

Chapter 5



Integral Hardy Inequalities on Homogeneous Groups

In this chapter we discuss the integral form of Hardy inequalities where instead of estimating a function by its gradient, we estimate the integral by the function itself. This stems from the original version of Hardy's inequality [Har20]:

$$\int_b^\infty \left(\frac{\int_b^x f(t) dt}{x} \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_b^\infty f(x)^p dx, \tag{5.1}$$

where $p > 1$, $b > 0$, and $f \geq 0$ is a non-negative function. We analyse the weighted versions of such inequalities in the setting of general homogeneous groups. Most of the results of this chapter have been obtained in [RY18a] and here we follow the presentation of this paper.

5.1 Two-weight integral Hardy inequalities

Here we discuss the weighted Hardy inequalities in the integral form extending that in (5.1). It turns out that one can actually derive the necessary and sufficient conditions on weights for these inequalities to hold.

Theorem 5.1.1 (Integral Hardy inequalities for $p \leq q$). *Let \mathbb{G} be a homogeneous group of homogeneous dimension Q and let $1 < p \leq q < \infty$. Let $\phi_1 > 0$, $\phi_2 > 0$ be positive functions on \mathbb{G} . Then we have the following properties:*

(1) *The inequality*

$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(z) dz \right)^q \phi_1(x) dx \right)^{1/q} \leq C_1 \left(\int_{\mathbb{G}} (f(x))^p \psi_1(x) dx \right)^{1/p} \tag{5.2}$$

holds for all $f \geq 0$ a.e. on \mathbb{G} if and only if

$$A_1 := \sup_{R>0} \left(\int_{\{|x|\geq R\}} \phi_1(x)dx \right)^{1/q} \left(\int_{\{|x|\leq R\}} (\psi_1(x))^{-(p'-1)} dx \right)^{1/p'} < \infty. \tag{5.3}$$

(2) *The inequality*

$$\left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} f(z)dz \right)^q \phi_2(x)dx \right)^{1/q} \leq C_2 \left(\int_{\mathbb{G}} (f(x))^p \psi_2(x)dx \right)^{1/p} \tag{5.4}$$

holds for all $f \geq 0$ a.e. on \mathbb{G} if and only if

$$A_2 := \sup_{R>0} \left(\int_{\{|x|\leq R\}} \phi_2(x)dx \right)^{1/q} \left(\int_{\{|x|\geq R\}} (\psi_2(x))^{-(p'-1)} dx \right)^{1/p'} < \infty. \tag{5.5}$$

(3) *If $\{C_i\}_{i=1}^2$ are the smallest constants for which (5.2) and (5.4) hold, then*

$$A_i \leq C_i \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_i, \quad i = 1, 2. \tag{5.6}$$

Before we prove this theorem let us give a few comments.

Remark 5.1.2.

1. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, if we take $p = q > 1$, and

$$\phi_1(x) = |B(0, |x|)|^{-p} \text{ and } \psi_1(x) = 1$$

in (5.2), then we have $A_1 = (p - 1)^{-1/p}$ and

$$\left(\int_{\mathbb{R}^n} \left| \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f(z)dz \right|^p dx \right)^{1/p} \leq \frac{p}{p-1} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}, \tag{5.7}$$

where $|B(0, |x|)|$ is the volume of the ball $B(0, |x|)$. The inequality (5.7) was obtained in [CG95].

2. Theorem 5.1.1 was obtained in [RY18a] and here we follow the proof from that paper. However, due to the fact that the formulations do not make use of the differential structure, the statement can be actually extended to general metric measure spaces with polar decomposition. More specifically, consider a metric space \mathbb{X} with a Borel measure dx allowing for the following *polar decomposition* at $a \in \mathbb{X}$: we assume that there is a locally integrable function $\lambda \in L^1_{\text{loc}}$ such that for all $f \in L^1(\mathbb{X})$ we have

$$\int_{\mathbb{X}} f(x)dx = \int_0^\infty \int_{\Sigma} f(r, \omega)\lambda(r, \omega)d\omega dr, \tag{5.8}$$

for some set $\Sigma \subset \mathbb{X}$ with a measure on it denoted by $d\omega$, and $(r, \omega) \rightarrow a$ as $r \rightarrow 0$. In the case of homogeneous groups such a polar decomposition is given in Proposition 1.2.10.

Let us denote by $B(a, r)$ the ball in \mathbb{X} with centre a and radius r , i.e.,

$$B(a, r) := \{x \in \mathbb{X} : d(x, a) < r\},$$

where d is the metric on \mathbb{X} . Once and for all we will fix some point $a \in \mathbb{X}$, and we will write

$$|x|_a := d(a, x).$$

Then the following result was obtained in [RV18] which we record here without proof.

Theorem 5.1.3 (Integral Hardy inequality in metric measure spaces). *Let $1 < p \leq q < \infty$ and let $s > 0$. Let \mathbb{X} be a metric measure space with a polar decomposition (5.8) at a . Let $u, v > 0$ be measurable functions positive a.e. in \mathbb{X} such that $u \in L^1(\mathbb{X} \setminus \{a\})$ and $v^{1-p'} \in L^1_{\text{loc}}(\mathbb{X})$. Denote*

$$U(x) := \int_{\mathbb{X} \setminus B(a, |x|_a)} u(y) dy \quad \text{and} \quad V(x) := \int_{B(a, |x|_a)} v^{1-p'}(y) dy$$

Then the inequality

$$\left(\int_{\mathbb{X}} \left(\int_{B(a, |x|_a)} |f(y)| dy \right)^q u(x) dx \right)^{1/q} \leq C \left\{ \int_{\mathbb{X}} |f(x)|^p v(x) dx \right\}^{1/p} \quad (5.9)$$

holds for all measurable functions $f : \mathbb{X} \rightarrow \mathbb{C}$ if and only if any of the following equivalent conditions hold:

1. $\mathcal{D}_1 := \sup_{x \neq a} \left\{ U^{\frac{1}{q}}(x) V^{\frac{1}{p'}}(x) \right\} < \infty$.
2. $\mathcal{D}_2 := \sup_{x \neq a} \left\{ \int_{\mathbb{X} \setminus B(a, |x|_a)} u(y) V^{q(\frac{1}{p'} - s)}(y) dy \right\}^{1/q} V^s(x) < \infty$.
3. $\mathcal{D}_3 := \sup_{x \neq a} \left\{ \int_{B(a, |x|_a)} u(y) V^{q(\frac{1}{p'} + s)}(y) dy \right\}^{1/q} V^{-s}(x) < \infty$,
provided that $u, v^{1-p'} \in L^1(\mathbb{X})$.
4. $\mathcal{D}_4 := \sup_{x \neq a} \left\{ \int_{B(a, |x|_a)} v^{1-p'}(y) U^{p'(\frac{1}{q} - s)}(y) dy \right\}^{1/p'} U^s(x) < \infty$.
5. $\mathcal{D}_5 := \sup_{x \neq a} \left\{ \int_{\mathbb{X} \setminus B(a, |x|_a)} v^{1-p'}(y) U^{p'(\frac{1}{q} + s)}(y) dy \right\}^{1/p'} U^{-s}(x) < \infty$,
provided that $u, v^{1-p'} \in L^1(\mathbb{X})$.

Moreover, the constant C for which (5.9) holds and quantities \mathcal{D}_1 – \mathcal{D}_5 are related by

$$\mathcal{D}_1 \leq C \leq \mathcal{D}_1(p')^{\frac{1}{p'}} p^{\frac{1}{q}}, \tag{5.10}$$

and

$$\mathcal{D}_1 \leq (\max(1, p's))^{\frac{1}{q}} \mathcal{D}_2, \quad \mathcal{D}_2 \leq \left(\max(1, \frac{1}{p's}) \right)^{1/q} \mathcal{D}_1,$$

$$\left(\frac{sp'}{1+p's} \right)^{1/q} \mathcal{D}_3 \leq \mathcal{D}_1 \leq (1+sp')^{\frac{1}{q}} \mathcal{D}_3,$$

$$\mathcal{D}_1 \leq (\max(1, qs))^{\frac{1}{p'}} \mathcal{D}_4, \quad \mathcal{D}_4 \leq \left(\max(1, \frac{1}{qs}) \right)^{1/p'} \mathcal{D}_1,$$

$$\left(\frac{sq}{1+qs} \right)^{1/p'} \mathcal{D}_5 \leq \mathcal{D}_1 \leq (1+sq)^{\frac{1}{p'}} \mathcal{D}_5.$$

3. As such, Theorem 5.1.3 is an extension of (5.1) to the setting of metric measures spaces \mathbb{X} with the polar decomposition (5.8): in particular, for $p = q$ and real-valued non-negative measurable $f \geq 0$, inequality (5.9) becomes

$$\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y) dy \right)^p u(x) dx \leq C \int_{\mathbb{X}} f(x)^p v(x) dx,$$

as an extension of (5.1). Indeed, in this case we can take $u(x) = \frac{1}{x^p}$, $v(x) = 1$, $\mathbb{X} = [b, \infty)$, $a = b$, so that Theorem 5.1.3 implies (5.1).

4. Let us give an application of Theorem 5.1.3 in the setting of homogeneous groups, recovering a two-weighted result obtained in [RV18]:

Corollary 5.1.4 (Characterization for homogeneous weights). *Let \mathbb{G} be a homogeneous group of homogeneous dimension Q , equipped with a homogeneous quasi-norm $|\cdot|$. Let $1 < p \leq q < \infty$ and let $\alpha, \beta \in \mathbb{R}$. Then the inequality*

$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} |f(y)| dy \right)^q |x|^\alpha dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{G}} |f(x)|^p |x|^\beta dx \right)^{\frac{1}{p}} \tag{5.11}$$

holds for all measurable functions $f : \mathbb{G} \rightarrow \mathbb{C}$ if and only if $\alpha + Q < 0$, $\beta(1 - p') + Q > 0$ and $\frac{\alpha+Q}{q} + \frac{\beta(1-p')+Q}{p'} = 0$. Moreover, the best constant C for (5.11) satisfies

$$\frac{\sigma^{\frac{1}{q} + \frac{1}{p'}}}{|\alpha + Q|^{\frac{1}{q}} (\beta(1 - p') + Q)^{\frac{1}{p'}}} \leq C \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} \frac{\sigma^{\frac{1}{q} + \frac{1}{p'}}}{|\alpha + Q|^{\frac{1}{q}} (\beta(1 - p') + Q)^{\frac{1}{p'}}},$$

where σ is the area of the unit sphere in \mathbb{G} with respect to the quasi-norm $|\cdot|$.

