1 < p = q = r < \infty$, $a = -\frac{n-2p}{p}$, $b = -\frac{n}{p}$ and $c = -\frac{n-\delta p}{p}$. Then by (6.58), for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ and all $1 < p < \infty$, $0 \leq \delta \leq 1$, we have the inequality
$$\left\|\frac{f}{|x|^{\frac{n-\delta p}{p}}}\right\|_{L^p(\mathbb{R}^n)} \leq \left\|\frac{\nabla f}{|x|^{\frac{n-2p}{p}}}\right\|^{\delta}_{L^p(\mathbb{R}^n)} \left\|\frac{f}{|x|^{\frac{n}{p}}}\right\|^{1-\delta}_{L^p(\mathbb{R}^n)}, \tag{6.59}$$

where ∇ is the standard gradient in $\mathbb{R}^n$. Since we have

$$\frac{1}{q}+\frac{b}{n}=\frac{1}{p}+\frac{1}{n}\left(-\frac{n}{p}\right)=0,$$

we see that (3.99) fails, so that the inequality (6.59) is not covered by Theorem 3.3.1.

Proof of Theorem 6.7.1. Case $\delta=0$. Notice that in this case we have $q=r$ and $b=c$ by $\frac{\delta r}{p}+\frac{(1-\delta)r}{q}=1$ and $c=\delta(a-1)+b(1-\delta)$, respectively. Then, the inequality (6.57) is reduced to

$$\||x'|^b f\|_{L^q(\mathbb{G})}\leq \left\||x'|^b f\right\|_{L^q(\mathbb{G})},$$

which is trivial.

Case $\delta=1$. In this case we have $p=r$ and $a-1=c$. By (6.53), for $N+cp\neq 0$ we obtain

$$\||x'|^c f\|_{L^r(\mathbb{G})}\leq \left|\frac{p}{N+cp}\right|\||x'|^{c+1}\nabla_H f\|_{L^r(\mathbb{G})}.$$

The constants in (6.53) is sharp, therefore, in this case the constant in (6.57) is sharp.

Case $\delta\in(0,1)\cap\left[\frac{r-q}{r},\frac{p}{r}\right]$. By using $c=\delta(a-1)+b(1-\delta)$, a direct calculation gives

$$\||x'|^c f\|_{L^r(\mathbb{G})}=\left(\int_{\mathbb{G}}|x'|^{cr}|f(x)|^r dx\right)^{1/r}=\left(\int_{\mathbb{G}}\frac{|f(x)|^{\delta r}}{|x'|^{\delta r(1-a)}}\frac{|f(x)|^{(1-\delta)r}}{|x'|^{-br(1-\delta)}}dx\right)^{1/r}.$$

Since we have $\delta\in(0,1)\cap\left[\frac{r-q}{r},\frac{p}{r}\right]$ and $p+q\geq r$, then by using Hölder's inequality for $\frac{\delta r}{p}+\frac{(1-\delta)r}{q}=1$, we obtain

$$\begin{aligned}\||x'|^c f\|_{L^r(\mathbb{G})}&\leq\left(\int_{\mathbb{G}}\frac{|f(x)|^p}{|x'|^{p(1-a)}}dx\right)^{\delta/p}\left(\int_{\mathbb{G}}\frac{|f(x)|^q}{|x'|^{-bq}}dx\right)^{(1-\delta)/q}\\&=\left\|\frac{f}{|x'|^{1-a}}\right\|^{\delta}_{L^p(\mathbb{G})}\left\|\frac{f}{|x'|^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{G})}.\end{aligned}\tag{6.60}$$

When $p=q$ and $a-b=1$, the Hölder equality condition is satisfied for all compactly supported smooth functions. We also note that in the case $p\neq q$ the function

$$h_2(x)=|x'|^{\frac{1}{(p-q)}(p(1-a)+bq)}\tag{6.61}$$

satisfies the Hölder equality condition:

$$\frac{|h_2(x)|^p}{|x'|^{p(1-a)}}=\frac{|h_2(x)|^q}{|x'|^{-bq}}.$$

If $N \neq p(1-a)$, then by (6.53), we have

$$\left\|\frac{f}{|x'|^{1-a}}\right\|_{L^p(\mathbb{G})}^{\delta} \leq \left|\frac{p}{N+p(a-1)}\right|^{\delta} \left\|\frac{\nabla_H f}{|x'|^{-a}}\right\|_{L^p(\mathbb{G})}^{\delta}. \tag{6.62}$$

Combining this with (6.60), we get

$$\||x'|^c f\|_{L^r(\mathbb{G})} \leq \left|\frac{p}{N+p(a-1)}\right|^{\delta} \left\|\frac{\nabla_H f}{|x'|^{-a}}\right\|_{L^p(\mathbb{G})}^{\delta} \left\|\frac{f}{|x'|^{-b}}\right\|_{L^q(\mathbb{G})}^{1-\delta}.$$

When we prove (6.62), in the same way as in the proof of Theorem 6.6.1, we note that

$$h_3(x) = |x'|^C, \quad C \neq 0, \tag{6.63}$$

satisfies the Hölder equality condition. Therefore, in the case $p = q$, $a - b = 1$ the Hölder equality condition of the inequalities (6.60) and (6.62) holds true for $h_3(x)$ in (6.63). Moreover, in the case $p \neq q$ and $p(1-a) + bq \neq 0$ the Hölder equality condition of the inequalities (6.60) and (6.62) holds true for $h_2(x)$ in (6.61). Therefore, the constant in (6.57) is sharp when $p = q$, $a - b = 1$ or $p \neq q$, $p(1-a) + bq \neq 0$. $\square$

6.8 Horizontal Hardy–Rellich type inequalities for *p*-sub-Laplacians

We prove the following Hardy–Rellich type inequalities for p-sub-Laplacians on the stratified group $\mathbb{G}$. As usual, N is the dimension of the first stratum and $|\cdot|$ is the Euclidean norm on it, identified with $\mathbb{R}^N$.

Theorem 6.8.1 (Horizontal Hardy–Rellich inequalities for p-sub-Laplacian)**.** *Let $1 < p < N$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha, \beta \in \mathbb{R}$ be such that*

$$\frac{p-N}{p-1} \leq \gamma := \alpha + \beta + 1 \leq 0.$$

Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have

$$\frac{N+\gamma(p-1)-p}{p} \left\|\frac{\nabla_H f}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{G})}^{p} \leq \left\|\frac{1}{|x'|^{\alpha}}\mathcal{L}_p f\right\|_{L^p(\mathbb{G})} \left\|\frac{\nabla_H f}{|x'|^{\beta}}\right\|_{L^q(\mathbb{G})}, \tag{6.64}$$

where $\mathcal{L}_p$ is the p-sub-Laplacian operator defined by

$$\mathcal{L}_p f := \mathrm{div}_H(|\nabla_H f|^{p-2}\nabla_H f). \tag{6.65}$$

Remark 6.8.2.

1. For $\beta = 0$, $\alpha = -1$ and $q = \frac{p}{p-1}$, the inequality (6.64) gives a stratified group Rellich type inequality for the p-sub-Laplacian $\mathcal{L}_p$:

$$\|\nabla_H f\|^p_{L^p(\mathbb{G})} \le \frac{p}{N-p} \, \||x'|\mathcal{L}_p f\|_{L^p(\mathbb{G})} \, \|\nabla_H f\|_{L^{\frac{p}{p-1}}(\mathbb{G})}, \quad 1 < p < N, \tag{6.66}$$

   for all $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$.

2. For $\alpha = 0$, $\beta = -1$, the inequality (6.64) implies the following Heisenberg–Pauli–Weyl type uncertainty principle for the p-sub-Laplacian $\mathcal{L}_p$: for $1 < p < N$ and for all $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have

$$\|\nabla_H f\|^p_{L^p(\mathbb{G})} \le \frac{p}{N-p} \, \|\mathcal{L}_p f\|_{L^p(\mathbb{G})} \, \||x'|\nabla_H f\|_{L^q(\mathbb{G})}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{6.67}$$

Proof of Theorem 6.8.1. As in the proof of Theorem 6.1.1 we have

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx &= \frac{1}{N-\gamma} \int_{\mathbb{G}} |\nabla_H f(x)|^p \mathrm{div}_H \left(\frac{x'}{|x'|^\gamma}\right) dx \\ &= -\frac{1}{N-\gamma} \int_{\mathbb{G}} \frac{p}{2} |\nabla_H f(x)|^{p-2} \frac{x' \cdot \nabla_H |\nabla_H f(x)|^2}{|x'|^\gamma} dx \\ &= \frac{p}{2(\gamma - N)} \int_{\mathbb{G}} |\nabla_H f(x)|^{p-2} \frac{x' \cdot \nabla_H |\nabla_H f(x)|^2}{|x'|^\gamma} dx.\end{aligned} \tag{6.68}$$

Moreover, we have

$$\begin{aligned}\int_{\mathbb{G}} \frac{\mathcal{L}_p f}{|x'|^\gamma} x' \cdot \nabla_H f(x) dx &= \int_{\mathbb{G}} \frac{\mathrm{div}_H(|\nabla_H f(x)|^{p-2} \nabla_H f(x))}{|x'|^\gamma} x' \cdot \nabla_H f(x) dx \\ &= -\int_{\mathbb{G}} |\nabla_H f(x)|^{p-2} \nabla_H f(x) \cdot \nabla_H \left(\frac{x' \cdot \nabla_H f(x)}{|x'|^\gamma}\right) dx\end{aligned}$$

$$= -\int_{\mathbb{G}} |\nabla_H f(x)|^{p-2} \left(\frac{|\nabla_H f(x)|^2}{|x'|^\gamma} + \frac{x' \cdot \nabla_H |\nabla_H f(x)|^2}{2|x'|^\gamma} - \frac{\gamma \left|x' \cdot \nabla_H f(x)\right|^2}{|x'|^{\gamma+2}}\right) dx,$$

that is,

$$\begin{aligned}&\int_{\mathbb{G}} \frac{|\nabla_H f(x)|^{p-2}}{|x'|^\gamma} x' \cdot \nabla_H |\nabla_H f(x)|^2 dx \\ &\quad = 2\gamma \int_{\mathbb{G}} |\nabla_H f(x)|^{p-2} \frac{\left|x' \cdot \nabla_H f(x)\right|^2}{|x'|^{\gamma+2}} dx - 2 \int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx \\ &\qquad - 2 \int_{\mathbb{G}} \frac{\mathcal{L}_p f}{|x'|^\gamma} x' \cdot \nabla_H f(x) dx.\end{aligned}$$

