Chapter 4

Fractional Hardy Inequalities

In this chapter we present results concerning fractional forms of Hardy inequalities. Such a topic is well investigated in the Abelian Euclidean setting and we will be providing relevant references in the sequel. For a general survey of fractional Laplacians in the Euclidean setting see, e.g., [Gar17]. However, as usual, the general approach based on homogeneous groups allows one to get insights also in the Abelian case, for example, from the point of view of the possibility of choosing an arbitrary quasi-norm. Moreover, another application of the setting of homogeneous groups is that the results can be equally applied to both elliptic and subelliptic problems.

We start by discussing fractional Sobolev and Hardy inequalities on the homogeneous groups. As a consequence of these inequalities, we derive a Lyapunov type inequality for the fractional p-sub-Laplacian, which also implies an estimate of the first eigenvalue in a quasi-ball for the Dirichlet fractional p-sub-Laplacian. We also extend this analysis to systems of fractional p-sub-Laplacians and to Riesz potential operators.

4.1 Gagliardo seminorms and fractional p-sub-Laplacians

Throughout this chapter $\mathbb{G}$ will be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. We start with the definition of the fractional p-sub-Laplacian.

Definition 4.1.1 (Fractional p-sub-Laplacian)**.** Let $p > 1$ and let $s \in (0,1)$. For a measurable and compactly supported function u the *fractional p-sub-Laplacian* $(-\Delta_{p,|\cdot|})^s = (-\Delta_p)^s$ on $\mathbb{G}$ is defined by the formula

$$(-\Delta_p)^s u(x) := 2 \lim_{\delta \searrow 0} \int_{\mathbb{G}\setminus B(x,\delta)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|y^{-1}x|^{Q+sp}} dy, \quad x \in \mathbb{G}, \quad (4.1)$$

M. Ruzhansky, D. Suragan, *Hardy Inequalities on Homogeneous Groups*,
Progress in Mathematics 327, https://doi.org/10.1007/978-3-030-02895-4_5

where $B(x,\delta) = B_{|\cdot|}(x,\delta)$ is a quasi-ball with respect to the quasi-norm $|\cdot|$, with radius δ centred at $x \in \mathbb{G}$.

Definition 4.1.2 (Gagliardo seminorm and fractional Sobolev spaces)**.** For a measurable function $u : \mathbb{G} \to \mathbb{R}$, its *Gagliardo seminorm* is defined as

$$[u]_{s,p,|\cdot|} = [u]_{s,p} := \left(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{1/p}. \tag{4.2}$$

For $p \geq 1$ and $s \in (0,1)$, the functional space

$$W^{s,p}(\mathbb{G}) = \{u \in L^p(\mathbb{G}) : u \text{ is measurable and } [u]_{s,p} < +\infty\}, \tag{4.3}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{G})} := (\|u\|^p_{L^p(\mathbb{G})} + [u]^p_{s,p})^{1/p}, \quad u \in W^{s,p}(\mathbb{G}), \tag{4.4}$$

is called the *fractional Sobolev space on* $\mathbb{G}$. Sometimes, to emphasize the dependence on a particular quasi-norm, we may write $[u]_{s,p,|\cdot|}$ and $W^{s,p,|\cdot|}$ but we note that the space $W^{s,p,|\cdot|}$ is independent of a particular choice of a quasi-norm due to their equivalence, see Proposition 1.2.3.

Similarly, if $\Omega \subset \mathbb{G}$ is a Haar measurable set, we define the *fractional Sobolev space* $W^{s,p}(\Omega)$ on Ω by

$$W^{s,p}(\Omega) = \Big\{ u \in L^p(\Omega) : u \text{ is measurable and } \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{1/p} < +\infty \Big\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{1/p}. \tag{4.5}$$

Moreover, the Sobolev space $W_0^{s,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{s,p}(\Omega)}$.

4.2 Fractional Hardy inequalities on homogeneous groups

In the present section we establish fractional Hardy inequalities on homogeneous groups.