Let us show how Theorem 5.1.3 implies Corollary 5.1.4. If we take $a = 0$, and the power weights

$$u(x) = |x|^\alpha \quad \text{and} \quad v(x) = |x|^\beta,$$

then the inequality (5.9) holds for $1 < p \leq q < \infty$ if and only if

$$\mathcal{D}_1 = \sup_{r>0} \left(\sigma \int_r^\infty \rho^\alpha \rho^{Q-1} d\rho \right)^{1/q} \left(\sigma \int_0^r \rho^{\beta(1-p')} \rho^{Q-1} d\rho \right)^{1/p'} < \infty,$$

where σ is the area of the unit sphere in \mathbb{G} with respect to the quasi-norm $|\cdot|$. For this supremum to be well defined we need to have $\alpha + Q < 0$ and $\beta(1 - p') + Q > 0$. Consequently, we can calculate

$$\begin{aligned} \mathcal{D}_1 &= \sigma^{\left(\frac{1}{q} + \frac{1}{p'}\right)} \sup_{r>0} \left(\int_r^\infty \rho^{\alpha+Q-1} d\rho \right)^{1/q} \left(\int_0^r \rho^{\beta(1-p')+Q-1} d\rho \right)^{1/p'} \\ &= \sigma^{\left(\frac{1}{q} + \frac{1}{p'}\right)} \sup_{r>0} \frac{r^{\frac{\alpha+Q}{q}}}{|\alpha + Q|^{\frac{1}{q}}} \frac{r^{\frac{\beta(1-p')+Q}{p'}}}{(\beta(1 - p') + Q)^{\frac{1}{p'}}}, \end{aligned}$$

which is finite if and only if the power of r is zero. Consequently, Corollary 5.1.4 follows from Theorem 5.1.3.

Proof of Theorem 5.1.1. We will prove Part (1) of the theorem since the proof of Part (2) is similar. The obtained estimates will also show the corresponding part of the statement in Part (3).

Thus, let us first show that (5.3) implies (5.2). Using the polar decomposition in Proposition 1.2.10 and denoting $r = |x|$, we write

$$\begin{aligned} &\int_{\mathbb{G}} \phi_1(x) \left[\int_{B(0,r)} f(z) dz \right]^q dx \\ &= \int_0^\infty \int_{\varphi} r^{Q-1} \phi_1(ry) \left[\int_0^r \int_{\varphi} s^{Q-1} f(sy) d\sigma(y) ds \right]^q d\sigma(y) dr. \end{aligned} \tag{5.12}$$

Denoting

$$g(r) := \left\{ \int_{\varphi} \int_0^r s^{Q-1} (\psi_1(sy))^{1-p'} ds d\sigma(y) \right\}^{1/(pp')}, \tag{5.13}$$

and using Hölder's inequality, we can estimate

$$\begin{aligned} &\int_0^r \int_{\varphi} s^{Q-1} f(sy) d\sigma(y) ds \\ &= \int_{\varphi} \int_0^r s^{(Q-1)/p} f(sy) (\psi_1(sy))^{1/p} g(s) s^{(Q-1)/p'} \\ &\quad \times \left((\psi_1(sy))^{1/p} g(s) \right)^{-1} ds d\sigma(y) \end{aligned}$$

$$\begin{aligned} &\leq \left(\int_{\varphi} \int_0^r s^{Q-1} \left[f(sy)(\psi_1(sy))^{1/p} g(s) \right]^p ds d\sigma(y) \right)^{1/p} \\ &\quad \times \left(\int_{\varphi} \int_0^r s^{Q-1} \left[(\psi_1(sy))^{1/p} g(s) \right]^{-p'} ds d\sigma(y) \right)^{1/p'}. \end{aligned} \quad (5.14)$$

Let us introduce the following notations:

$$U(s) := \int_{\varphi} s^{Q-1} \left(f(sy)(\psi_1(sy))^{1/p} g(s) \right)^p d\sigma(y), \quad (5.15)$$

$$V(r) := \int_0^r \int_{\varphi} s^{Q-1} \left((\psi_1(sy))^{1/p} g(s) \right)^{-p'} d\sigma(y) ds, \quad (5.16)$$

$$W_1(r) := \int_{\varphi} r^{Q-1} \phi_1(ry) d\sigma(y), \quad (5.17)$$

for $s, r > 0$. Plugging (5.14) into (5.12) we obtain

$$\begin{aligned} &\int_{\mathbb{G}} \phi_1(x) \left(\int_{B(0,r)} f(z) dz \right)^q dx \\ &\leq \int_0^\infty W_1(r) \left(\int_0^r U(s) ds \right)^{q/p} (V(r))^{q/p'} dr. \end{aligned} \quad (5.18)$$

We now recall the following continuous version of the *Minkowski inequality* (see, e.g., [DHK97, Formula 2.1]):

Let $\theta \geq 1$. Then for all $f_1(x), f_2(x) \geq 0$ on $(0, \infty)$, we have

$$\int_0^\infty f_1(x) \left(\int_0^x f_2(z) dz \right)^\theta dx \leq \left(\int_0^\infty f_2(z) \left(\int_z^\infty f_1(x) dx \right)^{1/\theta} dz \right)^\theta. \quad (5.19)$$

Using this inequality with $\theta = q/p \geq 1$ in the right-hand side of (5.18), we can estimate

$$\int_{\mathbb{G}} \phi_1(x) \left(\int_{B(0,r)} f(z) dz \right)^q dx \leq \left(\int_0^\infty U(s) \left(\int_s^\infty W_1(r) (V(r))^{q/p'} dr \right)^{p/q} ds \right)^{q/p}. \quad (5.20)$$

Let us introduce one more temporary notation

$$T(s) := \int_{\varphi} s^{Q-1} (\psi_1(sy))^{1-p'} d\sigma(y).$$

Using (5.13), (5.16), the integration by parts, (5.3) and (5.17), we compute

$$\begin{aligned}
 V(r) &= \int_{\varphi} \int_0^r s^{Q-1} (\psi_1(sy))^{1-p'} \left(\int_0^s \int_{\varphi} t^{Q-1} (\psi_1(tw))^{1-p'} d\sigma(w) dt \right)^{-1/p} ds d\sigma(y) \\
 &= \int_0^r T(s) \left(\int_0^s T(t) dt \right)^{-1/p} ds = p' \int_0^r \frac{d}{ds} \left(\int_0^s T(t) dt \right)^{1/p'} ds \\
 &= p' \left(\int_0^r T(s) ds \right)^{1/p'} = p' \left(\int_0^r \int_{\varphi} s^{Q-1} (\psi_1(sy))^{1-p'} d\sigma(y) ds \right)^{1/p'} \\
 &\leq p' A_1 \left(\int_r^{\infty} s^{Q-1} \int_{\varphi} \phi_1(sw) d\sigma(w) ds \right)^{-1/q} = p' A_1 \left(\int_r^{\infty} W_1(s) ds \right)^{-1/q}.
 \end{aligned}$$

Similarly, applying the integration by parts and (5.3), this implies

$$\begin{aligned}
 \int_s^{\infty} W_1(r) (V(r))^{q/p'} dr &= (p' A_1)^{q/p'} \int_s^{\infty} W_1(r) \left(\int_r^{\infty} W_1(s) ds \right)^{-1/p'} dr \\
 &= (p' A_1)^{q/p'} p \left(\int_s^{\infty} W_1(r) dr \right)^{1/p} \\
 &= (p' A_1)^{q/p'} p \left(\int_s^{\infty} \int_{\varphi} r^{Q-1} \phi_1(ry) d\sigma(y) dr \right)^{1/p} \\
 &\leq (p' A_1)^{q/p'} p A_1^{q/p} \left(\int_0^s r^{Q-1} \int_{\varphi} (\psi_1(ry))^{1-p'} d\sigma(y) dr \right)^{-q/(p'p)} \\
 &= A_1^q (p')^{q/p'} p (g(s))^{-q},
 \end{aligned} \tag{5.21}$$

where we have used (5.13) in the last line. Putting (5.21) in (5.20) and recalling (5.15), we obtain

$$\begin{aligned}
 \int_{\mathbb{G}} \phi_1(x) \left(\int_{B(0,r)} f(z) dz \right)^q dx &\leq \left(\int_0^{\infty} U(s) A_1^p (p')^{p-1} p^{p/q} (g(s))^{-p} ds \right)^{q/p} \\
 &= A_1^q (p')^{q/p'} p \left(\int_0^{\infty} U(s) (g(s))^{-p} ds \right)^{q/p} \\
 &= A_1^q (p')^{q/p'} p \left(\int_0^{\infty} \int_{\varphi} s^{Q-1} (f(sy))^p \psi_1(sy) d\sigma(y) ds \right)^{q/p} \\
 &= A_1^q (p')^{q/p'} p \left(\int_{\mathbb{G}} \psi_1(x) (f(x))^p dx \right)^{q/p},
 \end{aligned} \tag{5.22}$$

yielding (5.2) with $C_1 = A_1(p')^{1/p'} p^{1/q}$.

We now show the converse, namely, that (5.2) implies (5.3). For that, we set

$$f(x) := (\psi_1(x))^{1-p'} \chi_{(0,R)}(|x|),$$

with $R > 0$. For this f we observe the equality

$$\begin{aligned} & \left(\int_{\mathbb{G}} \psi_1(x) (f(x))^p dx \right)^{1/p} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &= \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{1/p} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} = 1. \end{aligned} \tag{5.23}$$

Consequently, by (5.2) we have

$$\begin{aligned} C &= C \left(\int_{\mathbb{G}} \psi_1(x) (f(x))^p dx \right)^{1/p} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &\geq \left(\int_{\mathbb{G}} \phi_1(x) \left(\int_{|z| \leq |x|} f(z) dz \right)^q dx \right)^{1/q} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &\geq \left(\int_{|x| \geq R} \phi_1(x) \left(\int_{|z| \leq |x|} f(z) dz \right)^q dx \right)^{1/q} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &= \left(\int_{|x| \geq R} \phi_1(x) dx \right)^{1/q} \left(\int_{|z| \leq R} (\psi_1(z))^{1-p'} dz \right)^{1/p'}. \end{aligned} \tag{5.24}$$

Combining (5.23) and (5.24), we obtain (5.3) with $C \geq A_1$. □

The next case is the version of Theorem 5.1.1 for the indices $p > q$:

Theorem 5.1.5 (Integral Hardy inequalities for $p > q$). *Let \mathbb{G} be a homogeneous group of homogeneous dimension Q and let $1 < q < p < \infty$ and $1/\delta = 1/q - 1/p$. Let ϕ_3 and ϕ_4 be positive functions on \mathbb{G} . Then we have the following properties:*

(1) *The inequality*

$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(z) dz \right)^q \phi_3(x) dx \right)^{1/q} \leq C_1 \left(\int_{\mathbb{G}} (f(x))^p \psi_3(x) dx \right)^{1/p} \tag{5.25}$$

holds for all $f \geq 0$ if and only if

$$\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} \phi_3(z) dz \right)^{\delta/q} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^{\delta/q'} (\psi_3(x))^{1-p'} dx < \infty. \tag{5.26}$$

(2) *The inequality*

$$\left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} f(z) dz \right)^q \phi_4(x) dx \right)^{1/q} \leq C_2 \left(\int_{\mathbb{G}} (f(x))^p \psi_4(x) dx \right)^{1/p} \quad (5.27)$$

holds for all $f \geq 0$ if and only if

$$\int_{\mathbb{G}} \left(\int_{B(0,|x|)} \phi_4(z) dz \right)^{\delta/q} \left(\int_{\mathbb{G} \setminus B(0,|x|)} (\psi_4(z))^{1-p'} dz \right)^{\delta/q'} (\psi_4(x))^{1-p'} dx < \infty. \quad (5.28)$$