Putting this in the right-hand side of (6.68) we obtain

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx = \frac{p\gamma}{\gamma - N} \int_{\mathbb{G}} |\nabla_H f(x)|^{p-2} \frac{|x' \cdot \nabla_H f(x)|^2}{|x'|^{\gamma+2}} dx \\ - \frac{p}{\gamma - N} \int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx - \frac{p}{\gamma - N} \int_{\mathbb{G}} \frac{\mathcal{L}_p f}{|x'|^\gamma} x' \cdot \nabla_H f(x) dx.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{G}} \frac{\mathcal{L}_p f}{|x'|^\gamma} x' \cdot \nabla_H f(x) dx = \frac{N-p-\gamma}{p} \int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx \\ + \gamma \int_{\mathbb{G}} |\nabla_H f(x)|^{p-2} \frac{|x' \cdot \nabla_H f(x)|^2}{|x'|^{\gamma+2}} dx.\end{aligned}$$

Since $\gamma \leq 0$, applying the Cauchy–Schwarz inequality to the last integrants we get

$$\begin{aligned}&\int_{\mathbb{G}} \frac{\mathcal{L}_p f}{|x'|^\gamma} x' \cdot \nabla_H f(x) dx \\ &= \frac{N-p-\gamma}{p} \int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx + \gamma \int_{\mathbb{G}} |\nabla_H f(x)|^{p-2} \frac{|x' \cdot \nabla_H f(x)|^2}{|x'|^{\gamma+2}} dx \\ &\geq \frac{N-p-\gamma}{p} \int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx + \gamma \int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx \\ &= \frac{N+\gamma(p-1)-p}{p} \int_{\mathbb{G}} \frac{|\nabla_H f(x)|^p}{|x'|^\gamma} dx.\end{aligned} \tag{6.69}$$

Moreover, again applying the Cauchy–Schwarz inequality and the Hölder inequality we obtain

$$\begin{aligned}\int_{\mathbb{G}} \frac{\mathcal{L}_p f}{|x'|^\gamma} x' \cdot \nabla_H f(x) dx &\leq \int_{\mathbb{G}} \frac{\mathcal{L}_p f}{|x'|^{\gamma-1}} |\nabla_H f(x)| \, dx \\ &\leq \left(\int_{\mathbb{G}} \left| \frac{\mathcal{L}_p f}{|x'|^\alpha} \right|^p dx \right)^{1/p} \left(\int_{\mathbb{G}} \left| \frac{\nabla_H f}{|x'|^\beta} \right|^q dx \right)^{1/q}.\end{aligned}$$

Combining it with (6.69), the proof of Theorem 6.8.1 is complete. □

6.8.1 Inequalities for weighted p-sub-Laplacians

In this section, for a non-negative function $0 \leq \rho \in C^1(\mathbb{G})$ we consider the corresponding weighted p-sub-Laplacian

$$\mathcal{L}_{p,\rho} f = \operatorname{div}_H \left(\rho(x) |\nabla_H f|^{p-2} \nabla_H f \right), \quad 1 < p < \infty. \tag{6.70}$$

Depending on the function ρ, it satisfies the following inequalities.

Theorem 6.8.3 (Inequalities for weighted p-sub-Laplacian)**.** *Let $0 < F \in C^\infty(\mathbb{G})$ and $0 \le \eta \in L^1_{\mathrm{loc}}(\mathbb{G})$ be such that*

$$\eta F^{p-1} \le -\mathcal{L}_{p,\rho} F \tag{6.71}$$

holds almost everywhere in $\mathbb{G}$. Then for each $2 \le p < \infty$ there is a positive constant $C_p > 0$ such that we have

$$\|\eta^{\frac{1}{p}} f\|^p_{L^p(\mathbb{G})} + C_p \left\| \rho^{\frac{1}{p}} F \nabla_H \frac{f}{F} \right\|^p_{L^p(\mathbb{G})} \le \|\rho^{\frac{1}{p}} \nabla_H f\|^p_{L^p(\mathbb{G})}, \tag{6.72}$$

for all real-valued functions $f \in C_0^\infty(\mathbb{G})$.

Proof of Theorem 6.8.3. We observe first that for all $x, y \in \mathbb{R}^n$ there exists a positive number C_p such that

$$|x|^p + C_p |y|^p + p|x|^{p-2} x \cdot y \le |x+y|^p, \quad 2 \le p < \infty. \tag{6.73}$$

Therefore, we have the estimate

$$\begin{aligned} |g|^p |\nabla_H F|^p + C_p F^p |\nabla_H g|^p + F |\nabla_H F|^{p-2} \nabla_H F \cdot \nabla_H |g|^p \\ \le |g \nabla_H F + F \nabla_H g|^p = |\nabla_H f|^p, \end{aligned}$$

with $g = \frac{f}{F}$. This implies that

$$\begin{aligned} \int_{\mathbb{G}} \rho(x) |\nabla_H f(x)|^p dx \ge & \int_{\mathbb{G}} \rho(x) |\nabla_H F(x)|^p |g(x)|^p dx \\ & + C_p \int_{\mathbb{G}} \rho(x) |\nabla_H g(x)|^p |F(x)|^p dx \\ & - \int_{\mathbb{G}} \mathrm{div}_H (\rho(x) F(x) |\nabla_H F(x)|^{p-2} \nabla_H F(x)) |g(x)|^p dx \\ \ge & \, C_p \int_{\mathbb{G}} \rho(x) |\nabla_H g(x)|^p |F(x)|^p dx \\ & + \int_{\mathbb{G}} -\mathrm{div}_H (\rho(x) |\nabla_H F(x)|^{p-2} \nabla_H F(x)) F(x) |g(x)|^p dx. \end{aligned}$$

Using the assumption (6.71) it follows that

$$\int_{\mathbb{G}} \eta(x) |g(x)|^p |F(x)|^p dx + C_p \int_{\mathbb{G}} \rho(x) |\nabla_H g(x)|^p |F(x)|^p dx \le \int_{\mathbb{G}} \rho(x) |\nabla_H f(x)|^p dx.$$

Since $g = \frac{f}{F}$ we obtain

$$\|\eta^{\frac{1}{p}} f\|^p_{L^p(\mathbb{G})} + C_p \left\| \rho^{\frac{1}{p}} F \nabla_H \left(\frac{f}{F} \right) \right\|^p_{L^p(\mathbb{G})} \le \|\rho^{\frac{1}{p}} \nabla_H f\|^p_{L^p(\mathbb{G})},$$

proving (6.72). $\square$

Remark 6.8.4.

1. For $p = 2$, the inequality (6.73) becomes an equality with $C_2 = 1$. Therefore, the proof yields a remainder formula for $p = 2$ in the form

$$\left\|\rho^{\frac{1}{2}} F\nabla_H \frac{f}{F}\right\|^2_{L^2(\mathbb{G})} = \|\rho^{\frac{1}{2}}\nabla_H f\|^2_{L^2(\mathbb{G})} - \|\eta^{\frac{1}{2}} f\|^2_{L^2(\mathbb{G})}. \tag{6.74}$$

2. In the case of $1 < p < 2$ the inequality (6.73) can be also stated in the form that for all $x, y \in \mathbb{R}^n$ there exists a positive constant $C_p > 0$ such that

$$|x|^p + C_p \frac{|y|^p}{(|x| + |y|)^{2-p}} + p|x|^{p-2} x\cdot y \le |x+y|^p, \quad 1 < p < 2, \tag{6.75}$$

see, e.g., [Lin90, Lemma 4.2]. Thus, from the proof it then follows that we have

$$\|\eta^{\frac{1}{p}} f\|^p_{L^p(\mathbb{G})} + C_p \left\|\rho^{\frac{1}{2}}\left(\left|\frac{f}{F}\nabla_H F\right| + F\left|\nabla_H\left(\frac{f}{F}\right)\right|\right)^{\frac{p-2}{2}} |F|\nabla_H\left(\frac{f}{F}\right)\right\|^2_{L^2(\mathbb{G})} \le \|\rho^{\frac{1}{p}}\nabla_H f\|^p_{L^p(\mathbb{G})}, \tag{6.76}$$

for all real-valued functions $f \in C_0^\infty(\mathbb{G})$.