Theorem 4.2.1 (Fractional Hardy inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $p > 1$, $s \in (0,1)$ and $Q > sp$. Then for all $u \in C_0^\infty(\mathbb{G})$ we have*

$$2\mu(\gamma)\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{ps}}dx \leq [u]^p_{s,p,|\cdot|}, \tag{4.6}$$

where $\mu(\gamma)$ is defined in (4.10).

Remark 4.2.2. In [AB17] the authors studied the weighted fractional p-Laplacian and established the following weighted fractional L^p-Hardy inequality:

$$C\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|_E^{ps+2\beta}}dx \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|_E^{N+ps}|x|_E^\beta |y|_E^\beta}dxdy, \tag{4.7}$$

where $\beta < \frac{N-ps}{2}$, $u \in C_0^\infty(\mathbb{R}^N)$, $C > 0$ is a positive constant, and $|\cdot|_E$ is the Euclidean distance in $\mathbb{R}^N$.

Before we prove Theorem 4.2.1, let us establish the following two lemmas that will be instrumental in the proof.

Lemma 4.2.3. *We fix a homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$. Let $\omega \in W_0^{s,p}(\Omega)$ and assume that $w > 0$ in $\Omega \subset \mathbb{G}$. Assume that $(-\Delta_p)^s\omega = \nu > 0$ with $\nu \in L^1_{\mathrm{loc}}(\Omega)$. Then for all $u \in C_0^\infty(\Omega)$, we have*

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+ps}}dxdy \geq \left\langle (-\Delta_p)^s\omega, \frac{|u|^p}{\omega^{p-1}}\right\rangle.$$

Proof of Lemma 4.2.3. Using the notations

$$v := \frac{|u|^p}{|\omega|^{p-1}} \text{ and } k(x,y) := \frac{1}{|y^{-1}x|^{Q+ps}},$$

we get

$$\begin{aligned}\langle(-\Delta_p)^s\omega(x), v(x)\rangle &= \int_\Omega v(x)dx\int_\Omega |\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dy\\ &= \int_\Omega \frac{|u(x)|^p}{|\omega(x)|^{p-1}}dx\int_\Omega |\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dy.\end{aligned}$$

Let us show that $k(x,y)$ is symmetric, that is, $k(x,y)=k(y,x)$ for all $x,y\in\mathbb{G}$. This readily follows, since by the definition of the quasi-norm we have $|x^{-1}|=|x|$ for all $x\in\mathbb{G}$. So, since $k(x,y)$ is symmetric, we obtain

$$\begin{aligned}&\langle(-\Delta_p)^s\omega(x),v(x)\rangle\\&=\frac{1}{2}\int_\Omega\int_\Omega\left(\frac{|u(x)|^p}{|\omega(x)|^{p-1}}-\frac{|u(y)|^p}{|\omega(y)|^{p-1}}\right)|\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dydx.\end{aligned}$$

Let $g:=\frac{u}{\omega}$ and

$$R(x,y):=|u(x)-u(y)|^p-(|g(x)|^p\omega(x)-|g(y)|^p\omega(y))|\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y)).$$

Then we have

$$\langle(-\Delta_p)^s\omega,v\rangle+\frac{1}{2}\int_\Omega\int_\Omega R(x,y)k(x,y)dydx=\frac{1}{2}\int_\Omega\int_\Omega|u(x)-u(y)|^pk(x,y)dydx.$$

By the symmetry argument, we can assume that $\omega(x)\geq\omega(y)$. By using the inequality (see, e.g., [FS08, Lemma 2.6])

$$|a-t|^p\geq(1-t)^{p-1}(|a|^p-t),\ p>1,\ t\in[0,1],\ a\in\mathbb{C},\tag{4.8}$$

with $t=\frac{\omega(y)}{\omega(x)}$ and $a=\frac{g(x)}{g(y)}$, we see that $R(x,y)\geq 0$. Therefore, we have proved the inequality

$$\langle(-\Delta_p)^s\omega,v\rangle\leq\frac{1}{2}\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dydx,$$

completing the proof of Lemma 4.2.3. ☐

Lemma 4.2.4. *Let $p>1$ and $\gamma\in\left(0,\frac{Q-ps}{p-1}\right)$. Then there exists a positive constant $\mu(\gamma)>0$ such that*

$$(-\Delta_p)^s(|x|^{-\gamma})=\mu(\gamma)\frac{1}{|x|^{ps+\gamma(p-1)}}\quad a.e.\ in\ \mathbb{G}\setminus\{0\}.\tag{4.9}$$