Proof of Theorem 5.1.5. We will prove Part (1) of the theorem since Part (2) is similar. We will denote

$$A_3 := \int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} \phi_3(z) dz \right)^{\delta/q} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^{\delta/q'} (\psi_3(x))^{1-p'} dx.$$

First we prove that if $A_3 < \infty$, then we have inequality (5.25). Denote

$$W_2(r) := \int_{\wp} r^{Q-1} \phi_3(rw) d\sigma(w) \quad (5.29)$$

and

$$G(s) := \int_{\wp} s^{Q-1} h(sy) (\psi_3(sy))^{1-p'} d\sigma(y) \quad (5.30)$$

for $h \geq 0$ on \mathbb{G} to be chosen later. Using the polar decomposition in Proposition 1.2.10 we have the following equalities:

$$\begin{aligned} & \int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z) (\psi_3(z))^{1-p'} dz \right)^q dx \\ &= \int_0^\infty \int_{\wp} r^{Q-1} \phi_3(rw) d\sigma(w) \left(\int_0^r \int_{\wp} s^{Q-1} h(sy) (\psi_3(sy))^{1-p'} d\sigma(y) ds \right)^q dr \\ &= \int_0^\infty W_2(r) \left(\int_0^r G(s) ds \right)^q dr \\ &= q \int_0^\infty G(s) \left(\int_0^s G(r) dr \right)^{q-1} \left(\int_s^\infty W_2(r) dr \right) ds \\ &= q \int_{\wp} \int_0^\infty s^{Q-1} h(sy) (\psi_3(sy))^{1-p'} \left(\int_0^s \int_{\wp} r^{Q-1} h(rw) (\psi_3(rw))^{1-p'} d\sigma(w) dr \right)^{q-1} \\ & \quad \times \left(\int_s^\infty W_2(r) dr \right) ds d\sigma(y) \end{aligned}$$

$$\begin{aligned}
 &= q \int_{\wp} \int_0^\infty s^{Q-1} h(sy) (\psi_3(sy))^{(1-p')(\frac{1}{p} + \frac{q-1}{p} + \frac{p-q}{p})} \\
 &\quad \times \left(\frac{\int_{\wp} \int_0^s r^{Q-1} h(rw) (\psi_3(rw))^{1-p'} dr d\sigma(w)}{\int_{\wp} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w)} \right)^{q-1} \\
 &\quad \times \left(\left(\int_{\wp} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{q-1} \left(\int_s^\infty W_2(r) dr \right) \right) ds d\sigma(y).
 \end{aligned}$$

Here, using Hölder’s inequality with three factors with indices $\frac{1}{p} + \frac{q-1}{p} + \frac{p-q}{p} = 1$, we can estimate

$$\int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z) (\psi_3(z))^{1-p'} dz \right)^q dx \leq qK_1K_2K_3, \tag{5.31}$$

where

$$\begin{aligned}
 K_1 &= \left(\int_{\wp} \int_0^\infty s^{Q-1} (h(sy))^p (\psi_3(sy))^{1-p'} ds d\sigma(y) \right)^{1/p} \\
 &= \left(\int_{\mathbb{G}} (h(x))^p (\psi_3(x))^{1-p'} dx \right)^{1/p}, \tag{5.32}
 \end{aligned}$$

$$\begin{aligned}
 K_2 &= \left(\int_{\wp} \int_0^\infty s^{Q-1} (\psi_3(sy))^{1-p'} \right. \\
 &\quad \times \left. \left(\frac{\int_{\wp} \int_0^s r^{Q-1} h(rw) (\psi_3(rw))^{1-p'} dr d\sigma(w)}{\int_{\wp} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w)} \right)^p ds d\sigma(y) \right)^{\frac{q-1}{p}} \tag{5.33}
 \end{aligned}$$

and

$$\begin{aligned}
 K_3 &= \left(\int_{\wp} \int_0^\infty s^{Q-1} (\psi_3(sy))^{1-p'} \left(\int_{\wp} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{\frac{(q-1)p}{p-q}} \right. \\
 &\quad \times \left. \left(\int_s^\infty W_2(r) dr \right)^{\frac{p}{p-q}} ds d\sigma(y) \right)^{\frac{p-q}{p}}. \tag{5.34}
 \end{aligned}$$

Leaving K_1 as it is, we will estimate K_2 and K_3 . We rewrite K_2 as

$$K_2 = \left(\int_{\mathbb{G}} \frac{(\psi_3(x))^{1-p'}}{\left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^p} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} h(z) dz \right)^p dx \right)^{(q-1)/p}.$$

We want to apply (5.2) to K_2 with indices $p = q$, and with functions $f(x) = (\psi_3(x))^{1-p'} h(x)$ and

$$\phi_1(x) = \frac{(\psi_3(x))^{1-p'}}{\left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^p}, \quad \psi_1(x) = (\psi_3(x))^{(1-p')(1-p)}.$$

For that, we will check the condition that

$$\begin{aligned}
 A_1(R) &= \left(\int_{|x| \geq R} (\psi_3(x))^{1-p'} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^{-p} dx \right)^{1/p} \\
 &\quad \times \left(\int_{|x| \leq R} (\psi_3(x))^{1-p'} dx \right)^{1/p'} < \infty
 \end{aligned} \tag{5.35}$$

holds uniformly for all $R > 0$. Assuming (5.35) uniformly in $R > 0$ for a moment, the inequality (5.2) would imply that

$$\begin{aligned}
 K_2 &\leq C \left(\int_{\mathbb{G}} (\psi_3(x))^{(1-p')(1-p+p)} (h(x))^p dx \right)^{(q-1)/p} \\
 &= C \left(\int_{\mathbb{G}} (h(x))^p (\psi_3(x))^{1-p'} dx \right)^{(q-1)/p},
 \end{aligned} \tag{5.36}$$

which is something we will use later. So, let us check (5.35). For this, we denote

$$S(s) := \int_{\wp} s^{Q-1} (\psi_3(sw))^{1-p'} d\sigma(w).$$

Using integration by parts we have

$$\begin{aligned}
 A_1(R) &= \left(\int_{\wp} \int_R^\infty r^{Q-1} (\psi_3(rw))^{1-p'} \left(\int_0^r S(s) ds \right)^{-p} dr d\sigma(w) \right)^{\frac{1}{p}} \left(\int_0^R S(s) ds \right)^{\frac{1}{p'}} \\
 &= \left(\int_R^\infty \left(\int_0^r S(s) ds \right)^{-p} S(r) dr \right)^{1/p} \left(\int_0^R S(s) ds \right)^{1/p'} \\
 &\leq \left(\frac{1}{p-1} \left(\int_0^R S(s) ds \right)^{1-p} \right)^{1/p} \left(\int_0^R S(s) ds \right)^{1/p'} = (p-1)^{-1/p} < \infty,
 \end{aligned}$$

so that (5.36) is confirmed. Next, for K_3 , taking into account

$$\frac{1}{\delta} = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq}$$

and using (5.26), we have

$$\begin{aligned}
 K_3 &= \left(\int_0^\infty \int_{\wp} \left(\int_s^\infty W_2(r) dr \right)^{\delta/q} \left(\int_{\wp} \int_0^r r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{\delta/q'} \right. \\
 &\quad \left. \times s^{Q-1} (\psi_3(sy))^{1-p'} d\sigma(y) ds \right)^{(p-q)/p}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} \phi_3(z) dz \right)^{\frac{\delta}{q}} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^{\frac{\delta}{q'}} (\psi_3(x))^{1-p'} dx \right)^{\frac{p-q}{p}} \\
 &= A_3^{\frac{p-q}{p}} < \infty.
 \end{aligned} \tag{5.37}$$

Now, plugging (5.32), (5.36) and (5.37) into (5.31), we obtain

$$\begin{aligned}
 &\int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z)(\psi_3(z))^{1-p'} dz \right)^q dx \\
 &\leq C A_3^{\frac{p-q}{p}} \left(\int_{\mathbb{G}} (h(x))^p (\psi_3(x))^{1-p'} dx \right)^{\frac{1}{p} + \frac{q-1}{p}},
 \end{aligned}$$

which implies (5.25) after taking $h := f\psi_3^{p'-1}$.

Let us now show the converse, namely, that (5.25) implies (5.26). For this, we consider a sequence of functions

$$\begin{aligned}
 f_k(x) &:= \left(\int_{|y| \geq |x|} \phi_3(z) dz \right)^{\delta/(pq)} \left(\int_{\alpha_k \leq |z| \leq |x|} (\psi_3(z))^{1-p'} dz \right)^{\delta/(pq')} \\
 &\quad \times (\psi_3(x))^{1-p'} \chi_{(\alpha_k, \beta_k)}(|x|), \quad k = 1, 2, \dots
 \end{aligned}$$

Inserting these functions in the place of $f(x)$ in (5.25), we obtain (5.26), if we take $0 < \alpha_k < \beta_k$ with $\alpha_k \searrow 0$ and $\beta_k \nearrow \infty$ for $k \rightarrow \infty$. □

5.2 Convolution Hardy inequalities

In this section we discuss integral Hardy inequalities in the convolution form. Such inequalities are particularly useful if we make particular choices of the convolution kernels. For example, by taking the Riesz kernels of hypoelliptic differential operators on graded groups, such inequalities can be used to derive a number of hypoelliptic versions of Hardy inequalities. While this topic falls outside the scope of this book, we refer to [RY18a] for such applications. The inequalities that we will present here have been established in [RY18a] and we follow the proofs there in our exposition.

Theorem 5.2.1 (Convolution Hardy inequality). *Let \mathbb{G} be a homogeneous Lie group of homogeneous dimension Q and with a homogeneous quasi-norm $|\cdot|$. Let $1 < p \leq q < \infty$, $0 < a < Q/p$, $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Assume that there is $C_2 = C_2(a, Q) > 0$ such that*

$$|T_a^{(1)}(x)| \leq C_2 |x|^{a-Q} \tag{5.38}$$

holds for all $x \neq 0$. Then there exists a positive constant $C_1 = C_1(p, q, a, b) > 0$ such that

$$\left\| \frac{f * T_a^{(1)}}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C_1 \|f\|_{L^p(\mathbb{G})} \tag{5.39}$$

holds for all $f \in L^p(\mathbb{G})$.

The critical case $b = Q$ of Theorem 5.2.1 will be shown in Theorem 5.2.5.

Remark 5.2.2 (Riesz kernels). Let us briefly describe a typical situation when condition (5.38) is satisfied. Without going much into detail, let us assume that \mathcal{R} is a positive homogeneous left invariant hypoelliptic differential operator on \mathbb{G} of homogeneous degree ν . The existence of such an operator implies that the group \mathbb{G} is graded, see [FR16, Section 4.1]. The operators \mathcal{R} satisfying the above properties are called *Rockland operators*.