As a special case, we can apply Theorem 6.8.3 to the usual p-sub-Laplacian by taking the function $\rho \equiv 1$. In turn, this gives another proof of the L^p-Hardy inequality (6.6):

Corollary 6.8.5 (Horizontal L^p-Hardy inequality)**.** *For $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left\|\frac{f}{|x'|}\right\|_{L^p} \le \frac{p}{N-p}\|\nabla_H f\|_{L^p}, \quad 1 < p < N. \tag{6.77}$$

Proof of Corollary 6.8.5. In Theorem 6.8.3 setting $\rho = 1$ and

$$F_\epsilon = |x'_\epsilon|^{-\frac{\theta-p-2}{p}} = \left((x'_1+\epsilon)^2 + \cdots + (x'_n+\epsilon)^2\right)^{-\frac{\theta-p-2}{2p}},$$

for a given $\epsilon > 0$, using the identity (1.72) we obtain

$$\begin{aligned}
-\mathcal{L}_{p,1}F_\epsilon &= -\mathrm{div}_H\left(|\nabla_H F_\epsilon|^{p-2}\nabla_H F_\epsilon\right)\\
&= -\mathrm{div}_H\left(|\nabla_H |x'_\epsilon|^{-\frac{\theta-p-2}{p}}|^{p-2}\nabla_H |x'_\epsilon|^{-\frac{\theta-p-2}{p}}\right)\\
&= \frac{\theta-p-2}{p}\left|\frac{\theta-p-2}{p}\right|^{p-2}\left(\frac{\theta-p-2}{p} - \theta + 2 + N\right)|x'_\epsilon|^{-\frac{(\theta-p-2)(p-1)}{p}-p}
\end{aligned}$$

$$= \left(\left| \frac{\theta - p - 2}{p} \right|^p + \frac{\theta - p - 2}{p} \left| \frac{\theta - p - 2}{p} \right|^{p-2} (-\theta + 2 + N) \right) |x'_\epsilon|^{-\frac{(\theta-p-2)(p-1)}{p} - p}. \tag{6.78}$$

If $1 < p < \theta - 2$ and $\theta \leq 2 + N$, then (6.78) gives

$$-\mathcal{L}_{p,1} F_\epsilon \geq \left| \frac{\theta - p - 2}{p} \right|^p \frac{1}{|x'_\epsilon|^p} F_\epsilon^{p-1},$$

that is, according to the assumption in Theorem 6.8.3, we can set

$$\eta(x) = \left| \frac{\theta - p - 2}{p} \right|^p \frac{1}{|x'_\epsilon|^p}.$$

It follows that (6.72) (and also (6.76)) implies

$$\left\| \frac{f}{|x'|} \right\|_{L^p} \leq \frac{p}{\theta - p - 2} \|\nabla_H f\|_{L^p}, \quad 1 < p < \theta - 2, \ \theta \leq 2 + N, \ \theta \in \mathbb{R}.$$

Optimizing with respect to θ we obtain (6.77). □

Remark 6.8.6.

1 A version of Theorem 6.8.3 in the Euclidean case was shown in [Yen16]. In the presentation of this section we followed [RS17e].

2. The Heisenberg group version of (6.77) was shown in [D'A04b]. Here it is worth to recall that on the Heisenberg group we have $Q = N + 2$.

3. We have included Corollary 6.8.5 as a consequence of Theorem 6.8.3 to demonstrate that this method actually also yields best constants in some inequalities, as this constant in the L^p-Hardy inequality (6.6) was sharp.

6.9 Horizontal Rellich inequalities for sub-Laplacians with drift

In this section, we discuss (weighted) Rellich inequalities for sub-Laplacians with drift. For this, we assume all the notation of Section 1.4.6 where sub-Laplacians with drift have been discussed.

In this section we will discuss the horizontal versions, that is, with the weights being the powers of $|x'|$. In Section 7.4, we will discuss a version with weights in terms of the $\mathcal{L}$-gauge but that analysis is currently available only in the setting of polarizable Carnot groups. In the presentation of this section as well as of Section 7.4 we follow [RY18b].

The following result shows that the drift allows one to improve over the Rellich inequality without drift, given in Theorem 6.2.2.

Theorem 6.9.1 (Horizontal Rellich inequalities for sub-Laplacians with drift)**.** *Let $\mathbb{G}$ be a stratified group with $N \geq 3$ being the dimension of the first stratum. Let $\delta \in \mathbb{R}$ with $-N/2 \leq \delta \leq -1$. Then for all functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$\begin{aligned} \left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|^2_{L^2(\mathbb{G},\mu_X)} \geq & \left(\frac{(N-2\delta-4)(N+2\delta)}{4} \right)^2 \left\| \frac{f}{|x'|^{\delta+2}} \right\|^2_{L^2(\mathbb{G},\mu_X)} \\ & + \gamma^2 b_X^2 \frac{(N-2\delta-2)(N+2\delta-2)}{2} \left\| \frac{f}{|x'|^{\delta+1}} \right\|^2_{L^2(\mathbb{G},\mu_X)} \\ & + \gamma^4 b_X^4 \left\| \frac{f}{|x'|^\delta} \right\|^2_{L^2(\mathbb{G},\mu_X)}, \end{aligned} \tag{6.79}$$

where $\mathcal{L}_X$ and b_X are defined in (1.93) *and* (1.95)*, respectively. If $(N+2\delta)(N+2\delta-2) \neq 0$, then the constants in* (6.79) *are sharp. Moreover, when $\delta = 0$ and $N > 4$, for all functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$\begin{aligned} \|\mathcal{L}_X f\|^2_{L^2(\mathbb{G},\mu_X)} \geq & \left(\frac{N(N-4)}{4} \right)^2 \left\| \frac{f}{|x'|^2} \right\|^2_{L^2(\mathbb{G},\mu_X)} \\ & + \gamma^2 b_X^2 \frac{(N-2)^2}{2} \left\| \frac{f}{|x'|} \right\|^2_{L^2(\mathbb{G},\mu_X)} + \gamma^4 b_X^4 \|f\|^2_{L^2(\mathbb{G},\mu_X)}, \end{aligned} \tag{6.80}$$

with sharp constants. The constants in (6.79) *and* (6.80) *are sharp in the sense that there is a sequence of functions such that the equalities in* (6.79) *and* (6.80) *are attained in the limit of this sequence of functions, respectively.*

Remark 6.9.2.

1. The improvement in Rellich inequalities with drift compared to the standard ones as in Theorem 6.2.2 can be seen since for $(N-2\delta-2)(N+2\delta-2) \geq 0$, by dropping positive terms in (6.79) we get the following 'standard' Rellich type inequality for all functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$

$$\left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|^2_{L^2(\mathbb{G},\mu_X)} \geq \left(\frac{(N-2\delta-4)(N+2\delta)}{4} \right)^2 \left\| \frac{f}{|x'|^{\delta+2}} \right\|^2_{L^2(\mathbb{G},\mu_X)}, \tag{6.81}$$

where $\delta \in \mathbb{R}$ with $-N/2 \leq \delta \leq -1$ and $N \geq 3$.

Similarly, from (6.80) we obtain for $N > 4$ and for all functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ the inequality

$$\|\mathcal{L}_X f\|_{L^2(\mathbb{G},\mu_X)} \geq \frac{N(N-4)}{4} \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G},\mu_X)}, \tag{6.82}$$

which can be compared to the Rellich inequality in Corollary 6.5.2.

2. In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the usual full gradient, and setting

$$X = \sum_{i=1}^{n} a_i \partial_{x_i}$$

for $a_i \in \mathbb{R}$ for $i = 1, \ldots, n$, and $\delta = -\alpha$, $\gamma \in \mathbb{R}$, (6.79) implies, for $\alpha \geq 1$ and $n \geq \max\{3, 2\alpha\}$, $n + 2\alpha - 4 > 0$, that for all functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ we have

$$\begin{aligned}
&\left\| |x|^\alpha \left(\Delta + \gamma \sum_{i=1}^{n} a_i \partial_{x_i} \right) f \right\|^2_{L^2(\mathbb{R}^n, \mu_X)} \\
&\quad \geq \frac{(n + 2\alpha - 4)^2 (n - 2\alpha)^2}{16} \left\| |x|_E^{\alpha - 2} f \right\|^2_{L^2(\mathbb{R}^n, \mu_X)} \\
&\quad\quad + \gamma^2 b_X^2 \frac{(n + 2\alpha - 2)(n - 2\alpha - 2)}{2} \left\| |x|_E^{\alpha - 1} f \right\|^2_{L^2(\mathbb{R}^n, \mu_X)} \\
&\quad\quad + \gamma^4 b_X^4 \left\| |x|_E^{\alpha} f \right\|^2_{L^2(\mathbb{R}^n, \mu_X)},
\end{aligned} \tag{6.83}$$

with the measure μ_X on $\mathbb{R}^n$ given by

$$d\mu_X = e^{-\gamma \sum_{i=1}^{n} a_i x_i} dx,$$

where dx is the Lebesgue measure, and

$$b_X = \frac{1}{2} \left(\sum_{j=1}^{n} a_j^2 \right)^{1/2}.$$

If $(n - 2\alpha)(n - 2\alpha - 2) \neq 0$ with $\alpha \geq 1$ and $n \geq 2\alpha$, then the constants in (6.83) are sharp, in the sense that there is a sequence of functions such that the equality in (6.83) is attained in the limit of this sequence of functions.

In particular, for $\alpha = 0$, in the Euclidean setting of $\mathbb{R}^n$ with $n \geq 5$, for all $a_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and $\gamma \in \mathbb{R}$, and all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ we have a family of inequalities

$$\begin{aligned}
&\left\| \left(\Delta + \gamma \sum_{i=1}^{n} a_i \partial_{x_i} \right) f \right\|^2_{L^2(\mathbb{R}^n, \mu_X)} \\
&\quad \geq \frac{n^2 (n - 4)^2}{16} \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{R}^n, \mu_X)} + \gamma^4 b_X^4 \| f \|^2_{L^2(\mathbb{R}^n, \mu_X)} \\
&\quad\quad + \gamma^2 b_X^2 \frac{(n - 2)^2}{2} \left\| \frac{f}{|x|} \right\|^2_{L^2(\mathbb{R}^n, \mu_X)}.
\end{aligned} \tag{6.84}$$

All the constants in (6.84) are sharp in the sense that there is a sequence of functions such that the equality in (6.84) is attained in the limit of this sequence of functions.