Proof of Lemma 4.2.4. Let us denote $\omega(x):=|x|^{-\gamma}$. We set $r=|x|$ and $\rho=|y|$ with $x=rx'$ and $y=\rho y'$ where $|x'|=|y'|=1$. Then we have

$$\begin{aligned}(-\Delta_p)^s\omega&=\int_0^{+\infty}||x|^{-\gamma}-|y|^{-\gamma}|^{p-2}(|x|^{-\gamma}-|y|^{-\gamma})|y|^{Q-1}\left(\int_{|y'|=1}\frac{d\sigma(y)}{|y^{-1}x|^{Q+ps}}\right)d|y|\\&=\frac{1}{|x|^{ps+\gamma(p-1)}}\int_0^{+\infty}\left|1-\frac{|y|^{-\gamma}}{|x|^{-\gamma}}\right|^{p-2}\\&\quad\times\left(1-\frac{|y|^{-\gamma}}{|x|^{-\gamma}}\right)\frac{|y|^{Q-1}}{|x|^{Q-1}}\left(\int_{|y'|=1}\frac{d\sigma(y)}{\left|\left(\frac{|y|}{|x|}y'\right)^{-1}\circ x'\right|^{Q+ps}}\right)d|y|.\end{aligned}$$

Let $\rho = \frac{|y|}{|x|}$ and $L(\rho) = \int_{|y'|=1} \frac{d\sigma(y)}{|(\rho y')^{-1} \circ x'|^{Q+ps}}$. Then we have

$$(-\Delta_p)^s \omega = \frac{1}{|x|^{ps+\gamma(p-1)}} \int_0^{+\infty} |1-\rho^{-\gamma}|^{p-2}(1-\rho^{-\gamma}) L(\rho) \rho^{Q-1} d\rho.$$

We then have (4.9) with

$$\mu(\gamma) := \int_0^{+\infty} \phi(\rho) d\rho \tag{4.10}$$

for

$$\phi(\rho) = |1-\rho^{-\gamma}|^{p-2}(1-\rho^{-\gamma}) L(\rho) \rho^{Q-1}.$$

Now it remains to show that $\mu(\gamma)$ is positive and bounded. Firstly, let us show that $\mu(\gamma)$ is bounded. We have

$$\mu(\gamma) = \int_0^1 \phi(\rho) d\rho + \int_1^{+\infty} \phi(\rho) d\rho = I_1 + I_2. \tag{4.11}$$

Using the new variable $\zeta = \frac{1}{\rho}$ we have $L(\rho) = L\left(\frac{1}{\zeta}\right) = \zeta^{Q+ps} L(\zeta)$ for any $\zeta > 0$. Thus, we get that

$$\mu(\gamma) = \int_1^{+\infty} (\rho^{-\gamma} - 1)^{p-1} (\rho^{Q-1-\gamma(p-1)} - \rho^{ps-1}) L(\rho) d\rho. \tag{4.12}$$

For $\rho \to 1$ we have

$$(\rho^{-\gamma} - 1)^{p-1} (\rho^{Q-1-\gamma(p-1)} - \rho^{ps-1}) L(\rho) \simeq (\rho - 1)^{-1-ps+p} \in L^1(1,2). \tag{4.13}$$

Similarly, for $\rho \to \infty$ we get

$$(\rho^{-\gamma} - 1)^{p-1} (\rho^{Q-1-\gamma(p-1)} - \rho^{ps-1}) L(\rho) \simeq \rho^{-1-ps} \in L^1(2,\infty). \tag{4.14}$$

These properties show that $\mu(\gamma)$ is bounded. On the other hand, by (4.12) with $\gamma \in \left(0, \frac{Q-ps}{p-1}\right)$, we see that $\mu(\gamma)$ is positive.