Let h_t denote the heat kernel associated to the operator \mathcal{R} , see [FR16, Section 4.3.4] for a thorough treatment of this and for the proof of the following notes. This heat kernel satisfies the following properties, see [FR16, Theorem 4.2.7 and Lemma 4.3.8]:

Theorem 5.2.3 (Heat kernels). *Let \mathcal{R} be homogeneous left invariant hypoelliptic differential operator on \mathbb{G} of homogeneous degree ν and let h_t be the associated heat kernel. Then each h_t is Schwartz and we have*

$$\forall s, t > 0 \quad h_t * h_s = h_{t+s}, \tag{5.40}$$

$$\forall x \in \mathbb{G}, r, t > 0 \quad h_{r^\nu t}(rx) = r^{-Q} h_t(x), \tag{5.41}$$

$$\forall x \in \mathbb{G} \quad h_t(x) = \overline{h_t(x^{-1})}, \tag{5.42}$$

$$\int_{\mathbb{G}} h_t(x) dx = 1. \tag{5.43}$$

Moreover, we have

$$\exists C = C_{\alpha, N, \ell} > 0 \quad \forall t \in (0, 1] \quad \sup_{|x|=1} |\partial_t^\ell X^\alpha h_t(x)| \leq C_{\alpha, N} t^N \tag{5.44}$$

for any $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and $\ell \in \mathbb{N}_0$.

Furthermore, for any multi-index $\alpha \in \mathbb{N}_0^n$ and any real number a with $0 < a < (Q + [\alpha])/\nu$ there exists a positive constant $C > 0$ such that

$$\int_0^\infty t^{a-1} |X^\alpha h_t(x)| dt \leq C |x|^{-Q - [\alpha] + \nu a}. \tag{5.45}$$

The fractional powers $\mathcal{R}^{-a/\nu}$ for $\{a \in \mathbb{R}, 0 < a < Q\}$ and $(I + \mathcal{R})^{-a/\nu}$ for $a \in \mathbb{R}_+$ are called *Riesz and Bessel potentials*, respectively, and they are well

defined, see [FR16, Chapter 4.2]. Let us denote their respective kernels by \mathcal{I}_a and \mathcal{B}_a . Then we have the relations

$$\mathcal{I}_a(x) = \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} h_t(x) dt \quad (5.46)$$

for $0 < a < Q$ with $a \in \mathbb{R}$, and

$$\mathcal{B}_a(x) = \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \quad (5.47)$$

for $a > 0$, where Γ denotes the Gamma function. Consequently, it can be shown (see [FR16, Section 4.3.4]) that for any $0 < a < Q$ there exists a positive constant $C = C(Q, a)$ such that

$$|\mathcal{I}_a(x)| \leq C|x|^{-(Q-a)} \quad (5.48)$$

holds for all $x \neq 0$. Therefore, the Riesz kernel \mathcal{I}_a gives a typical example of an operator satisfying condition (5.38).

Proof of Theorem 5.2.1. We split the integral in the left-hand side of (5.39) into three parts:

$$\int_{\mathbb{G}} |(f * T_a^{(1)})(x)|^q \frac{dx}{|x|^b} \leq 3^q (M_1 + M_2 + M_3), \quad (5.49)$$

with

$$M_1 := \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |T_a^{(1)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{|x|^b},$$

$$M_2 := \int_{\mathbb{G}} \left(\int_{\{|x| \leq 2|y| < 4|x|\}} |T_a^{(1)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{|x|^b}$$

and

$$M_3 := \int_{\mathbb{G}} \left(\int_{\{|y| > 2|x|\}} |T_a^{(1)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{|x|^b}.$$

First, let us *estimate* M_1 . We can assume without loss of generality that $|\cdot|$ is a norm (such a norm always exists, see Proposition 1.2.4, Part (2)) since replacing the seminorm by an equivalent one only changes the appearing constants.

Observe that by the reverse triangle inequality and the assumption $2|y| < |x|$ we have

$$|y^{-1}x| \geq |x| - |y| > |x| - \frac{|x|}{2} = \frac{|x|}{2}, \quad (5.50)$$

which is $|x| < 2|y^{-1}x|$. Taking into account this and that $T_a^{(1)}(x)$ is bounded by a radial function which is non-increasing with respect to $|x|$, we can estimate

$$\begin{aligned} M_1 &\leq \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |f(y)| dy \right)^q \left(\sup_{\{|x| < 2|z|\}} |T_a^{(1)}(z)| \right)^q \frac{dx}{|x|^b} \\ &\leq C \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |f(y)| dy \right)^q \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b}. \end{aligned} \tag{5.51}$$

We will now apply Theorem 5.1.1, Part (1), to estimate M_1 . For this we need to check condition (5.3), that is, that

$$\sup_{R>0} \left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} < \infty. \tag{5.52}$$

To check this, we consider two cases: $R \geq 1$ and $0 < R < 1$. For $R \geq 1$, we can estimate

$$\begin{aligned} &\left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} \\ &\leq CR^{\frac{Q}{p'}} \left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \\ &\leq CR^{\frac{Q}{p'}} \left(\int_{\{2R < |x|\}} |x|^{(a-Q)q-b} dx \right)^{\frac{1}{q}} \\ &\leq CR^{\frac{Q}{p'}} R^{\frac{(a-Q)q-b+Q}{q}} \\ &\leq C, \end{aligned} \tag{5.53}$$

which is uniformly bounded since $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$ and $(a-Q)q-b+Q = -\frac{Qq}{p'} \neq 0$. Now let us check the condition (5.52) for $0 < R < 1$. Here, taking into account that $(a-Q)q-b+Q = -\frac{Qq}{p'} \neq 0$ we have

$$\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \leq CR^{(a-Q)q-b+Q}. \tag{5.54}$$

It follows with $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$ that

$$\left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} \leq CR^{a-Q-\frac{b}{q}+\frac{Q}{q}} R^{Q/p'} \leq C \tag{5.55}$$

holds for any $0 < R < 1$. Thus, we have checked (5.52). Applying Theorem 5.1.1, Part (1), we obtain

$$M_1^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_1 \|f\|_{L^p(\mathbb{G})}. \quad (5.56)$$

Let us now *estimate* M_2 . For this, we decompose M_2 as

$$M_2 = \sum_{k \in \mathbb{Z}} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_a^{(1)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{|x|^b}.$$

Since $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| < 2^{k+1}$, we have $2^{k-1} \leq |y| < 2^{k+2}$. As in (5.50), assuming $|\cdot|$ is the norm and using the triangle inequality, we have

$$3|x| = |x| + 2|x| \geq |x| + |y| \geq |y^{-1}x|, \quad (5.57)$$

which implies $0 \leq |y^{-1}x| \leq 3|x| < 3 \cdot 2^{k+1}$. If we denote

$$\tilde{I}_a(x) := C_2|x|^{a-Q},$$

then by the assumption we have

$$|T_a^{(1)}(x)| \leq \tilde{I}_a(x).$$

Taking into account these observations and applying Young's inequality in Proposition 1.2.13 with $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, $r \in [1, \infty]$, we can estimate M_2 by

$$\begin{aligned} M_2 &\leq \sum_{k \in \mathbb{Z}} 2^{-kb} \int_{\mathbb{G}} (([f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}] * \tilde{I}_a)(x))^q dx \\ &= \sum_{k \in \mathbb{Z}} 2^{-kb} \|[f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}] * \tilde{I}_a\|_{L^q(\mathbb{G})}^q \\ &\leq \sum_{k \in \mathbb{Z}} 2^{-kb} \|\tilde{I}_a \cdot \chi_{\{0 \leq |\cdot| < 3 \cdot 2^{k+1}\}}\|_{L^r(\mathbb{G})}^q \|f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\ &= C_2 \sum_{k \in \mathbb{Z}} 2^{-kb} \left(\int_{|x| < 3 \cdot 2^{k+1}} |x|^{(a-Q)r} dx \right)^{\frac{q}{r}} \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \quad (5.58) \\ &\leq C \sum_{k \in \mathbb{Z}} 2^{-kb} (3 \cdot 2^{k+1})^{\left(\frac{(a-Q)pq}{pq+p-q} + Q\right) \frac{pq+p-q}{p}} \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\ &= C \sum_{k \in \mathbb{Z}} 2^{-kb} (3 \cdot 2^{k+1})^b \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\ &\leq C \sum_{k \in \mathbb{Z}} \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\ &\leq C \|f\|_{L^p(\mathbb{G})}^q, \end{aligned}$$

since $\frac{(a-Q)pq}{pq+p-q} + Q = \frac{bp}{pq+p-q} > 0$ and $q \geq p$.

Now let us estimate M_3 . Without loss of generality, we may assume again that $|\cdot|$ is a norm. Then, similarly to (5.50) we note that $2|x| < |y|$ implies $|y| < 2|y^{-1}x|$. Consequently we can estimate M_3 as

$$M_3 \leq \int_{\mathbb{G}} \left(\int_{\{|y|>2|x\}} \left(\frac{|y|}{2} \right)^{(a-Q)} |f(y)| dy \right)^q \frac{dx}{|x|^b}.$$

We will apply Theorem 5.1.1, Part (2), to estimate M_3 . For this, we need to check that

$$\sup_{R>0} \left(\int_{\{|x|<R\}} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} < \infty. \tag{5.59}$$

To verify this, we consider two cases: $R \geq 1$ and $0 < R < 1$. First, for $R \geq 1$, using the assumption $|T_a^{(1)}(x)| \leq C|x|^{a-Q}$ and that $Q \neq ap$, one gets

$$\left(\int_{\{2R<|x\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq C \left(\int_{\{2R<|x\}} |x|^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq CR^{a-\frac{Q}{p}}. \tag{5.60}$$

Since $R \geq 1$ we can estimate

$$\left(\int_{\{|x|<R\}} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq CR^{a-\frac{Q}{p}+\frac{Q-b}{q}} \leq C,$$

since $b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Now let us check the condition (5.59) for the range $0 < R < 1$. In this case, noting that $ap - Q < 0$ we have

$$\int_{\{2R<|x\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \leq C \int_{\{2R<|x\}} |x|^{(a-Q)p'} dx \leq CR^{(a-Q)p'+Q}. \tag{5.61}$$

Since

$$\int_{\{|x|<R\}} \frac{dx}{|x|^b} \leq CR^{Q-b},$$

we have

$$\left(\int_{\{|x|<R\}} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq CR^{\frac{Q-b}{q}} R^{\frac{(a-Q)p'+Q}{p'}} \leq C, \tag{5.62}$$

since $Q > b$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Thus, we have checked (5.59). Consequently, the application of Theorem 5.1.1, Part (2), to M_3 yields

$$M_3^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_2 \|f\|_{L^p(\mathbb{G})}. \tag{5.63}$$

Thus, (5.56), (5.63) and (5.58) complete the proof of Theorem 5.2.1. □

Remark 5.2.4 (Schur test argument). In the case $p = q$, we can also prove Theorem 5.2.1 by using Schur’s test ([FR75]) as in the proof of Theorem 4.7.1. For the Riesz kernels of the sub-Laplacian on stratified groups such an argument was used in [CCR15], and the argument below was given in [RY18a].

For $p = q$, the condition $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$ in Theorem 5.2.1 implies that $b = ap$, so we are interested in the L^p -boundedness of the operator

$$S_a f := |x|^{-b/p}(f * |x|^{a-Q}) = |x|^{-a}(f * |x|^{a-Q}).$$

The adjoint, defined by $(f, S_a^* g) = (S_a f, g)$, is given by $S_a^* g := (|x|^{-a} g) * |x|^{a-Q}$.