Proof of Theorem 6.9.1. We denote by χ the positive character on $\mathbb{G}$ that appeared in Proposition 1.4.14. Let $g = g(x) \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ be such that $f = \chi^{-1/2}g$. Since the mapping (1.101) is an isomorphism we have

$$\left\|\frac{\mathcal{L}_X f}{|x'|^\delta}\right\|_{L^2(\mathbb{G},\mu_X)} = \left\|\frac{\chi^{1/2}\mathcal{L}_X f}{|x'|^\delta}\right\|_{L^2(\mathbb{G},\mu)} = \left\|\frac{\chi^{1/2}\mathcal{L}_X(\chi^{-1/2}g)}{|x'|^\delta}\right\|_{L^2(\mathbb{G},\mu)}.$$

By this, (1.100) and integration by parts, we have the equalities

$$\begin{aligned}
\left\|\frac{\mathcal{L}_X f}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu_X)} &= \left\|\frac{(\mathcal{L}_0+\gamma^2 b_X^2)g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} \\
&= \left\|\frac{\mathcal{L}_0 g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \mathrm{Re}\int_{\mathbb{G}} \frac{\mathcal{L}_0 g(x)\overline{g(x)}}{|x'|^{2\delta}}dx + \gamma^4 b_X^4 \left\|\frac{g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} \\
&= \left\|\frac{\mathcal{L}_0 g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} - 2\gamma^2 b_X^2 \mathrm{Re}\sum_{j=1}^N\int_{\mathbb{G}} \frac{X_j^2 g(x)\overline{g(x)}}{|x'|^{2\delta}}dx + \gamma^4 b_X^4 \left\|\frac{g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} \\
&= \left\|\frac{\mathcal{L}_0 g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \int_{\mathbb{G}} \frac{|\nabla_H g(x)|^2}{|x'|^{2\delta}} \\
&\quad - 4\delta\gamma^2 b_X^2 \mathrm{Re}\sum_{j=1}^N\int_{\mathbb{G}} \frac{x_j' X_j g(x)\overline{g(x)}}{|x'|^{2\delta+2}}dx + \gamma^4 b_X^4 \left\|\frac{g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)}.
\end{aligned} \tag{6.85}$$

Since we also have the equality

$$\begin{aligned}
&\mathrm{Re}\sum_{j=1}^N\int_{\mathbb{G}} \frac{x_j' X_j g(x)\overline{g(x)}}{|x'|^{2\delta+2}}dx \\
&\quad = (2\delta+2-N)\int_{\mathbb{G}} \frac{|g(x)|^2}{|x'|^{2\delta+2}}dx - \mathrm{Re}\sum_{j=1}^N\int_{\mathbb{G}} \frac{x_j' g(x)\overline{X_j g(x)}}{|x'|^{2\delta+2}}dx,
\end{aligned}$$

we obtain

$$\mathrm{Re}\sum_{j=1}^N\int_{\mathbb{G}} \frac{x_j' X_j g(x)\overline{g(x)}}{|x'|^{2\delta+2}}dx = \frac{2\delta+2-N}{2}\int_{\mathbb{G}} \frac{|g(x)|^2}{|x'|^{2\delta+2}}dx.$$

If we plug this into (6.85) we get

$$\begin{aligned}
\left\|\frac{\mathcal{L}_X f}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu_X)} &= \left\|\frac{\mathcal{L}_0 g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \left\|\frac{\nabla_H g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)} \\
&\quad + 2\delta(N-2\delta-2)\gamma^2 b_X^2 \left\|\frac{g}{|x'|^{\delta+1}}\right\|^2_{L^2(\mathbb{G},\mu)} + \gamma^4 b_X^4 \left\|\frac{g}{|x'|^\delta}\right\|^2_{L^2(\mathbb{G},\mu)}.
\end{aligned} \tag{6.86}$$

Using the Rellich (6.20) and Hardy (6.21) inequalities, we get from (6.86) that

$$\begin{aligned}\left\|\frac{\mathcal{L}_X f}{|x'|^{\delta}}\right\|^2_{L^2(\mathbb{G},\mu_X)} \geq &\left(\frac{(N-2\delta-4)(N+2\delta)}{4}\right)^2 \left\|\frac{g}{|x'|^{\delta+2}}\right\|^2_{L^2(\mathbb{G},\mu)} \\ &+ \gamma^4 b_X^4 \left\|\frac{g}{|x'|^{\delta}}\right\|^2_{L^2(\mathbb{G},\mu)} + 2\gamma^2 b_X^2 \left(\frac{N-2\delta-2}{2}\right)^2 \left\|\frac{g}{|x'|^{\delta+1}}\right\|^2_{L^2(\mathbb{G},\mu)} \\ &+ 2\delta(N-2\delta-2)\gamma^2 b_X^2 \left\|\frac{g}{|x'|^{\delta+1}}\right\|^2_{L^2(\mathbb{G},\mu)}.\end{aligned}$$

It follows then that

$$\begin{aligned}\left\|\frac{\mathcal{L}_X f}{|x'|^{\delta}}\right\|^2_{L^2(\mathbb{G},\mu_X)} \geq &\left(\frac{(N-2\delta-4)(N+2\delta)}{4}\right)^2 \left\|\frac{f}{|x'|^{\delta+2}}\right\|^2_{L^2(\mathbb{G},\mu_X)} \\ &+ \gamma^4 b_X^4 \left\|\frac{f}{|x'|^{\delta}}\right\|^2_{L^2(\mathbb{G},\mu_X)} \\ &+ \gamma^2 b_X^2 \frac{(N-2\delta-2)(N+2\delta-2)}{2} \left\|\frac{f}{|x'|^{\delta+1}}\right\|^2_{L^2(\mathbb{G},\mu_X)}.\end{aligned}$$

As we have discussed in the proof of Theorem 6.2.2, since the same function satisfies the equality conditions in Hölder's inequalities, the constants in (6.79) are sharp.

To obtain (6.80), that is the unweighted case $\delta = 0$, we use the inequality (6.21) and (6.43) in Corollary 6.5.2 that gives the inequality

$$\|\mathcal{L}f\|_{L^2(\mathbb{G})} \geq \frac{N(N-4)}{4} \left\|\frac{f}{|x'|^2}\right\|_{L^2(\mathbb{G})}, \quad N \geq 5, \tag{6.87}$$

for $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$. Since it is known from Corollary 6.5.2 that the constant $\frac{N(N-4)}{4}$ is sharp in (6.87), using the same argument as for the constants in (6.79), we obtain the sharpness of the constants in (6.80). $\square$

6.10 Horizontal anisotropic Hardy and Rellich inequalities

In this section we discuss the anisotropic versions of horizontal Hardy and Rellich inequalities. These inequalities appear in the analysis of anisotropic p-sub-Laplacians. The presentation of this section follows [RSS18a]. To put the notions in perspective, we start by recalling the Euclidean counterparts of the appearing objects.

The anisotropic Laplacian (on $\mathbb{R}^N$) is defined by

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right), \tag{6.88}$$

for $p_i > 1$, with $i = 1, \ldots, N$. Note that choosing $p_i = 2$ or $p_i = p$ for all i in (6.88) we get the Laplacian and the pseudo-p-Laplacian, respectively.

A subelliptic analogue of the operator in (6.88) is the anisotropic p-sub-Laplacian on stratified groups which is the operator of the form

$$\mathcal{L}_p f := \sum_{i=1}^{N} X_i\left(|X_i f|^{p_i-2} X_i f\right), \quad 1 < p_i < \infty,$$

where X_i, $i = 1, \ldots, N$, are the generators of the first stratum of a stratified Lie group.

Following the classical scheme for the analysis of such operators, first, we present the horizontal versions of the so-called Picone type identities. As a consequence, Hardy and Rellich type inequalities for anisotropic sub-Laplacians can be obtained.

6.10.1 Horizontal Picone identities

First, we discuss the horizontal Picone type identity on a stratified group $\mathbb{G}$.

Lemma 6.10.1 (Horizontal Picone identity)**.** *Let $\Omega \subset \mathbb{G}$ be an open set of a stratified group $\mathbb{G}$, and let N be the dimension of the first stratum of $\mathbb{G}$. Let u, v be differentiable a.e. in Ω, $v > 0$ a.e. in Ω and $u \geq 0$. Denote*

$$R(u,v) := \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i v|^{p_i-2} X_i v, \tag{6.89}$$

and

$$L(u,v) := \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u + \sum_{i=1}^{N}(p_i-1)\frac{u^{p_i}}{v^{p_i}}|X_i v|^{p_i}, \tag{6.90}$$

where $p_i > 1$, $i = 1, \ldots, N$. Then we have

$$L(u, v) = R(u, v) \geq 0. \tag{6.91}$$

In addition, we have $L(u, v) = 0$ a.e. in Ω if and only if $u = cv$ a.e. in Ω with a positive constant c.

Remark 6.10.2.

1. The Euclidean case of Lemma 6.10.1 was obtained by Feng and Cui [FC17].
2. Our proof of Lemma 6.10.1 follows [RSS18a] and is based on the method of Allegretto and Huang [AH98] for the (Euclidean) p-Laplacian, see also [NZW01].

Proof of Lemma 6.10.1. A direct computation gives

$$\begin{aligned}
R(u, v) &= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i \left(\frac{u^{p_i}}{v^{p_i - 1}} \right) |X_i v|^{p_i - 2} X_i v \\
&= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} \frac{p_i u^{p_i - 1} X_i u v^{p_i - 1} - u^{p_i}(p_i - 1) v^{p_i - 2} X_i v}{(v^{p_i - 1})^2} |X_i v|^{p_i - 2} X_i v \\
&= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} |X_i v|^{p_i - 2} X_i v X_i u + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i} \\
&= L(u, v).
\end{aligned}$$

This proves the equality in (6.91). Now we rewrite $L(u, v)$ to see that $L(u, v) \geq 0$, that is, we write

$$\begin{aligned}
L(u, v) = &\sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} |X_i v|^{p_i - 1} |X_i u| + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i} \\
&+ \sum_{i=1}^{N} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} |X_i v|^{p_i - 2} \left(|X_i v| |X_i u| - X_i v X_i u \right) = S_1 + S_2,
\end{aligned}$$

where we denote

$$\begin{aligned}
S_1 := &\sum_{i=1}^{N} p_i \left[\frac{1}{p_i} |X_i u|^{p_i} + \frac{p_i - 1}{p_i} \left(\left(\frac{u}{v} |X_i v| \right)^{p_i - 1} \right)^{\frac{p_i}{p_i - 1}} \right] \\
&- \sum_{i=1}^{N} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} |X_i v|^{p_i - 1} |X_i u|,
\end{aligned}$$

and

$$S_2 := \sum_{i=1}^{N} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} |X_i v|^{p_i - 2} \left(|X_i v| |X_i u| - X_i v X_i u \right).$$

We can see that $S_2 \geq 0$ due to $|X_i v||X_i u| \geq X_i v X_i u$. To check that we also have $S_1 \geq 0$, we will use Young's inequality for $a \geq 0$ and $b \geq 0$:

$$ab \leq \frac{a^{p_i}}{p_i} + \frac{b^{q_i}}{q_i}, \tag{6.92}$$

for $p_i > 1$, $q_i > 1$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for all $i = 1, \ldots, N$. The equality in (6.92) holds if and only if $a^{p_i} = b^{q_i}$, that is, if $a = b^{\frac{1}{p_i - 1}}$.