Lemma 4.2.4 is proved. □

Proof of Theorem 4.2.1. Let $u \in C_0^\infty(\mathbb{G})$ and $\gamma < \frac{Q-ps}{p-1}$. By Lemma 4.2.4 and Lemma 4.2.3 we readily obtain that

$$\begin{aligned} \frac{1}{2}[u]_{s,p}^p &= \frac{1}{2} \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+ps}} dx dy \\ &\geq \left\langle (-\Delta_p)^s (|x|^{-\gamma}), \frac{|u(x)|^p}{|x|^{-\gamma(p-1)}} \right\rangle = \mu(\gamma) \int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{ps}} dx. \end{aligned}$$

This completes the proof of Theorem 4.2.1. □

4.3 Fractional Sobolev inequalities on homogeneous groups

In this section we establish fractional Sobolev inequalities on homogeneous groups.

Theorem 4.3.1 (Fractional Sobolev inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let us fix a homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$. Let $p>1$, $s\in(0,1)$ and $Q>sp$, and let us set $p^*:=\frac{Qp}{Q-sp}$. Then there is a positive constant $C=C(Q,p,s,|\cdot|)$ such that we have*

$$\|u\|_{L^{p^*}(\mathbb{G})}\leq C[u]_{s,p,|\cdot|}, \tag{4.15}$$

for all measurable compactly supported functions $u:\mathbb{G}\to\mathbb{R}$.

Remark 4.3.2. In [DNPV12] the authors obtained the fractional Sobolev inequality in the case $N>sp$, $1<p<\infty$, and $s\in(0,1)$. Namely, for all measurable and compactly supported functions u one has

$$\|u\|_{L^{p^*}(\mathbb{R}^N)}\leq C[u]_{s,p}, \tag{4.16}$$

where $p^*=\frac{Np}{N-sp}$, $C=C(N,p,s)>0$ is a suitable constant independent of u, and

$$[u]^p_{s,p}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|_E^{N+sp}}dxdy,$$

with $|\cdot|_E$ being the Euclidean distance in $\mathbb{R}^N$.

To prove the above analogue of the fractional Sobolev inequality, first we present the following two lemmas.

Lemma 4.3.3. *Let $p>1$, $s\in(0,1)$, and let $K\subset\mathbb{G}$ be a Haar measurable set. Fix $x\in\mathbb{G}$ and a quasi-norm $|\cdot|$ on $\mathbb{G}$. Then we have*

$$\int_{K^c}\frac{dy}{|y^{-1}x|^{Q+sp}}\geq C|K|^{-sp/Q}, \tag{4.17}$$

where $C=C(Q,s,p,|\cdot|)$ is a positive constant, $K^c:=\mathbb{G}\setminus K$, and $|K|$ is the Haar measure of K.

Proof of Lemma 4.3.3. Let $\delta:=\left(\frac{|K|}{\omega_Q}\right)^{1/Q}$, where ω_Q is a surface measure of the unit quasi-ball on $\mathbb{G}$. For the corresponding quasi-ball $B(x,\delta)=B_{|\cdot|}(x,\delta)$ centred at x with radius δ, we have

$$\begin{aligned}|K^c\cap B(x,\delta)|&=|B(x,\delta)|-|K\cap B(x,\delta)|\\&=|K|-|K\cap B(x,\delta)|=|K\cap B^c(x,\delta)|,\end{aligned} \tag{4.18}$$

where $|\cdot|$ (by abuse of notation, only in this proof) is the Haar measure on $\mathbb{G}$. Then,

$$\begin{aligned}\int_{K^c} \frac{dy}{|y^{-1}x|^{Q+sp}} &= \int_{K^c\cap B(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}} + \int_{K^c\cap B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}} \\ &\geq \int_{K^c\cap B(x,\delta)} \frac{dy}{\delta^{Q+sp}} + \int_{K^c\cap B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}} \\ &= \frac{|K^c\cap B(x,\delta)|}{\delta^{Q+sp}} + \int_{K^c\cap B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}}.\end{aligned}$$