We now recall again the Schur test:

Assume that the integral operator S has a positive integral kernel, and that there exist a positive function h and constants A_p and B_p such that

$$S(h^{p'})(x) \leq A_p(h(x))^{p'} \quad \text{and} \quad S^*(h^p)(x) \leq B_p(h(x))^p$$

hold for almost all $x \in \mathbb{G}$. Then we have

$$\|S_a f\|_{L^p(\mathbb{G})} \leq A_p^{1/p'} B_p^{1/p} \|f\|_{L^p(\mathbb{G})}$$

for all $f \in L^p(\mathbb{G})$.

Let us now take $h_c(x) := |x|^{c-Q}$ with $c > 0$ and consider the convolution integrals

$$h_c^{p'} * |x|^{a-Q} \quad \text{and} \quad (|x|^{-b/p} h_c^p) * |x|^{a-Q},$$

which arise in the computation of $S_a(h_c^{p'})$ and $S_a^*(h_c^p)$. We see that the homogeneity orders of $h_c^{p'}$ and $|x|^{-b/p} h_c^p$ are $(c-Q)p'$ and $(c-Q)p - b/p$, respectively. Then, the homogeneity orders of $h_c^{p'} * |x|^{a-Q}$ and $(|x|^{-b/p} h_c^p) * |x|^{a-Q}$ are $a - Q + (c-Q)p'$ and $a - Q + (c-Q)p - b/p$, respectively. Therefore, these convolution integrals converge absolutely in $\mathbb{G} \setminus \{0\}$ if and only if $0 < (c-Q)p' + Q < Q - a$ and $0 < (c-Q)p - b/p + Q < Q - a$, that is, if

$$\max\left(\frac{Q}{p}, \frac{a}{p} + \frac{Q}{p'}\right) < c < Q - \frac{a}{p}$$

since $b = ap$. This condition is true if $0 < a < Q/p$.

Thus, it follows from Schur’s test that

$$\||x|^{-a}(f * |x|^{a-Q})\|_{L^p(\mathbb{G})} \leq A_{a,p}^{1/p'} B_{a,p}^{1/p} \|f\|_{L^p(\mathbb{G})},$$

where $0 < a < Q/p$, $1 < p < \infty$, and $f \in L^p(\mathbb{G})$.

Taking into account this and $|T_a^{(1)}(x)| \leq C|x|^{a-Q}$, we obtain

$$\begin{aligned} \left\| \frac{f * T_a^{(1)}}{|x|^{\frac{b}{p}}} \right\|_{L^p(\mathbb{G})} &\leq C \left\| \frac{|f| * |T_a^{(1)}|}{|x|^{\frac{b}{p}}} \right\|_{L^p(\mathbb{G})} \\ &\leq C \||x|^{-a}(|f| * |x|^{a-Q})\|_{L^p(\mathbb{G})} \leq C \|f\|_{L^p(\mathbb{G})}, \end{aligned} \tag{5.64}$$

which proves Theorem 5.2.1 in the case $p = q$.

Let us now show the critical case $b = Q$ of Theorem 5.2.1.

Theorem 5.2.5 (Critical convolution Hardy inequality). *Let \mathbb{G} be a homogeneous Lie group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $1 < p < r < \infty$ and $p < q < (r - 1)p'$, where $1/p + 1/p' = 1$. Assume that for $a = Q/p$ we have*

$$|T_a^{(2)}(x)| \leq C_2 \begin{cases} |x|^{a-Q}, & \text{for } x \in \mathbb{G} \setminus \{0\}, \\ |x|^{-Q}, & \text{for } x \in \mathbb{G} \text{ with } |x| \geq 1, \end{cases} \tag{5.65}$$

for some positive $C_2 = C_2(a, Q)$. Then there exists a positive constant $C_1 = C_1(p, q, r, Q) > 0$ such that

$$\left\| \frac{f * T_{Q/p}^{(2)}}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \leq C_1 \|f\|_{L^p(\mathbb{G})} \tag{5.66}$$

holds for all $f \in L^p(\mathbb{G})$.

Remark 5.2.6. We note that compared to the condition (5.38), the decay assumption in (5.65) for large x is stronger. Continuing with the notation of Remark 5.2.2, we observe that the Bessel kernel (5.47) of the operator $(I + \mathcal{R})^{-a/\nu}$ for $0 < a < Q$ satisfies (5.65): there exists a positive constant $C = C(Q, a) > 0$ such that we have, in particular,

$$|\mathcal{B}_a(x)| \leq \begin{cases} C|x|^{-(Q-a)}, & \text{for } x \in \mathbb{G} \setminus \{0\}, \\ C|x|^{-Q}, & \text{for } x \in \mathbb{G} \text{ with } |x| \geq 1. \end{cases} \tag{5.67}$$

We refer to [RY18a] for further details, as well as to the original proof of Theorem 5.2.5 that we follow here.

Proof of Theorem 5.2.5. Let us split the integral in the left-hand side of (5.66) into three parts,

$$\int_{\mathbb{G}} |(f * T_{Q/p}^{(2)})(x)|^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q} \leq 3^q (N_1 + N_2 + N_3), \tag{5.68}$$

where

$$N_1 := \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q},$$

$$N_2 := \int_{\mathbb{G}} \left(\int_{\{|x| \leq 2|y| < 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q}$$

and

$$N_3 := \int_{\mathbb{G}} \left(\int_{\{|y|>2|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)|dy \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}.$$

We begin by *estimating* N_1 . Similar to the argument in (5.50), in the region $2|y| < |x|$ we have

$$|y^{-1}x| \geq |x| - |y| > |x| - \frac{|x|}{2} = \frac{|x|}{2}, \tag{5.69}$$

which is $|x| < 2|y^{-1}x|$. Denote

$$|T_a^{(2)}(x)| \leq \tilde{B}_a(x) := C_2 \begin{cases} |x|^{a-Q}, & \text{for } x \in \mathbb{G} \setminus \{0\}, \\ |x|^{-Q}, & \text{for } x \in \mathbb{G} \text{ with } |x| \geq 1. \end{cases} \tag{5.70}$$

Since $T_{Q/p}^{(2)}(x)$ is bounded by $\tilde{B}_{Q/p}(x)$ which is non-increasing with respect to $|x|$, then using (5.69) we get

$$\begin{aligned} N_1 &\leq \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)|dy \right)^q \left(\sup_{\{|x|<2|z|\}} |T_{Q/p}^{(2)}(z)| \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \\ &\leq \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)|dy \right)^q \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}. \end{aligned}$$

We will now apply Theorem 5.1.1, Part (1), to estimate N_1 . For this we have to check the condition (5.3), that is, that

$$\left(\int_{\{2R<|x|\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx \right)^{\frac{1}{p'}} \leq A_1 \tag{5.71}$$

holds uniformly for all $R > 0$. To verify this uniform boundedness, we consider two cases: $R \geq 1$ and $0 < R < 1$. First, for $R \geq 1$, using the second equality in (5.70), we can estimate

$$\begin{aligned} &\left(\int_{\{2R<|x|\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx \right)^{\frac{1}{p'}} \\ &\leq CR^{\frac{Q}{p'}} \left(\int_{\{2R<|x|\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{|x|^Q} \right)^{\frac{1}{q}} = CR^{\frac{Q}{p'}} \left(\int_{\{2R<|x|\}} |x|^{-Qq-Q} dx \right)^{\frac{1}{q}} \\ &\leq CR^{-Q}R^{\frac{Q}{p'}} \leq C. \end{aligned} \tag{5.72}$$

Next, let us check (5.71) for $0 < R < 1$. We split the integral into two terms,

$$\begin{aligned} & \int_{\{2R < |x|\}} \left(\tilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q} \\ &= \int_{\{2R < |x| < 2\}} \left(\tilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q} \\ &+ \int_{\{|x| \geq 2\}} \left(\tilde{B}_{Q/p}\left(\frac{x}{2}\right)\right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q}. \end{aligned} \tag{5.73}$$

We note that the second integral in the right-hand side of (5.73) is finite by the second equality in (5.70). For the first integral, using the first equality in (5.70), we can estimate

$$\begin{aligned} & \int_{\{2R < |x| < 2\}} \left|\tilde{B}_{Q/p}\left(\frac{x}{2}\right)\right|^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q} \\ & \leq \int_{\{2R < |x| < 2\}} \left|\tilde{B}_{Q/p}\left(\frac{x}{2}\right)\right|^q \frac{dx}{|x|^Q} \\ & \leq C \int_{\{2R < |x| < 2\}} |x|^{-Qq/p' - Q} dx \\ & \leq CR^{-Qq/p'}. \end{aligned}$$

Combining this with (5.73), we obtain

$$\begin{aligned} & \left(\int_{\{2R < |x|\}} \left|\tilde{B}_{Q/p}\left(\frac{x}{2}\right)\right|^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q}\right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx\right)^{\frac{1}{p'}} \\ & \leq C(R^{-Q/p'} + 1)R^{Q/p'} \leq C \end{aligned}$$

uniformly for all $0 < R < 1$. Thus, we have verified (5.71), so that applying Theorem 5.1.1, Part (1), to N_1 we obtain

$$N_1^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_1 \|f\|_{L^p(\mathbb{G})}. \tag{5.74}$$

Now let us estimate N_2 . We decompose it into the sum

$$\begin{aligned} N_2 &= \sum_{k \in \mathbb{Z}} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy\right)^q \\ & \times \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q}. \end{aligned}$$

Since the function $\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q$ is non-decreasing with respect to $|x|$ near the origin, there exists an integer $k_0 \in \mathbb{Z}$ with $k_0 \leq -3$ such that this function is non-decreasing in $|x| \in (0, 2^{k_0+1})$. Fixing this k_0 , we decompose N_2 further as

$$N_2 = N_{21} + N_{22}, \tag{5.75}$$

where

$$N_{21} := \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \times \frac{dx}{\left| \log\left(e + \frac{1}{|x|}\right) \right|^r |x|^Q}$$

and

$$N_{22} := \sum_{k=k_0+1}^{\infty} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \times \frac{dx}{\left| \log\left(e + \frac{1}{|x|}\right) \right|^r |x|^Q}.$$

Let us first estimate N_{22} . Since $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| < 2^{k+1}$, we must also have $2^{k-1} \leq |y| < 2^{k+2}$. Before starting to estimate N_{22} , using (5.65) and $q > p$, let us show that

$$\int_{\mathbb{G}} |T_{Q/p}^{(2)}(x)|^{\tilde{r}} dx = \int_{|x| < 1} |T_{Q/p}^{(2)}(x)|^{\tilde{r}} dx + \int_{|x| \geq 1} |T_{Q/p}^{(2)}(x)|^{\tilde{r}} dx \leq C_2 \left(\int_{|x| < 1} |x|^{-\frac{Qq(p-1)}{pq+p-q}} dx + \int_{|x| \geq 1} |x|^{-\frac{Qpq}{pq+p-q}} dx \right) < \infty, \tag{5.76}$$

where $\tilde{r} \in [1, \infty]$ is such that $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$.