Let us now take $a = |X_i u|$ and $b = \left(\frac{u}{v}|X_i v|\right)^{p_i - 1}$ and apply (6.92) to get

$$p_i |X_i u| \left(\frac{u}{v}|X_i v|\right)^{p_i - 1} \leq p_i \left[\frac{1}{p_i}|X_i u|^{p_i} + \frac{p_i - 1}{p_i}\left(\left(\frac{u}{v}|X_i v|\right)^{p_i - 1}\right)^{\frac{p_i}{p_i - 1}}\right]. \tag{6.93}$$

From this we see that $S_1 \geq 0$ which proves that $L(u, v) = S_1 + S_2 \geq 0$.

It is easy to see that $u = cv$ implies $R(u, v) = 0$. Now let us prove that $L(u, v) = 0$ implies $u = cv$. Due to $u(x) \geq 0$ and since $L(u, v)(x_0) = 0$, $x_0 \in \Omega$, we can consider two cases $u(x_0) > 0$ and $u(x_0) = 0$.

(a) For the case $u(x_0) > 0$ we conclude from $L(u, v)(x_0) = 0$ that $S_1 = 0$ and $S_2 = 0$. Then $S_1 = 0$ implies

$$|X_i u| = \frac{u}{v}|X_i v|, \quad i = 1, \ldots, N, \tag{6.94}$$

and $S_2 = 0$ implies

$$|X_i v||X_i u| - X_i v X_i u = 0, \quad i = 1, \ldots, N. \tag{6.95}$$

The combination of (6.94) and (6.95) gives

$$\frac{X_i u}{X_i v} = \frac{u}{v} = c, \quad \text{with} \quad c \neq 0, \quad i = 1, \ldots, N. \tag{6.96}$$

(b) Let us denote

$$\Omega^* := \{x \in \Omega : \, u(x) = 0\}.$$

If $\Omega^* \neq \Omega$, then suppose that $x_0 \in \partial \Omega^*$. Then there exists a sequence $x_k \notin \Omega^*$ such that $x_k \to x_0$. In particular, $u(x_k) \neq 0$, and hence by Case (a) we have $u(x_k) = cv(x_k)$. Passing to the limit we get $u(x_0) = cv(x_0)$. Since $u(x_0) = 0$ and $v(x_0) \neq 0$, we get that $c = 0$. But then by Case (a) again, since $u = cv$ and $u \neq 0$ in $\Omega \backslash \Omega^*$, it is impossible to have $c = 0$. This contradiction implies that $\Omega^* = \Omega$.

This completes the proof of Lemma 6.10.1. □

The following consequence of Lemma 6.10.1 will be instrumental in the proof of the horizontal anisotropic Hardy inequality in Theorem 6.10.5.

Lemma 6.10.3. *Let $\Omega \subset \mathbb{G}$ be an open set of a stratified group $\mathbb{G}$, and let N be the dimension of the first stratum of $\mathbb{G}$. Let constants $K_i > 0$ and functions $H_i(x)$ with $i = 1, \ldots, N$, be such that for an a.e. differentiable function v, such that $v > 0$ a.e. in Ω, we have*

$$- X_i(|X_i v|^{p_i-2} X_i v) \geq K_i H_i(x) v^{p_i-1}, \quad i = 1, \ldots, N. \tag{6.97}$$

Then, for all non-negative functions $u \in C^1(\Omega)$ we have

$$\sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx \geq \sum_{i=1}^{N} K_i \int_{\Omega} H_i(x) u^{p_i} dx. \tag{6.98}$$

Proof of Lemma 6.10.3. In view of (6.91) and (6.97) we have

$$\begin{aligned}
0 \leq \int_{\Omega} L(u,v) dx &= \int_{\Omega} R(u,v) dx \\
&= \sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx - \sum_{i=1}^{N} \int_{\Omega} X_i \left(\frac{u^{p_i}}{v^{p_i-1}} \right) |X_i v|^{p_i-2} X_i v dx \\
&= \sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx + \sum_{i=1}^{N} \int_{\Omega} \frac{u^{p_i}}{v^{p_i-1}} X_i \left(|X_i v|^{p_i-2} X_i v \right) dx \\
&\leq \sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx - \sum_{i=1}^{N} K_i \int_{\Omega} H_i(x) u^{p_i} dx,
\end{aligned}$$

proving the statement. □

We now present the second-order horizontal Picone type identity that will be instrumental in the proof of Theorem 6.10.6 giving the Rellich type inequality for the anisotropic sub-Laplacians.

Lemma 6.10.4 (Second-order horizontal Picone identity)**.** *Let $\Omega \subset \mathbb{G}$ be an open set of a stratified group $\mathbb{G}$, and let N be the dimension of the first stratum of $\mathbb{G}$. Let u, v be twice differentiable a.e. in Ω and satisfying the following conditions: $u \geq 0$, $v > 0$, $X_i^2 v < 0$ a.e. in Ω for $p_i > 1$, $i = 1, \ldots, N$. Then we have*

$$L_1(u,v) = R_1(u,v) \geq 0, \tag{6.99}$$

where

$$R_1(u,v) := \sum_{i=1}^{N} |X_i^2 u|^{p_i} - \sum_{i=1}^{N} X_i^2 \left(\frac{u^{p_i}}{v^{p_i-1}} \right) |X_i^2 v|^{p_i-2} X_i^2 v,$$

and

$$L_1(u,v) := \sum_{i=1}^{N} |X_i^2 u|^{p_i} - \sum_{i=1}^{N} p_i \left(\frac{u}{v} \right)^{p_i-1} X_i^2 u X_i^2 v |X_i^2 v|^{p_i-2}$$

$$+\sum_{i=1}^{N}(p_i-1)\left(\frac{u}{v}\right)^{p_i}|X_i^2v|^{p_i}$$
$$-\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left(X_iu-\frac{u}{v}X_iv\right)^2.$$

Proof of Lemma 6.10.4. A direct computation gives

$$\begin{aligned}
X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right) &= X_i\left(p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_iu-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_iv\right)\\
&= p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-2}}\left(\frac{(X_iu)v-u(X_iv)}{v^2}\right)X_iu+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u\\
&\quad -p_i(p_i-1)\frac{u^{p_i-1}}{v^{p_i-1}}\left(\frac{(X_iu)v-u(X_iv)}{v^2}\right)X_iv-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v\\
&= p_i(p_i-1)\left(\frac{u^{p_i-2}}{v^{p_i-1}}|X_iu|^2-2\frac{u^{p_i-1}}{v^{p_i}}X_ivX_iu+\frac{u^{p_i}}{v^{p_i+1}}|X_iv|^2\right)\\
&\quad +p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v\\
&= p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}\left(X_iu-\frac{u}{v}X_iv\right)^2+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v,
\end{aligned}$$

which yields (6.99). By Young's inequality (6.92) we have

$$\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2uX_i^2v|X_i^2v|^{p_i-2}\le\frac{|X_i^2u|^{p_i}}{p_i}+\frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i},\quad i=1,\dots,N,$$

where $p_i>1$, $q_i>1$, $\frac{1}{p_i}+\frac{1}{q_i}=1$. Since $X_i^2v<0$ we arrive at

$$\begin{aligned}
L_1(u,v) &\ge \sum_{i=1}^{N}|X_i^2u|^{p_i}+\sum_{i=1}^{N}(p_i-1)\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}-\sum_{i=1}^{N}p_i\left(\frac{|X_i^2u|^{p_i}}{p_i}+\frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}\right)\\
&\quad -\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left|X_iu-\frac{u}{v}X_iv\right|^2\\
&= \sum_{i=1}^{N}\left(p_i-1-\frac{p_i}{q_i}\right)\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}\\
&\quad -\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left|X_iu-\frac{u}{v}X_iv\right|^2\ge 0.
\end{aligned}$$

This completes the proof of Lemma 6.10.4. $\square$

6.10.2 Horizontal anisotropic Hardy type inequality

As a consequence of the horizontal Picone type identity in Lemma 6.10.1 we can obtain the Hardy type inequality for the anisotropic sub-Laplacian on stratified Lie groups. We recall that for $x \in \mathbb{G}$ we write customarily

$$x = (x', x''),$$

with coordinates x' corresponding to the first stratum of $\mathbb{G}$.

Theorem 6.10.5 (Horizontal anisotropic Hardy type inequality)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of its first stratum, and let $\Omega \subset \mathbb{G}\backslash\{x' = 0\}$ be an open set. Let $1 < p_i < N$ for all $i = 1, \ldots, N$. Then we have*

$$\sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx \geq \sum_{i=1}^{N} \left(\frac{p_i - 1}{p_i} \right)^{p_i} \int_{\Omega} \frac{|u|^{p_i}}{|x_i'|^{p_i}} dx, \tag{6.100}$$

for all $u \in C^1(\Omega)$.