By using (4.18) we obtain

$$\begin{aligned}\int_{K^c} \frac{dy}{|y^{-1}x|^{Q+sp}} &\geq \frac{|K^c\cap B(x,\delta)|}{\delta^{Q+sp}} + \int_{K^c\cap B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}} \\ &= \frac{|K\cap B^c(x,\delta)|}{\delta^{Q+sp}} + \int_{K^c\cap B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}} \\ &\geq \int_{K\cap B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}} + \int_{K^c\cap B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}} \\ &= \int_{B^c(x,\delta)} \frac{dy}{|y^{-1}x|^{Q+sp}}.\end{aligned}$$

Now using the polar decomposition formula in Proposition 1.2.10 we obtain that

$$\int_{K^c} \frac{dy}{|y^{-1}x|^{Q+sp}} \geq C|K|^{-sp/Q}, \tag{4.19}$$

completing the proof. □

We now establish a useful technical estimate for the Gagliardo seminorm $[u]_{s,p}$ defined in (4.2).

Lemma 4.3.4. *Let $p>1$, $s\in(0,1)$ and $Q>sp$. Let $u\in L^\infty(\mathbb{G})$ be compactly supported and denote $a_k := |\{|u|>2^k\}|$ for any $k\in\mathbb{Z}$. Then we have*

$$C\sum_{k\in\mathbb{Z},\, a_k\neq 0} a_{k+1}a_k^{-sp/Q}2^{kp} \leq [u]^p_{s,p}, \tag{4.20}$$

where $C=C(Q,p,s,|\cdot|)$ is a positive constant.

Proof of Lemma 4.3.4. We define

$$A_k := \{|u|>2^k\},\ k\in\mathbb{Z}, \tag{4.21}$$

and

$$D_k := A_k\setminus A_{k+1} = \{2^k<|u|\leq 2^{k+1}\} \text{ and } d_k := |D_k|, \tag{4.22}$$

the Haar measure of D_k. Since $A_{k+1} \subseteq A_k$, it follows that

$$a_{k+1} \leq a_k. \tag{4.23}$$

By the assumption $u \in L^\infty(\mathbb{G})$ is compactly supported, a_k and d_k are bounded and vanish when k is large enough. Also, we notice that the D_k's are disjoint, therefore,

$$\bigcup_{l\in\mathbb{Z},\ l\leq k} D_l = A_{k+1}^c \tag{4.24}$$

and

$$\bigcup_{l\in\mathbb{Z},\ l\geq k} D_l = A_k. \tag{4.25}$$

From (4.25) it follows that

$$\sum_{l\in\mathbb{Z},\ l\geq k} d_l = a_k \tag{4.26}$$

and

$$d_k = a_k - \sum_{l\in\mathbb{Z},\ l\geq k+1} d_l. \tag{4.27}$$

Since a_k and d_k are bounded and vanish when k is large enough, (4.26) and (4.27) are convergent. We define the convergent series

$$S := \sum_{l\in\mathbb{Z},\ a_{l-1}\neq 0} 2^{lp} a_{l-1}^{-sp/Q} d_l. \tag{4.28}$$

We have that $D_k \subseteq A_k \subseteq A_{k-1}$, therefore, $a_{i-1}^{-sp/Q} d_l \leq a_{i-1}^{-sp/Q} a_{l-1}$. Thus,

$$\{(i,l) \in \mathbb{Z} \text{ s.t. } a_{i-1} \neq 0 \text{ and } a_{i-1}^{-sp/Q} d_l \neq 0\} \subseteq \{(i,l) \in \mathbb{Z} \text{ s.t. } a_{l-1} \neq 0\}. \tag{4.29}$$