Then, (5.76) and Young’s inequality in Proposition 1.2.13 with $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$ and $\tilde{r} \in [1, \infty]$ imply that

$$\begin{aligned} N_{22} &\leq C \sum_{k=k_0+1}^{\infty} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q dx \\ &\leq C \| [f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}] * T_{Q/p}^{(2)} \|_{L^q(\mathbb{G})}^q \\ &\leq C \| T_{Q/p}^{(2)} \|_{L^{\tilde{r}}(\mathbb{G})}^q \sum_{k=k_0+1}^{\infty} \| f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}} \|_{L^p(\mathbb{G})}^q \\ &= C \sum_{k=k_0+1}^{\infty} \left(\int_{\{2^k \leq |x| < 2^{k+1}\}} |f(x)|^p dx \right)^{q/p} \end{aligned}$$

$$\begin{aligned} &\leq C \left(\sum_{k \in \mathbb{Z}} \int_{\{2^k \leq |x| < 2^{k+1}\}} |f(x)|^p dx \right)^{q/p} \\ &= C \|f\|_{L^p(\mathbb{G})}^q. \end{aligned} \tag{5.77}$$

Next, let us estimate N_{21} . As in (5.69), assuming $|\cdot|$ is the norm and using the triangle inequality and $|y| \leq 2|x|$, we can estimate

$$3|x| = |x| + 2|x| \geq |x| + |y| \geq |y^{-1}x|. \tag{5.78}$$

Since $\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q$ is non-decreasing in $|x| \in (0, 2^{k_0+1})$ and $3|x| \geq |y^{-1}x|$, we have

$$\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q \geq \left(\log\left(\frac{1}{|y^{-1}x/3|}\right)\right)^r \left|\frac{y^{-1}x}{3}\right|^Q.$$

Consequently, this and (5.65) yield

$$\begin{aligned} N_{21} &\leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |y^{-1}x|^{-\frac{Q}{p'}} |f(y)| dy \right)^q \\ &\quad \times \frac{dx}{\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q} \\ &= C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|y^{-1}x|^{-\frac{Q}{p'}} |f(y)|}{\left(\log\left(\frac{1}{|x|}\right)\right)^r |x|^Q} dy \right)^q dx \\ &\leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|y^{-1}x|^{-\frac{Q}{p'}} |f(y)|}{\left(\log\left(\frac{1}{|(y^{-1}x)/3|}\right)\right)^r |(y^{-1}x)/3|^Q} dy \right)^q dx. \end{aligned}$$

Since $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| < 2^{k+1}$ with $k \leq k_0$, we must also have $2^{k-1} \leq |y| < 2^{k+2}$ and $|y^{-1}x| \leq 3|x| < 3 \cdot 2^{k_0+1} \leq 3/4$, using (5.78) and $k_0 \leq -3$. Taking into account these and setting

$$g(x) := \frac{\chi_{B_{\frac{3}{4}}(0)}(x)}{\left(\log\left(\frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q} + \frac{Q}{p'}}$$

we have for N_{21} that

$$\begin{aligned} N_{21} &\leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|f(y)|}{\left(\log\left(\frac{1}{|y^{-1}x|\right)\right)^{\frac{r}{q}} |y^{-1}x|^{\frac{Q}{q} + \frac{Q}{p'}}} dy \right)^q dx \\ &\leq C \sum_{k=-\infty}^{k_0} \|[f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}] * g\|_{L^q(\mathbb{G})}^q. \end{aligned}$$

Since $p < q < (r - 1)p'$, we use Young's inequality in Proposition 1.2.13 with $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$ and $\tilde{r} \in [1, \infty)$, to get

$$N_{21} \leq C \|g\|_{L^{\tilde{r}}(\mathbb{G})}^q \sum_{k=-\infty}^{k_0} \|f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \leq C \|f\|_{L^p(\mathbb{G})}^q, \tag{5.79}$$

provided that $g \in L^{\tilde{r}}(\mathbb{G})$. Since $\left(\frac{Q}{q} + \frac{Q}{p'}\right) \tilde{r} = Q$, $\frac{r\tilde{r}}{q} = \frac{rp'}{p'+q}$ and $q < (r - 1)p'$, then changing variables, we obtain

$$\|g\|_{L^{\tilde{r}}(\mathbb{G})}^{\tilde{r}} = \int_{B(0,3/4)} \frac{dx}{\left(\log\left(\frac{1}{x}\right)\right)^{\frac{rp'}{p'+q}} |x|^Q} = C \int_{\log(\frac{4}{3})}^{\infty} \frac{dt}{t^{\frac{rp'}{p'+q}}} < \infty.$$

Let us estimate N_3 now. Without loss of generality, we may assume again that $|\cdot|$ is the norm. Similarly to (5.69) we obtain $|y| < 2|y^{-1}x|$ from $2|x| < |y|$. Then, we have for N_3 that

$$N_3 \leq \int_{\mathbb{G}} \left(\int_{\{|y|>2|x|\}} \left| \tilde{B}_{Q/p} \left(\frac{y}{2} \right) \right| |f(y)| dy \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}.$$

We will apply Theorem 5.1.1, Part (2), for the required estimate of N_3 . For this we have to check the following condition:

$$\left(\int_{\{|x|<R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \leq A_2. \tag{5.80}$$

To check this, let us consider the cases: $R \geq 1$ and $0 < R < 1$. Then, for $R \geq 1$ by the second equality in (5.70), we get

$$\left(\int_{\{2R<|x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \leq C \left(\int_{\{2R<|x|\}} |x|^{-Qp'} dx \right)^{\frac{1}{p'}} \leq CR^{-\frac{Q}{p}}. \tag{5.81}$$

Moreover, we have

$$\begin{aligned} \int_{\{|x|<R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} &= \int_{\{|x|<\frac{1}{2}\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \\ &\quad + \int_{\{\frac{1}{2} \leq |x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}, \end{aligned}$$

and we note that the first summand in the right-hand side of above is finite since $r > 1$. For the second term, we get

$$\int_{\{\frac{1}{2} \leq |x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \leq \int_{\{\frac{1}{2} \leq |x| < R\}} \frac{dx}{|x|^Q} \leq C(1 + \log R). \tag{5.82}$$

Combining (5.81) and (5.82), we have for $R \geq 1$ that

$$\begin{aligned} \left(\int_{\{|x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \\ \leq CR^{-\frac{Q}{p}} (1 + \log R)^{\frac{1}{q}} \leq C. \end{aligned}$$

Now let us check the condition (5.80) for $0 < R < 1$. We split the integral into two terms:

$$\begin{aligned} \int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \\ = \int_{\{2R < |x| < 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx + \int_{\{|x| \geq 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx. \end{aligned} \tag{5.83}$$

We note that the second integral in the right-hand side of above is finite by the second equality in (5.70). Then, using the first equality in (5.70) we get for the first integral that

$$\int_{\{2R < |x| < 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \leq C \int_{\{2R < |x| < 2\}} |x|^{-Q} dx \leq C \log \left(\frac{1}{R} \right).$$

Combined with (5.83), it follows that

$$\int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \leq C \left(1 + \log \left(\frac{1}{R} \right) \right). \tag{5.84}$$

Since

$$\int_{\{|x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \leq C \left(\log \left(e + \frac{1}{R} \right) \right)^{-(r-1)},$$

and (5.84), and taking into account $r > 1$ and $q < (r - 1)p'$ we obtain that

$$\begin{aligned} \left(\int_{\{|x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \\ \leq C \left(\log \left(e + \frac{1}{R} \right) \right)^{-\frac{r-1}{q}} \left(1 + \left(\log \left(\frac{1}{R} \right) \right)^{\frac{1}{p'}} \right) \leq C. \end{aligned} \tag{5.85}$$

Thus, we have checked (5.80). Consequently, applying Theorem 5.1.1, Part (2), for the term N_3 , we obtain

$$N_3^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_2 \|f\|_{L^p(\mathbb{G})}. \tag{5.86}$$

Finally, a combination of (5.74), (5.86), (5.75), (5.77), (5.79) and (5.68) completes the proof of Theorem 5.2.5. \square

5.3 Hardy–Littlewood–Sobolev inequalities on homogeneous groups

In this section we discuss the Hardy–Littlewood–Sobolev inequality on homogeneous groups. We show that it can be obtained as a simple consequence of the convolution Hardy inequality in Theorem 5.2.1. In fact, the argument implies a little more.

Theorem 5.3.1 (Hardy–Littlewood–Sobolev inequality). *Let \mathbb{G} be a homogeneous Lie group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $0 < \lambda < Q$ and $1 < p, q < \infty$ be such that*

$$1/p + 1/q + (\alpha + \lambda)/Q = 2$$

with $0 \leq \alpha < Q/p'$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha) > 0$ such that

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda} dx dy \right| \leq C \|f\|_{L^p(\mathbb{G})} \|g\|_{L^q(\mathbb{G})} \tag{5.87}$$

holds for all $f \in L^p(\mathbb{G})$ and $g \in L^q(\mathbb{G})$.

Remark 5.3.2 (Stein–Weiss inequality).

1. The original Hardy–Littlewood–Sobolev inequality goes back to the work of Hardy–Littlewood [HL27], [HL30] and Sobolev [Sob38]. More specifically, in [HL27], Hardy and Littlewood considered the one-dimensional fractional operator on $(0, \infty)$, given by

$$T_\lambda f(x) = \int_0^\infty \frac{f(y)}{|x - y|^\lambda} dy, \quad 0 < \lambda < 1, \tag{5.88}$$

and proved that if $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} + \lambda - 1$, then there is $C > 0$ such that

$$\|T_\lambda f\|_{L^q(0, \infty)} \leq C \|f\|_{L^p(0, \infty)},$$

holds for all $f \in L^p(0, \infty)$. The N -dimensional analogue of (5.88) can be written by the formula

$$I_\lambda f(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x - y|^\lambda} dy, \quad 0 < \lambda < N. \tag{5.89}$$

Consequently, it was shown by Sobolev in [Sob38] that if $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{N} - 1$, then there is $C > 0$ such that

$$\|I_\lambda f\|_{L^q(\mathbb{R}^N)} \leq C \|f\|_{L^p(\mathbb{R}^N)},$$

holds for all $f \in L^p(\mathbb{R}^N)$. In [SW58], Stein and Weiss obtained the following two-weight extension of the Hardy–Littlewood–Sobolev inequality, which is nowadays called the *Stein–Weiss inequality*. More specifically, they have shown that if $0 < \lambda < N$, $1 < p < \infty$, $\alpha < \frac{N(p-1)}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \geq 0$,

$\frac{1}{q} = \frac{1}{p} + \frac{\lambda + \alpha + \beta}{N} - 1$, and $1 < p \leq q < \infty$, then there is $C > 0$ such that we have

$$\| |x|^{-\beta} I_\lambda f \|_{L^q(\mathbb{R}^N)} \leq C \| |x|^\alpha f \|_{L^p(\mathbb{R}^N)}. \tag{5.90}$$