Proof of Theorem 6.10.5*.* The proof is based on the application of Lemma 6.10.3. For this, we introduce the auxiliary function

$$v := \prod_{j=1}^{N} |x_j'|^{\alpha_j} = |x_i'|^{\alpha_i} V_i, \tag{6.101}$$

where $V_i = \prod_{j=1, j\neq i}^{N} |x_j'|^{\alpha_j}$ and $\alpha_j = \frac{p_j - 1}{p_j}$. Then we have

$$\begin{aligned} X_i v &= \alpha_i V_i |x_i'|^{\alpha_i - 2} x_i', \\ |X_i v|^{p_i - 2} &= \alpha_i^{p_i - 2} V_i^{p_i - 2} |x_i'|^{\alpha_i p_i - 2\alpha_i - p_i + 2}, \\ |X_i v|^{p_i - 2} X_i v &= \alpha_i^{p_i - 1} V_i^{p_i - 1} |x_i'|^{\alpha_i p_i - \alpha_i - p_i} x_i'. \end{aligned}$$

Consequently, we also have

$$- X_i(|X_i v|^{p_i - 2} X_i v) = \left(\frac{p_i - 1}{p_i} \right)^{p_i} \frac{v^{p_i - 1}}{|x_i'|^{p_i}}. \tag{6.102}$$

To complete the proof of Theorem 6.10.5, we choose $K_i = \left(\frac{p_i - 1}{p_i} \right)^{p_i}$ and $H_i(x) = \frac{1}{|x_i'|^{p_i}}$, and use Lemma 6.10.3. □

6.10.3 Horizontal anisotropic Rellich type inequality

Now we present the horizontal anisotropic Rellich type inequality on stratified Lie groups.

Theorem 6.10.6 (Horizontal anisotropic Rellich type inequality)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of its first stratum, and let $\Omega \subset \mathbb{G}\backslash\{x'=0\}$ be an open set. Then for a function $u \geq 0$, $u \in C^2(\Omega)$, and $2 < \alpha_i < N-2$ we have the following inequality*

$$\sum_{i=1}^{N}\int_{\Omega}|X_i^2 u|^{p_i}dx \geq \sum_{i=1}^{N}C_i(\alpha_i,p_i)\int_{\Omega}\frac{|u|^{p_i}}{|x_i'|^{2p_i}}dx, \tag{6.103}$$

where $1 < p_i < N$ for $i=1,\ldots,N$, and

$$C_i(\alpha_i,p_i) = (\alpha_i(\alpha_i-1))^{p_i-1}(\alpha_i p_i - 2p_i - \alpha_i + 2)(\alpha_i p_i - 2p_i - \alpha_i + 1).$$

Proof of Theorem 6.10.6. We introduce the auxiliary function

$$v := \prod_{j=1}^{N}|x_j'|^{\alpha_j} = |x_i'|^{\alpha_i}V_i,$$

we choose α_j later, and where $V_i := \prod_{j=1,j\neq i}^{N}|x_j'|^{\alpha_j}$. Then we have

$$\begin{aligned}
X_i^2 v &= X_i(\alpha_i V_i |x_i'|^{\alpha_i-2}x_i') = \alpha_i(\alpha_i-1)V_i|x_i'|^{\alpha_i-2},\\
|X_i^2 v|^{p_i-2} &= (\alpha_i(\alpha_i-1))^{p_i-2}V_i^{p_i-2}|x_i'|^{\alpha_i p_i-2p_i-2\alpha_i+4},\\
|X_i^2 v|^{p_i-2}X_i^2 v &= (\alpha_i(\alpha_i-1))^{p_i-1}V_i^{p_i-1}|x_i'|^{\alpha_i p_i-2p_i-\alpha_i+2}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&X_i^2(|X_i^2 v|^{p_i-2}X_i^2 v)\\
&\quad= (\alpha_i(\alpha_i-1))^{p_i-1}V_i^{p_i-1}X_i^2(|x_i'|^{\alpha_i p_i-2p_i-\alpha_i+2})\\
&\quad= (\alpha_i(\alpha_i-1))^{p_i-1}(\alpha_i p_i-2p_i-\alpha_i+2)V_i^{p_i-1}X_i\left(|x_i'|^{\alpha_i p_i-2p_i-\alpha_i}x_i'\right)\\
&\quad= (\alpha_i(\alpha_i-1))^{p_i-1}(\alpha_i p_i-2p_i-\alpha_i+2)(\alpha_i p_i-2p_i-\alpha_i+1)\\
&\qquad\times V_i^{p_i-1}|x_i'|^{\alpha_i(p_i-1)-2p_i}.
\end{aligned}$$

Thus, for a twice differentiable function $v > 0$ a.e. in Ω with $X_i^2 v < 0$, we have

$$X_i^2(|X_i^2|^{p_i-2}X_i^2 v) = C_i(\alpha_i,p_i)\frac{v^{p_i-1}}{|x_i'|^{2p_i}} \tag{6.104}$$

a.e. in Ω. Using (6.104) we compute

$$\begin{aligned}
0 &\leq \int_{\Omega}L_1(u,v)dx = \int_{\Omega}R_1(u,v)dx\\
&= \sum_{i=1}^{N}\int_{\Omega}|X_i^2 u|^{p_i}dx - \sum_{i=1}^{N}\int_{\Omega}X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i^2 v|^{p_i-2}X_i^2 v dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_{\Omega} |X_i^2 u|^{p_i} dx - \sum_{i=1}^{N} \int_{\Omega} \frac{u^{p_i}}{v^{p_i-1}} X_i^2 \left(|X_i^2 v|^{p_i-2} X_i^2 v \right) dx \\
&= \sum_{i=1}^{N} \int_{\Omega} |X_i^2 u|^{p_i} dx - \sum_{i=1}^{N} C_i(\alpha_i, p_i) \int_{\Omega} \frac{|u|^{p_i}}{|x_i'|^{2p_i}} dx.
\end{aligned}
$$

The proof of Theorem 6.10.6 is complete. □

6.11 Horizontal Hardy inequalities with multiple singularities

In this section we obtain the analogue of the Hardy inequality with multiple singularities on stratified Lie groups. The singularities will be represented by a family of points $\{a_k\}_{k=1}^m \in \mathbb{G}$. We will be using the usual notation $a_k = (a_k', a_k'')$, with a_k' corresponding to the first stratum of $\mathbb{G}$. In turn, we can also write $a_k' = (a_{k1}', \ldots, a_{kN}')$. From (1.17) it follows that

$$(xa_k^{-1})' = x' - a_k'.$$

We denote by $(xa_k^{-1})_j' = x_j' - a_{kj}'$ the jth component of xa_k^{-1}.

Theorem 6.11.1 (Horizontal Hardy inequality with multiple singularities)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of its first stratum, and let $\Omega \subset \mathbb{G}$ be an open set. Let $N \geq 3$, $x = (x', x'') \in \mathbb{G}$ with $x' = (x_1', \ldots, x_N')$ being in the first stratum of $\mathbb{G}$, and let $a_k \in \mathbb{G}, k = 1, \ldots, m$, be the singularities. Then we have*

$$
\int_{\Omega} |\nabla_H u|^2 dx \geq \left(\frac{N-2}{2} \right)^2 \int_{\Omega} \frac{\sum_{j=1}^{N} \left| \sum_{k=1}^{m} \frac{(xa_k^{-1})_j'}{|(xa_k^{-1})'|^N} \right|^2}{\left(\sum_{k=1}^{m} \frac{1}{|(xa_k^{-1})'|^{N-2}} \right)^2} |u|^2 dx, \tag{6.105}
$$

for all $u \in C_0^\infty(\Omega)$.

Remark 6.11.2. The Euclidean case of the inequality (6.105) was obtained by Kapitanski and Laptev [KL16]. Theorem 6.11.1 was obtained in [RSS18a] and our presentation here follows the arguments there.

Proof of Theorem 6.11.1*.* Let us fix a vector-valued function

$$\mathcal{A}(x) = (\mathcal{A}_1(x), \ldots, \mathcal{A}_N(x))$$

to be specified later. Also let λ be a real parameter. We start with the inequality

$$
\begin{aligned}
0 &\leq \int_{\Omega} \sum_{j=1}^{N} (|X_j u - \lambda \mathcal{A}_j u|^2) dx \\
&= \int_{\Omega} \left(|\nabla_H u|^2 - 2\lambda \mathrm{Re} \sum_{j=1}^{N} \overline{\mathcal{A}_j u} X_j u + \lambda^2 \sum_{j=1}^{N} |\mathcal{A}_j|^2 |u|^2 \right) dx.
\end{aligned}
$$

By using the integration by parts we get

$$-\int_{\Omega}\left(\lambda^2\sum_{j=1}^{N}|\mathcal{A}_j|^2+\lambda\mathrm{div}_H\mathcal{A}\right)|u|^2dx\leq\int_{\Omega}|\nabla_H u|^2dx. \tag{6.106}$$

We differentiate the integral on the left-hand side with respect to λ to optimize it, yielding

$$2\lambda|\mathcal{A}|^2+\mathrm{div}_H\mathcal{A}=0,$$

for all $x\in\Omega$. This is the condition that we impose on $\mathcal{A}(x)$, that is, the quotient $\frac{\mathrm{div}_H\mathcal{A}(x)}{|\mathcal{A}(x)|^2}$ must be constant. For $\lambda=\frac{1}{2}$ we get

$$\mathrm{div}_H\mathcal{A}(x)=-|\mathcal{A}(x)|^2. \tag{6.107}$$

Then putting (6.107) in (6.106) we have the following Hardy inequality

$$\frac{1}{4}\int_{\Omega}\sum_{j=1}^{N}|\mathcal{A}_j(x)|^2|u|^2dx\leq\int_{\Omega}|\nabla_H u|^2dx. \tag{6.108}$$

Now if we assume that $\mathcal{A}=\nabla_H\phi$ for some function ϕ, then (6.107) becomes

$$\mathcal{L}\phi+|\nabla_H\phi|^2=0.$$

It follows that the function

$$w=e^{\phi}\geq 0$$

is harmonic with respect to the sub-Laplacian $\mathcal{L}$. Thus, w is a constant >0 or it has a singularity. Let us now take

$$w(x):=\sum_{k=1}^{m}\frac{1}{|(xa_k^{-1})'|^{N-2}},$$

and then also

$$\phi(x):=\ln(w(x)).$$

Therefore

$$\begin{aligned}\mathcal{A}(x)=\nabla_H(\ln w)&=\frac{1}{w}\nabla_H\left(\sum_{k=1}^{m}|(xa_k^{-1})'|^{2-N}\right)\\&=\frac{1}{w}\sum_{k=1}^{m}\nabla_H\left(\sum_{j=1}^{N}((xa_k^{-1})'_j)^2\right)^{\frac{2-N}{2}}\\&=-\frac{N-2}{w}\left(\sum_{k=1}^{m}\frac{(xa_k^{-1})'}{|(xa_k^{-1})'|^N}\right),\end{aligned}$$

and

$$|\mathcal{A}(x)|^2 = \sum_{j=1}^{N} |\mathcal{A}_j(x)|^2 = \left(\frac{N-2}{w}\right)^2 \sum_{j=1}^{N} \left| \sum_{k=1}^{m} \frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N} \right|^2 .$$

The inequality (6.105) now follows from (6.108), completing the proof of Theorem 6.11.1. □

We then also obtain the corresponding uncertainty principle.