By using (4.29) and (4.23), we can estimate

$$\begin{aligned}
\sum_{\substack{i\in\mathbb{Z},\\ a_{i-1}\neq 0}} \sum_{\substack{l\in\mathbb{Z},\\ l\geq i+1}} 2^{ip} a_{i-1}^{-sp/Q} d_l &= \sum_{\substack{i\in\mathbb{Z},\\ a_{i-1}\neq 0}} \sum_{\substack{l\in\mathbb{Z},\\ l\geq i+1,\\ a^{sp/Q} d_l\neq 0}} 2^{ip} a_{i-1}^{-sp/Q} d_l \\
\leq \sum_{i\in\mathbb{Z}} \sum_{\substack{l\in\mathbb{Z},\\ l\geq i+1,\\ a_{l-1}\neq 0}} 2^{ip} a_{i-1}^{-sp/Q} d_l &= \sum_{\substack{l\in\mathbb{Z},\\ a_{l-1}\neq 0}} \sum_{\substack{i\in\mathbb{Z},\\ i\leq l-1}} 2^{ip} a_{i-1}^{-sp/Q} d_l \\
\leq \sum_{\substack{l\in\mathbb{Z},\\ a_{l-1}\neq 0}} \sum_{\substack{i\in\mathbb{Z},\\ i\leq l-1}} 2^{ip} a_{l-1}^{-sp/Q} d_l &= \sum_{\substack{l\in\mathbb{Z},\\ a_{l-1}\neq 0}} \sum_{k=0}^{+\infty} 2^{p(l-1-k)} a_{l-1}^{-sp/Q} d_l \leq S.
\end{aligned} \tag{4.30}$$

Notice that

$$||u(x)| - |u(y)|| \leq |u(x) - u(y)|,$$

for any $x, y \in \mathbb{G}$. If we fix $i \in \mathbb{Z}$ and $x \in D_i$, then for any $j \in \mathbb{Z}$ with $j \leq i-2$, for any $y \in D_j$ using the above inequality, we obtain that

$$|u(x) - u(y)| \geq 2^i - 2^{j+1} \geq 2^i - 2^{i-1} \geq 2^{i-1}.$$

Then, using (4.24), we have

$$\begin{aligned} \sum_{j\in\mathbb{Z},\, j\leq i-2} \int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dy &\geq 2^{(i-1)p} \sum_{j\in\mathbb{Z},\, j\leq i-2} \int_{D_j} \frac{dy}{|y^{-1}x|^{Q+sp}} \\ &= 2^{(i-1)p} \int_{A_{i-1}^c} \frac{dy}{|y^{-1}x|^{Q+sp}}. \end{aligned} \tag{4.31}$$

Now using (4.31) and Lemma 4.3.3, we obtain that

$$\sum_{j\in\mathbb{Z},\, j\leq i-2} \int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dy \geq C 2^{ip} a_{i-1}^{-sp/Q},$$

with some positive constant C. That is, for any $i \in \mathbb{Z}$, we have

$$\sum_{j\in\mathbb{Z},\; j\leq i-2} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \geq C 2^{ip} a_{i-1}^{-sp/Q} d_i. \tag{4.32}$$

From (4.32) and (4.27) we get

$$\begin{aligned} &\sum_{j\in\mathbb{Z},\; j\leq i-2} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \\ &\geq C \left(2^{ip} a_{i-1}^{-sp/Q} a_i - \sum_{l\in\mathbb{Z},\, l\geq i+1} 2^{ip} a_{i-1}^{-sp/Q} d_l \right). \end{aligned} \tag{4.33}$$

By (4.32) and (4.28) it follows that

$$\begin{aligned} &\sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \; \sum_{j\in\mathbb{Z}, j\leq i-2} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \\ &\geq C \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} d_i \geq C\, S. \end{aligned} \tag{4.34}$$

Then, by using (4.30), (4.33) and (4.34), we obtain that

$$\begin{aligned} &\sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \; \sum_{j\in\mathbb{Z}, j\leq i-2} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \\ &\geq C \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} a_i - C \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \; \sum_{l\in\mathbb{Z}, l\geq i+1} 2^{ip} a_{i-1}^{-sp/Q} d_l \end{aligned}$$

$$\geq C \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} a_i - C\,S$$

$$\geq C \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} a_i - \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \sum_{j\in\mathbb{Z}, j\leq i-2} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy.$$