2. An extension of the Hardy–Littlewood–Sobolev inequality to the Heisenberg groups was considered in [FS74]. The sharp constants in the Hardy–Littlewood–Sobolev inequality in the cases of \mathbb{R}^N and the Heisenberg group were obtained in [Lie83] and [FL12], respectively.
3. In [GMS10] the analogues of the Stein–Weiss inequality were obtained on Carnot groups. Note that in [HLZ12] the authors also proved an analogue of the Stein–Weiss inequality on the Heisenberg groups.
4. On general homogeneous groups the statement of Theorem 5.3.1 will be here obtained as a consequence of the integral Hardy inequalities. The estimate (5.87) contains the Hardy–Littlewood–Sobolev inequality and half of the Stein–Weiss inequality. The full Stein–Weiss inequality on homogeneous groups was obtained in [KRS18b]: *Let*

$$I_\lambda u(x) := \int_{\mathbb{G}} \frac{u(y)}{|y^{-1}x|^\lambda} dy, \quad 0 < \lambda < Q. \tag{5.91}$$

Let $0 < \lambda < Q$, $1 < p < \infty$, $\alpha < \frac{Q}{p'}$, $\beta < \frac{Q}{q}$, $\alpha + \beta \geq 0$, $\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta + \lambda}{Q} - 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for $1 < p \leq q < \infty$ we have

$$\| |x|^{-\beta} I_\lambda u \|_{L^q(\mathbb{G})} \leq C \| |x|^\alpha u \|_{L^p(\mathbb{G})}. \tag{5.92}$$

5. **(Differential Stein–Weiss inequality on graded groups).** Continuing with the notation of Remark 5.2.2, let $\dot{L}_a^p(\mathbb{G})$ be the homogeneous Sobolev space over L^p of order a associated to a Rockland operator. Such spaces are well defined on graded groups and do not depend on a particular choice of a Rockland operator, we refer to [FR17] or to [FR16, Section 4.4] for the extensive analysis and exposition of their properties. The following differential version of the Stein–Weiss inequality was obtained in [RY18a]:

Theorem 5.3.3 (Differential Stein–Weiss inequality). *Let \mathbb{G} be a graded group of homogeneous dimension Q and let $|\cdot|$ be a quasi-norm on \mathbb{G} . Let $1 < p, q < \infty$, $0 \leq a < Q/p$ and $0 \leq b < Q/q$. Let $0 < \lambda < Q$, $0 \leq \alpha < a + Q/p'$ and $0 \leq \beta \leq b$ be such that $(Q - ap)/(pQ) + (Q - q(b - \beta))/(qQ) + (\alpha + \lambda)/Q = 2$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha, \beta, a, b)$ such that*

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda |y|^\beta} dx dy \right| \leq C \|f\|_{\dot{L}_a^p(\mathbb{G})} \|g\|_{\dot{L}_b^q(\mathbb{G})} \tag{5.93}$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$ and $g \in \dot{L}_b^q(\mathbb{G})$.

While the setting of graded groups falls outside the scope of this book, we follow [RY18a] in the proof of Theorem 5.3.1 below.

Proof of Theorem 5.3.1. Let $T_a(x) := |x|^{a-Q}$ with $0 < a < Q/r$ for some $1 < r < \infty$. Then, using Hölder’s inequality we have

$$\begin{aligned} \left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)g(y)}}{|x|^\alpha |y^{-1}x|^\lambda} dx dy \right| &= \left| \int_{\mathbb{G}} \overline{f(x)} \frac{(g * T_{Q-\lambda})(x)}{|x|^\alpha} dx \right| \\ &\leq \|f\|_{L^p(\mathbb{G})} \left\| \frac{g * T_{Q-\lambda}}{|x|^\alpha} \right\|_{L^{p'}(\mathbb{G})}. \end{aligned} \tag{5.94}$$

Note that the conditions $\alpha + \lambda \leq Q$ and $1/p + 1/q + (\alpha + \lambda)/Q = 2$ imply $q \leq p'$, while $0 < \lambda < Q$, $\alpha < Q/p'$ and $1/p + 1/q + (\alpha + \lambda)/Q = 2$ give

$$0 < Q - \lambda = Q - Q \left(2 - \frac{1}{p} - \frac{1}{q} \right) + \alpha < Q - Q \left(2 - \frac{1}{p} - \frac{1}{q} \right) + \frac{Q}{p'} = Q/q.$$

Since we have $1 < q \leq p' < \infty$, $0 \leq \alpha p' < Q$, $0 < Q - \lambda < Q/q$ and $(Q - \lambda)/Q = 1/q - 1/p' + \alpha/Q$, using Theorem 5.2.1 in (5.94) we obtain (5.87). \square

Remark 5.3.4 (Reversed Hardy–Littlewood–Sobolev inequality). Let us make some remarks concerning the reversed Hardy–Littlewood–Sobolev inequality on homogeneous groups. Namely, consider the inequality

$$\int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy \geq C_{Q,\lambda,p} \|f\|_{L^1(\mathbb{G})}^\theta \|f\|_{L^p(\mathbb{G})}^{2-\theta} \tag{5.95}$$

for any $0 \leq f \in L^1 \cap L^p(\mathbb{G})$ with $f \not\equiv 0$ and $0 < p < 1$, where $\lambda > 0$ and $\theta := (2Q - p(2Q + \lambda))/(Q(1 - p))$.

In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$, i.e., with $Q = n$, the case $p = 2n/(2n + \lambda)$ was investigated in [DZ15] and [NN17], while the case $p > n/(n + \lambda)$ was studied in [DFH18].

Following [RY18a], let us briefly recapture the argument in the setting of general homogeneous groups. Namely, let us show that in the case $0 < p \leq Q/(Q + \lambda)$ the inequality (5.95) is not valid, namely, (5.95) fails for any $C_{Q,\lambda,p} > 0$. In the Euclidean case this was shown in [CDP18] when $p < n/(n + \lambda)$ and in [DFH18] when $p \leq n/(n + \lambda)$.

Let f be a non-negative function with compact support, and let h be a non-negative smooth function such that $\int_{\mathbb{G}} h(x) dx = 1$. Then, for some $A > 0$, consider the function

$$f_\varepsilon(x) := f(x) + A\varepsilon^{-Q} h(x/\varepsilon).$$

Suppose now that (5.95) holds for some $C_{Q,\lambda,p} > 0$. Putting this f_ε in the inequality (5.95), we obtain

$$\begin{aligned} C_{Q,\lambda,p} &\leq \frac{\int_{\mathbb{G}} \int_{\mathbb{G}} f_\varepsilon(x) |y^{-1}x|^\lambda f_\varepsilon(y) dx dy}{\|f_\varepsilon\|_{L^1(\mathbb{G})}^\theta \|f_\varepsilon\|_{L^p(\mathbb{G})}^{2-\theta}} \\ &\rightarrow \frac{\int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy + 2A \int_{\mathbb{G}} |x|^\lambda f(x) dx}{(\int_{\mathbb{G}} f(x) dx + A)^\theta (\int_{\mathbb{G}} (f(x))^p dx)^{(2-\theta)/p}} \end{aligned} \tag{5.96}$$

as $\varepsilon \rightarrow 0_+$, where we have used that

- $\int_{\mathbb{G}} f_\varepsilon(x) dx = \int_{\mathbb{G}} f(x) dx + A$;
- when $\varepsilon \rightarrow 0_+$, we have

$$\int_{\mathbb{G}} (f_\varepsilon(x))^p dx \rightarrow \int_{\mathbb{G}} (f(x))^p dx;$$

- when $\varepsilon \rightarrow 0_+$, we have

$$\begin{aligned} & \int_{\mathbb{G}} \int_{\mathbb{G}} f_\varepsilon(x) |y^{-1}x|^\lambda f_\varepsilon(y) dx dy \\ &= \int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy + 2A \int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |(\varepsilon^{-1}y)^{-1}x|^\lambda h(y) dx dy \\ & \quad + A^2 \varepsilon^{-2Q} \int_{\mathbb{G}} \int_{\mathbb{G}} h\left(\frac{x}{\varepsilon}\right) h\left(\frac{y}{\varepsilon}\right) dx dy \\ & \rightarrow \int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy + 2A \int_{\mathbb{G}} |x|^\lambda f(x) dx, \end{aligned}$$

since $\int_{\mathbb{G}} h(x) dx = 1$.

Note that in (5.96) we can take also the limit as $A \rightarrow +\infty$ since it is valid for all $A > 0$. Then, when $\theta > 1$, that is, for $p < Q/(Q + \lambda)$, taking $A \rightarrow +\infty$ in (5.96) we see that $C_{Q,\lambda,p} = 0$. In the case $\theta = 1$, that is, for $p = Q/(Q + \lambda)$, taking the limit as $A \rightarrow +\infty$ in (5.96) we get

$$C_{Q,\lambda,p} \leq \frac{2 \int_{\mathbb{G}} |x|^\lambda f(x) dx}{\left(\int_{\mathbb{G}} (f(x))^p dx\right)^{1/p}}. \tag{5.97}$$

Finally, we show that the right-hand side of (5.97) goes to zero as $R \rightarrow \infty$ if we insert the function

$$f_R(x) = \begin{cases} |x|^{-(Q+\lambda)}, & \text{for } 1 \leq |x| \leq R, \\ 0, & \text{otherwise,} \end{cases} \tag{5.98}$$

for any $R > 1$. Indeed, in this case $p = Q/(Q + \lambda)$, and from (5.97) we obtain that

$$C_{Q,\lambda,p} \leq \frac{2 \int_{\mathbb{G}} |x|^\lambda f_R(x) dx}{\left(\int_{\mathbb{G}} (f_R(x))^p dx\right)^{1/p}} = 2(|\varphi| \log R)^{-\lambda/Q} \rightarrow 0 \tag{5.99}$$

as $R \rightarrow \infty$, where $|\varphi|$ is a $Q - 1$ -dimensional surface measure of the unit quasi-sphere in \mathbb{G} .

Summarizing, we conclude that for $0 < p \leq Q/(Q + \lambda)$ the reversed Hardy–Littlewood–Sobolev inequality (5.95) is not valid with any constant $C_{Q,\lambda,p} > 0$.

5.4 Maximal weighted integral Hardy inequality

Here we present a maximal Hardy inequality in the integral form, involving the maximal function

$$(\mathcal{M}f)(x) := \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f(z) dz.$$

Theorem 5.4.1 (Maximal integral weighted Hardy inequality). *Let \mathbb{G} be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let ϕ and ψ be positive functions defined on \mathbb{G} . Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{G}} \phi(x) \exp(\mathcal{M} \log f)(x) dx \leq C \int_{\mathbb{G}} \psi(x) f(x) dx \tag{5.100}$$

holds for all positive $f \geq 0$ if and only if

$$A := \sup_{R>0} R^Q \int_{|x| \geq R} \frac{\phi(x) \exp\left(\mathcal{M} \log \frac{1}{\psi}\right)(x)}{|x|^{2Q}} dx < \infty. \tag{5.101}$$

Remark 5.4.2. Inequalities of the type of those in Theorem 5.4.1 in the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ were studied in [HKK01] for the one-dimensional case $n = 1$, and in [DHK97] for the multidimensional case $n \geq 1$. Theorem 5.4.1 was proved in [RSY18a] and we follow the presentation there.