Corollary 6.11.3 (Uncertainty principle with multiple singularities)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of its first stratum, and let $\Omega \subset \mathbb{G}$ be an open set. Let $N \geq 3$, $x = (x', x'') \in \mathbb{G}$ with $x' = (x'_1, \ldots, x'_N)$ corresponding to the first stratum of $\mathbb{G}$. Let $a_k \in \mathbb{G}$, $k = 1, \ldots, m$, be the singularities, and let $1 < p_i < N$ for $i = 1, \ldots, N$. Then we have*

$$\frac{N-2}{2} \int_\Omega |u|^2 dx \leq \left(\int_\Omega |\nabla_H u|^2 dx \right)^{1/2} \left(\int_\Omega \frac{\left(\sum_{k=1}^m \frac{1}{|(xa_k^{-1})'|^{N-2}} \right)^2}{\sum_{j=1}^N \left| \sum_{k=1}^m \frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N} \right|^2} |u|^2 dx \right)^{1/2} ,$$

for all $u \in C_0^\infty(\Omega)$.

Proof of Corollary 6.11.3. By (6.105) and the Cauchy–Schwarz inequality we get

$$\begin{aligned}
&\int_\Omega |\nabla_H u|^2 dx \int_\Omega \frac{\left(\sum_{k=1}^m \frac{1}{|(xa_k^{-1})'|^{N-2}} \right)^2}{\sum_{j=1}^N \left| \sum_{k=1}^m \frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N} \right|^2} |u|^2 dx \\
&\geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{\sum_{j=1}^N \left| \sum_{k=1}^m \frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N} \right|^2}{\left(\sum_{k=1}^m \frac{1}{|(xa_k^{-1})'|^{N-2}} \right)^2} |u|^2 dx \\
&\quad \times \int_\Omega \frac{\left(\sum_{k=1}^m \frac{1}{|(xa_k^{-1})'|^{N-2}} \right)^2}{\sum_{j=1}^N \left| \sum_{k=1}^m \frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N} \right|^2} |u|^2 dx \\
&\geq \left(\frac{N-2}{2}\right)^2 \left(\int_\Omega |u|^2 dx \right)^2 .
\end{aligned}$$

The proof is complete. □

6.12 Horizontal many-particle Hardy inequality

In this section we discuss Hardy inequalities for $n \geq 1$ particles on stratified Lie groups. We denote by $\mathbb{G}^n$ the product

$$\mathbb{G}^n := \overbrace{\mathbb{G} \times \cdots \times \mathbb{G}}^{n}.$$

We consider the points $x = (x_1, \ldots, x_n) \in \mathbb{G}^n$, with $x_j \in \mathbb{G}$. The horizontal component of $x \in \mathbb{G}^n$ will be denoted by $x' = (x'_1, \ldots, x'_n)$, with $x'_i = (x'_{i1}, \ldots, x'_{iN})$ being the coordinates corresponding to the first stratum of $\mathbb{G}$ for $i = 1, \ldots, n$. The (horizontal) distance between particles $x_i, x_j \in \mathbb{G}$ can be defined by

$$r_{ij} := |(x_i x_j^{-1})'| = |x'_i - x'_j| = \sqrt{\sum_{k=1}^{N} (x'_{ik} - x'_{jk})^2}.$$

We will also use the notation

$$\nabla_{H_i} = (X_{i1}, \ldots, X_{iN})$$

for the horizontal gradient associated to the ith particle. We denote

$$\nabla_{H^n} := (\nabla_{H_1}, \ldots, \nabla_{H_n}), \qquad \text{and} \qquad \mathcal{L}_i := \sum_{k=1}^{N} X_{ik}^2,$$

the sub-Laplacian associated to the ith particle. We note that

$$\mathcal{L} = \sum_{i=1}^{N} \mathcal{L}_i.$$

We now recall a simple but crucial inequality on $\mathbb{R}^m$.

Lemma 6.12.1. *Let $m \geq 1$, and let*

$$\mathcal{A} = (\mathcal{A}_1(x), \ldots, \mathcal{A}_m(x))$$

be a mapping $\mathcal{A} : \mathbb{R}^m \to \mathbb{R}^m$ whose components and their first derivatives are uniformly bounded on $\mathbb{R}^m$. Then for every non-trivial $u \in C_0^1(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^m} \operatorname{div}\mathcal{A}|u|^2 dx\right)^2}{\int_{\mathbb{R}^m} |\mathcal{A}|^2 |u|^2 dx}. \tag{6.109}$$

Proof of Lemma 6.12.1. We have

$$\begin{aligned}
\left| \int_{\mathbb{R}^m} \operatorname{div}\mathcal{A}|u|^2 dx \right| &= 2 \left| \operatorname{Re} \int_{\mathbb{R}^m} \langle \mathcal{A}, \nabla u \rangle \overline{u} dx \right| \\
&\leq 2 \left(\int_{\mathbb{R}^m} |\mathcal{A}|^2 |u|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^m} |\nabla u|^2 dx \right)^{1/2},
\end{aligned}$$

using the Cauchy–Schwarz inequality in the last line. This implies (6.109). ☐

Theorem 6.12.2 (Horizontal many-particle Hardy inequality)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of its first stratum, and let $\Omega \subset \mathbb{G}^n$ be an open set. Let $N \geq 2$ and $n \geq 3$. Let $r_{ij} = |(x_i x_j^{-1})'| = |x_i' - x_j'|$. Then we have*

$$\int_\Omega |\nabla_{H^n} u|^2 dx \geq \frac{(N-2)^2}{n} \int_\Omega \sum_{1 \leq i < j \leq n} \frac{|u|^2}{r_{ij}^2} dx, \tag{6.110}$$

for all $u \in C^1(\Omega)$.

Remark 6.12.3. The Euclidean case of the inequality (6.110) was obtained by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, and J. Tidblom in [HOHOLT08]. The Euclidean case of the subsequent Theorem 6.12.4 was obtained by D. Lundholm [Lun15]. Theorem 6.12.2 and Theorem 6.12.4 were obtained in [RSS18a] and our presentation here follows the arguments there.

Proof of Theorem 6.12.2. Let us define a mapping $\mathcal{B}_1$ by the formula

$$\mathcal{B}_1(x_i', x_j') := \frac{(x_i x_j^{-1})'}{r_{ij}^2}, \quad 1 \leq i < j \leq n.$$

In the subsequent arguments we denote by $\operatorname{div}_{\mathbb{G}_i}$ the horizontal divergence on $\mathbb{G}_i$. Applying inequality (6.109) to the mapping $\mathcal{B}_1$ we have

$$\begin{aligned}
\int_\Omega |(\nabla_{H_i} - \nabla_{H_j})u|^2 dx &\geq \frac{1}{4} \frac{\left(\int_\Omega \left((\operatorname{div}_{H_i} - \operatorname{div}_{H_j})\mathcal{B}_1\right)|u|^2 dx\right)^2}{\int_\Omega |\mathcal{B}_1|^2 |u|^2 dx} \\
&= \frac{1}{4} \frac{\left(\int_\Omega \frac{2(N-2)}{|(x_i x_j^{-1})'|^2} |u|^2 dx\right)^2}{\int_\Omega \frac{|u|^2}{|(x_i x_j^{-1})'|^2} dx} \\
&= (N-2)^2 \int_\Omega \frac{|u|^2}{r_{ij}^2} dx.
\end{aligned} \tag{6.111}$$

Also, we define another mapping $\mathcal{B}_2$ by

$$\mathcal{B}_2(x) := \frac{\sum_{j=1}^n x_j'}{\left|\sum_{j=1}^n x_j'\right|^2}.$$

We can calculate

$$\begin{aligned}
\nabla_{H_i} \cdot \mathcal{B}_2 &= \sum_{k=1}^N X_{ik} \left(\frac{\sum_{j=1}^n x_{jk}'}{|\sum_{j=1}^n x_j'|^2} \right) \\
&= \frac{Nn|\sum_{j=1}^n x_j'|^2 - 2n\left((\sum_{j=1}^n x_{j1}')^2 + \cdots + (\sum_{j=1}^n x_{jN}')^2\right)}{|\sum_{j=1}^n x_j'|^4} = \frac{Nn - 2n}{|\sum_{j=1}^n x_j'|^2}.
\end{aligned}$$

Applying inequality (6.109) to the mapping $\mathcal{B}_2$ we obtain

$$\begin{aligned}\int_\Omega \left|\sum_{i=1}^n \nabla_{H_i} u\right|^2 dx &\geq \frac{1}{4}\frac{\left(\int_\Omega (\sum_{i=1}^n \operatorname{div}_{H_i}\mathcal{B}_2)|u|^2 dx\right)^2}{\int_\Omega |\mathcal{B}_2|^2|u|^2 dx}\\ &= \frac{1}{4}\frac{\left(\int_\Omega \sum_{i=1}^n \frac{Nn-2n}{|\sum_{j=1}^n x'_j|^2}|u|^2 dx\right)^2}{\int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x'_j\right|^2}dx}\\ &= \frac{(N-2)^2n^4}{4}\int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x'_j\right|^2}dx. \end{aligned} \tag{6.112}$$

Adding inequalities (6.111) and (6.112) and using the identity

$$n\sum_{i=1}^n |\nabla_{H_i}u|^2 = \sum_{1\leq i<j\leq n}\left|\nabla_{H_i}u - \nabla_{H_j}u\right|^2 + \left|\sum_{i=1}^n \nabla_{H_i}u\right|^2,$$

we arrive at

$$\sum_{i=1}^n \int_\Omega |\nabla_{H_i}u|^2 dx \geq \frac{(N-2)^2}{n}\int_\Omega \sum_{i<j}\frac{|u|^2}{r_{ij}^2}dx + \frac{(N-2)^2n^3}{4}\int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x'_j\right|^2}dx.$$

Because the last term on right-hand side is positive, we get

$$\sum_{i=1}^n \int_\Omega |\nabla_{H_i}u|^2 dx \geq \frac{(N-2)^2}{n}\int_\Omega \sum_{i<j}\frac{|u|^2}{r_{ij}^2}dx.$$

Also we have $\sum_{i=1}^n |\nabla_{H_i}u|^2 = |\nabla_{H^n}u|^2$. Putting everything together, the proof of Theorem 6.12.2 is complete. □

The following theorem deals with the total separation of $n \geq 2$ particles.

Theorem 6.12.4 (Total separation of many-particles)**.** *Let $\mathbb{G}$ be a stratified group with N being the dimension of its first stratum, and let $\Omega \subset \mathbb{G}^n$ be an open set. Let $\rho^2 := \sum_{i<j}|(x_i x_j^{-1})'|^2 = \sum_{i<j}|x'_i - x'_j|^2$ with $x'_i \neq x'_j$. Then we have*

$$\int_\Omega |\nabla_H u|^2 dx = n\left(\frac{(n-1)}{2}N - 1\right)^2\int_\Omega \frac{|u|^2}{\rho^2}dx + \int_\Omega |\nabla_H \rho^{-2\alpha}u|^2\rho^{4\alpha}dx \tag{6.113}$$

for all $u \in C_0^\infty(\Omega)$ with $\alpha = \frac{2-(n-1)N}{4}$.

The proof of Theorem 6.12.4 will rely on the following identity.

Proposition 6.12.5. *Let $\mathbb{G}$ be a stratified group with N being the dimension of its first stratum, and let $\Omega \subset \mathbb{G}^n$ be an open set. Let $f : \Omega \to (0,\infty)$ be twice differentiable. Then for any function $u \in C_0^\infty(\Omega)$ and $\alpha \in \mathbb{R}$, we have*

$$\int_\Omega |\nabla_H u|^2 dx = \int_\Omega \left(\alpha(1-\alpha)\frac{|\nabla_H f|^2}{f^2} - \alpha\frac{\mathcal{L}f}{f}\right)|u|^2 dx + \int_\Omega |\nabla_H v|^2 f^{2\alpha} dx,$$

where $v := f^{-\alpha}u$.

Proof of Proposition 6.12.5. Let us first observe that for $u = f^\alpha v$, we have

$$\nabla_H u = \alpha f^{\alpha-1}(\nabla_H f)v + f^\alpha \nabla_H v.$$

By squaring the above expression we get

$$\begin{aligned}|\nabla_H u|^2 &= \alpha^2 f^{2(\alpha-1)}|\nabla_H f|^2|v|^2 + \mathrm{Re}(2\alpha v f^{2\alpha-1}(\nabla_H f)\cdot(\nabla_H v)) + f^{2\alpha}|\nabla_H v|^2\\ &= \alpha^2 f^{2(\alpha-1)}|\nabla_H f|^2|v|^2 + \alpha f^{2\alpha-1}(\nabla_H f)\cdot\nabla_H|v|^2 + f^{2\alpha}|\nabla_H v|^2.\end{aligned}$$

By integrating this expression over Ω, we obtain

$$\begin{aligned}\int_\Omega |\nabla_H u|^2 dx = &\int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_H f|^2|v|^2 dx\\ &+ \int_\Omega \mathrm{Re}(\alpha f^{2\alpha-1}(\nabla_H f)\cdot\nabla_H|v|^2)dx + \int_\Omega f^{2\alpha}|\nabla_H v|^2 dx\\ = &\int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_H f|^2|v|^2 dx\\ &- \alpha\int_\Omega \nabla_H\cdot(f^{2\alpha-1}\nabla_H f)|v|^2 dx + \int_\Omega f^{2\alpha}|\nabla_H v|^2 dx.\end{aligned}$$

We have used integration by parts to the middle term on the right-hand side. Since

$$\nabla_H\cdot(f^{2\alpha-1}\nabla_H f) = (2\alpha-1)f^{2\alpha-2}|\nabla_H f|^2 + f^{2\alpha-1}\mathcal{L}f,$$

we get

$$\begin{aligned}\int_\Omega |\nabla_H u|^2 dx = &\int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_H f|^2|v|^2 dx - \int_\Omega \alpha f^{2\alpha-1}\mathcal{L}f|v|^2 dx\\ &- \int_\Omega \alpha(2\alpha-1)f^{2\alpha-2}|\nabla_H f|^2|v|^2 dx + \int_\Omega f^{2\alpha}|\nabla_H v|^2 dx.\end{aligned}$$

Putting back $v = f^{-\alpha}u$ and collecting the terms we arrive at the equality of Proposition 6.12.5. □

Proof of Theorem 6.12.4. With $\nabla_{H_k} = (X_{k1},\ldots,X_{kN})$, using the definition of ρ we have

$$\nabla_{H_k}\rho^2 = (X_{k1}\rho^2,\ldots,X_{kN}\rho^2) = 2\sum_{k\neq j}^{n}(x_k x_j^{-1})'.$$

Hence

$$\mathcal{L}\rho^2 = 2\sum_{k=1}^{n}\sum_{k\neq j}^{n} \nabla_{H_k}\cdot (x_k x_j^{-1})' = 2n(n-1)N, \tag{6.114}$$

$$\begin{aligned} |\nabla_H \rho^2|^2 &= 8 \sum_{1\le i<j\le n} |(x_k x_j^{-1})'|^2 \\ &\quad + 8\sum_{k=1}^{n}\sum_{1\le i<j\le n} (x_k x_i^{-1})'\cdot (x_k x_j^{-1})' = 4n\rho^2, \end{aligned} \tag{6.115}$$

where in the last step we used the identity

$$\sum_{k=1}^{n}\sum_{1\le i<j\le n} (x_k x_i^{-1})'\cdot (x_k x_j^{-1})' = \frac{n-2}{2}\sum_{1\le i<j\le n} |(x_i x_j^{-1})'|^2.$$

By putting (6.114) and (6.115) in the identity of Proposition 6.12.5 with $f=\rho^2$, we obtain

$$\int_\Omega |\nabla_H u|^2 dx = 4n\alpha\left(\frac{2-(n-1)N}{2}-\alpha\right)\int_\Omega \frac{|u|^2}{\rho^2}dx + \int_\Omega |\nabla_H \rho^{-2\alpha}u|^2\rho^{4\alpha}dx.$$

To optimize we differentiate the integral

$$4n\alpha\left(\frac{2-(n-1)N}{2}-\alpha\right)\int_\Omega \frac{|u|^2}{\rho^2}dx$$

with respect to α, then we have

$$\frac{2-(n-1)N}{2}-2\alpha = 0 \quad \text{and} \quad \alpha = \frac{2-(n-1)N}{4},$$

which completes the proof of Theorem 6.12.4. □

6.13 Hardy inequality with exponential weights

In this section, we discuss a horizontal Hardy inequality with exponential weights. In the Euclidean case such a type of inequalities is sometimes called two parabolic type Hardy inequalities, see Zhang [Zha17]. The following statement was obtained in [RSS18a].

Theorem 6.13.1 (Hardy inequality with exponential horizontal weights)**.** *Let $\mathbb{G}$ be a stratified group with $N\ge 3$ being the dimension of its first stratum, and let $\Omega\subset\mathbb{G}$ be an open set. Let $x_0\in\Omega$. Then we have*

$$\int_\Omega e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}\left(\frac{(N-2)^2}{4|x'|^2}-\frac{N}{4\alpha}+\frac{|(xx_0^{-1})'|^2}{16\lambda^2}\right)|u|^2dx \le \int_\Omega e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|\nabla_H u|^2dx$$

for all $u\in C^1(\Omega)$ and for all $\lambda>0$.

Proof of Theorem 6.13.1. We will use the horizontal Hardy inequality

$$\frac{(N-2)^2}{4}\int_\Omega \frac{|v|^2}{|x'|^2}dx \leq \int_\Omega |\nabla_H v|^2 dx, \tag{6.116}$$

see (6.6), valid for all $v \in C^1(\Omega)$, with the choice of $v = e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}u$. We note that

$$\nabla_H v = e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}\nabla_H u - \frac{(xx_0^{-1})'}{4\lambda}e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}u,$$

for all $v \in C^1(\Omega)$. Then by inequality (6.116) we have

$$\begin{aligned}
&\frac{(N-2)^2}{4}\int_\Omega e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}\frac{|u|^2}{|x'|^2}dx \\
&\leq \int_\Omega e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|\nabla_H u|^2 + \frac{|(xx_0^{-1})'|^2}{16\lambda^2}e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|u|^2 dx \\
&\quad - \frac{1}{2\lambda}\mathrm{Re}\int_\Omega (xx_0^{-1})' \cdot (\nabla_H u)ue^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}dx.
\end{aligned} \tag{6.117}$$

Integration by parts in the last term of the right-hand side of this inequality yields

$$\begin{aligned}
&\mathrm{Re}\int_\Omega (xx_0^{-1})' \cdot (\nabla_H u)ue^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}dx \\
&\quad = -\frac{1}{2}\int_\Omega \left(N - \frac{|(xx_0^{-1})'|^2}{2\lambda}\right)e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|u|^2 dx.
\end{aligned}$$

By using this in (6.117) and rearranging the terms, we complete the proof of Theorem 6.13.1. □