Proof of Theorem 5.4.1. Let us first show (5.101) implies (5.100) for all $f \geq 0$. Denoting

$$W_3(x) := \phi(x) \exp\left(\mathcal{M} \log \frac{1}{\psi}\right)(x), \tag{5.102}$$

as well as $u(x) := f(x)\psi(x)$, and $z = |x|\xi$, we have

$$\begin{aligned} & \int_{\mathbb{G}} \phi(x) \exp(\mathcal{M} \log f)(x) dx \\ &= \int_{\mathbb{G}} \phi(x) \exp\left(\frac{1}{|B(0, |x|)|} \left(\int_{B(0, |x|)} \log\left(\frac{1}{\phi}\right)(z) dz + \int_{|z| \leq |x|} \log(\phi f)(z) dz\right)\right) dx \\ &= \int_{\mathbb{G}} \phi(x) \exp\left(\mathcal{M} \log \frac{1}{\phi}\right)(x) \exp\left(\frac{1}{|B(0, |x|)|} \int_{|z| \leq |x|} \log(\phi(z) f(z)) dz\right) dx \\ &= \int_{\mathbb{G}} W_3(x) \exp\left(\frac{1}{|B(0, |x|)|} \int_{|z| \leq |x|} \log(u(z)) dz\right) dx \\ &= \int_{\mathbb{G}} W_3(x) \exp\left(\frac{1}{|x|^Q |B(0, 1)|} \int_{B(0, 1)} \log(u(|x|\xi)) |x|^Q d\xi\right) dx. \end{aligned} \tag{5.103}$$

Since

$$\int_{B(0, 1)} \log(|\xi|^Q) d\xi = Q \int_{\varphi} \int_0^1 r^{Q-1} \log r dr d\sigma(y) = -|B(0, 1)|,$$

and by using Jensen's inequality, we obtain

$$\begin{aligned}
 & \int_{\mathbb{G}} \phi(x) \exp(\mathcal{M} \log f)(x) dx \\
 &= \int_{\mathbb{G}} W_3(x) \exp \left(\frac{1}{|B(0, 1)|} \left(\int_{B(0,1)} \log(|\xi|^Q u(|x|\xi)) d\xi - \int_{B(0,1)} \log(|\xi|^Q) d\xi \right) \right) dx \\
 &= \int_{\mathbb{G}} W_3(x) \exp \left(\frac{1}{|B(0, 1)|} \int_{B(0,1)} \log(|\xi|^Q u(|x|\xi)) d\xi + 1 \right) dx \\
 &= e \int_{\mathbb{G}} W_3(x) \exp \left(\frac{1}{|B(0, 1)|} \int_{B(0,1)} \log(|\xi|^Q u(|x|\xi)) d\xi \right) dx \\
 &\leq \frac{e}{|B(0, 1)|} \int_{\mathbb{G}} W_3(x) \int_{B(0,1)} |\xi|^Q u(|x|\xi) d\xi dx \\
 &= \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^\infty r^{Q-1} W_3(rw) \int_0^1 s^{2Q-1} u(rsy) ds dr d\sigma(y) d\sigma(w),
 \end{aligned}$$

where $|\xi| = s$ and $|x| = r$. Furthermore, with $t = rs$ we get

$$\begin{aligned}
 & \int_{\mathbb{G}} \phi(x) \exp(\mathcal{M} \log f)(x) dx \\
 &\leq \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^\infty r^{Q-1} W_3(rw) \int_0^1 s^{2Q-1} u(rsy) ds dr d\sigma(y) d\sigma(w) \\
 &= \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^1 s^{2Q-1} \int_0^\infty W_3 \left(\frac{t}{s} w \right) \left(\frac{t}{s} \right)^{Q-1} u(ty) \frac{dt}{s} ds d\sigma(y) d\sigma(w) \\
 &= \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^1 s^{Q-1} \int_0^\infty W_3 \left(\frac{t}{s} w \right) t^{Q-1} u(ty) dt ds d\sigma(y) d\sigma(w) \\
 &= \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^\infty t^{Q-1} u(ty) \left(\int_0^1 s^{Q-1} W_3 \left(\frac{tw}{s} \right) ds \right) dt d\sigma(y) d\sigma(w) \\
 &= \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^\infty t^{Q-1} u(ty) \left(\int_t^\infty \left(\frac{t}{r} \right)^{Q-1} W_3(rw) t \frac{dr}{r^2} \right) dt d\sigma(y) d\sigma(w) \\
 &= \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^\infty t^{2Q-1} u(ty) \left(\int_t^\infty r^{-Q-1} W_3(rw) dr \right) dt d\sigma(y) d\sigma(w) \\
 &\leq \frac{e}{|B(0, 1)|} \int_{\varphi} \int_{\varphi} \int_0^\infty t^{Q-1} u(ty) t^Q \left(\int_t^\infty \frac{r^{Q-1} W_3(rw) dr}{r^{2Q}} \right) dt d\sigma(w) d\sigma(y) \\
 &= \frac{e}{|B(0, 1)|} \int_{\mathbb{G}} u(x) \left(|x|^Q \int_{|z| \geq |x|} \frac{W_3(z)}{|z|^{2Q}} dz \right) dx \leq A \frac{e}{|B(0, 1)|} \int_{\mathbb{G}} u(x) dx,
 \end{aligned}$$

yielding (5.100), where we have used (5.101) in the last line.

We now show that (5.100) implies (5.101). From (5.103) we notice that (5.100) is equivalent to

$$\int_{\mathbb{G}} W_3(x) \exp\left(\frac{1}{|B(0, |x|)|} \int_{|z| \leq |x|} \log(u(z)) dz\right) dx \leq C \int_{\mathbb{G}} u(x) dx. \quad (5.104)$$

Furthermore, for a function

$$u(x) = R^{-Q} \chi_{(0, R)}(|x|) + e^{-2Q} |x|^{-2Q} R^Q \chi_{(R, \infty)}(|x|), \quad x \in \mathbb{G}, \quad R > 0, \quad (5.105)$$

we have

$$\begin{aligned} & \int_{\mathbb{G}} W_3(x) \exp\left(\frac{1}{|B(0, |x|)|} \int_{|z| \leq |x|} \log(u(z)) dz\right) dx \\ & \leq C \int_{\mathbb{G}} u(x) dx = C \int_{\varphi} \int_0^{\infty} s^{Q-1} u(s) ds d\sigma(y) \\ & = C |\varphi| \left(\int_0^R s^{Q-1} R^{-Q} ds + \int_R^{\infty} e^{-2Q} s^{Q-1} R^Q s^{-2Q} ds \right) \\ & = C |\varphi| \left(\frac{1}{Q} + \frac{e^{-2Q}}{Q} \right) =: C(Q) < \infty, \end{aligned}$$

since χ is the cut-off function. Thus, from this, by plugging (5.105) into the left-hand side of (5.104) we calculate

$$\begin{aligned} \infty > C(Q) & \geq \int_{\mathbb{G}} W_3(x) \exp\left(\frac{1}{|B(0, |x|)|} \int_{|z| \leq |x|} \log(u(z)) dz\right) dx \\ & = \int_{\varphi} \int_0^{\infty} s^{Q-1} W_3(sy) \exp\left(\frac{1}{|B(0, s)|} \int_{\varphi} \int_0^s r^{Q-1} \log(u(r)) dr d\sigma(w)\right) ds d\sigma(y) \\ & = \int_{\varphi} \int_0^{\infty} s^{Q-1} W_3(sy) \exp\left(\frac{|\varphi|}{s^Q |B(0, 1)|} \int_0^s r^{Q-1} \log(u(r)) dr\right) ds d\sigma(y) \\ & = \int_{\varphi} \left(\int_0^R s^{Q-1} W_3(sy) \exp\left(\frac{|\varphi|}{s^Q |B(0, 1)|} \int_0^s r^{Q-1} \log(u(r)) dr\right) ds \right) d\sigma(y) \\ & \quad + \int_{\varphi} \left(\int_R^{\infty} s^{Q-1} W_3(sy) \exp\left(\frac{|\varphi|}{s^Q |B(0, 1)|} \left(\int_0^R r^{Q-1} \log(R^{-Q}) dr \right. \right. \right. \\ & \quad \left. \left. \left. + \int_R^s r^{Q-1} \log(e^{-2Q} r^{-2Q} R^Q) dr \right) \right) ds \right) d\sigma(y) \\ & \geq \int_{\varphi} \left(\int_R^{\infty} s^{Q-1} W_3(sy) \exp\left(\frac{|\varphi|}{s^Q |B(0, 1)|} \left(\int_0^R r^{Q-1} \log(R^{-Q}) dr \right. \right. \right. \\ & \quad \left. \left. \left. + \int_R^s r^{Q-1} \log(e^{-2Q} r^{-2Q} R^Q) dr \right) \right) ds \right) d\sigma(y) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\varphi} \left(\int_R^\infty s^{Q-1} W_3(sy) \exp \left(\frac{|\varphi|}{s^Q |B(0,1)|} \left(\int_0^R r^{Q-1} \log(R^{-Q}) dr \right. \right. \right. \\
 &\quad \left. \left. \left. + \int_R^s r^{Q-1} \log(e^{-2Q}) dr - 2Q \int_R^s r^{Q-1} (\log r) dr \right. \right. \right. \\
 &\quad \left. \left. \left. + \int_R^s r^{Q-1} \log(R^Q) dr \right) \right) ds \right) d\sigma(y) \\
 &= \int_{\varphi} \left(\int_R^\infty s^{Q-1} W_3(sy) \exp \left(\frac{|\varphi|}{s^Q |B(0,1)|} \left(\frac{R^Q \log(R^{-Q})}{Q} - 2Q \frac{s^Q - R^Q}{Q} \right. \right. \right. \\
 &\quad \left. \left. \left. - 2Q \left[\frac{r^Q \log r}{Q} - \frac{r^Q}{Q^2} \right]_R^s + \frac{s^Q - R^Q}{Q} \log(R^Q) \right) \right) ds \right) d\sigma(y) \\
 &= \int_{\varphi} \int_R^\infty s^{Q-1} W_3(sy) \\
 &\quad \exp \left(\frac{|\varphi|}{|B(0,1)|} \left(-2 + \frac{2R^Q}{s^Q} - 2 \log s + \frac{2}{Q} - \frac{2R^Q}{Qs^Q} + \log R \right) \right) ds d\sigma(y) \\
 &\geq e^{(2-2Q)} \int_{\varphi} \int_R^\infty s^{Q-1} W_3(sy) \frac{R^{\frac{|\varphi|}{|B(0,1)|}}}{s^{\frac{2|\varphi|}{|B(0,1)|}}} ds d\sigma(y) \\
 &= e^{2-2Q} R^Q \int_{|x| \geq R} \frac{W_3(x)}{|x|^{2Q}} dx = e^{2-2Q} R^Q \int_{|x| \geq R} \frac{\phi(x) \exp \left(\mathcal{M} \log \frac{1}{\psi} \right) (x)}{|x|^{2Q}} dx,
 \end{aligned}$$

which implies (5.101), where we have used $\frac{|\varphi|}{|B(0,1)|} = Q$, $\frac{2R^Q}{s^Q} - \frac{2R^Q}{Qs^Q} > 0$, and (5.102) in the last two lines. □

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