This means that

$$\begin{aligned}\sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \sum_{j\in\mathbb{Z}, j\leq i-2} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \\ \geq \frac{C}{2} \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} a_i,\end{aligned} \tag{4.35}$$

for some constant $C > 0$. By symmetry and using (4.35), we arrive at

$$\begin{aligned}[u]^p_{s,p,|\cdot|} &= \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy = \sum_{i,j\in\mathbb{Z}} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \\ &\geq 2 \sum_{i,j\in\mathbb{Z},\, j<i} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \\ &\geq 2 \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \sum_{j\in\mathbb{Z}, j\leq i-2} \int_{Di}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \\ &\geq C \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} a_i,\end{aligned}$$

completing the proof of Lemma 4.3.4. □

Proof of Theorem 4.3.1. Assume that Gagliardo's seminorm $[u]_{s,p}$ is bounded, i.e., that

$$[u]^p_{s,p} = \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy < +\infty. \tag{4.36}$$

Suppose also that $u \in L^\infty(\mathbb{G})$.

If (4.36) is satisfied for bounded functions, it holds also for the function u_n, obtained from u by cutting at levels $-n$ and n, that is, for

$$u_n := \max\{\min\{u(x), n\}, -n\},$$

for any $n \in \mathbb{R}$ and $x \in \mathbb{G}$. Thus, using the fact that

$$\lim_{n\to+\infty} \|u_n\|_{L^p(\mathbb{G})} = \|u\|_{L^p(\mathbb{G})},$$

$1 < p < \infty$, and by using (4.36) with the dominated convergence theorem, we obtain that

$$\begin{aligned}\lim_{n\to+\infty}[u_n]^p_{s,p} &= \lim_{n\to+\infty}\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u_n(x)-u_n(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\ &= \int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy = [u]^p_{s,p}.\end{aligned} \tag{4.37}$$

Defining a_k and A_k as in Lemma 4.3.4, we have

$$\begin{aligned}||u\|_{L^{p^*}(\mathbb{G})} &= \left(\sum_{k\in\mathbb{Z}}\int_{A_k\setminus A_{k+1}}|u(x)|^{p^*}dx\right)^{1/p^*} \\ &\leq \left(\sum_{k\in\mathbb{Z}}\int_{A_k\setminus A_{k+1}}2^{(k+1)p^*}dx\right)^{1/p^*} \\ &\leq \left(\sum_{k\in\mathbb{Z}}2^{(k+1)p^*}a_k\right)^{1/p^*}.\end{aligned}$$

Recall the following fact from [DNPV12, Lemma 6.2]: let $T, p > 1$ and $s \in (0,1)$ be such that $Q > sp$, $m \in \mathbb{Z}$, and assume that a_k is a bounded, decreasing, non-negative sequence with $a_k = 0$ for any $k \geq m$; then we have

$$\sum_{k\in\mathbb{Z}}a_k^{(Q-sp)/Q}T^k \leq C\sum_{k\in\mathbb{Z},\, a_k\neq 0}a_{k+1}a_k^{-sp/Q}T^k, \tag{4.38}$$

for some positive constant $C = C(Q, s, p, T)$.

Then, with $p/p^* = 1 - sp/Q < 1$ and $T = 2^p$, this fact yields

$$\begin{aligned}\|u\|^p_{L^{p^*}(\mathbb{G})} &\leq 2^p\left(\sum_{k\in\mathbb{Z}}2^{kp^*}a_k\right)^{p/p^*} \leq 2^p\sum_{k\in\mathbb{Z}}2^{kp}a_k^{(Q-sp)/Q} \\ &\leq C\sum_{k\in\mathbb{Z},\, a_k\neq 0}2^{kp}a_k^{-sp/Q}a_{k+1}\end{aligned} \tag{4.39}$$

for a positive constant $C = C(Q, p, s, |\cdot|)$. Finally, using Lemma 4.3.4 we arrive at

$$\begin{aligned}\|u\|^p_{L^{p^*}(\mathbb{G})} &\leq C\sum_{k\in\mathbb{Z},\, a_k\neq 0}2^{kp}a_k^{-sp/Q}a_{k+1} \\ &\leq C\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy = C[u]^p_{s,p,|\cdot|}.\end{aligned}$$

Theorem 4.3.1 is proved. □

4.4 Fractional Gagliardo–Nirenberg inequalities

In this section we discuss an analogue of the fractional Gagliardo–Nirenberg inequality on homogeneous groups. As it can be partly expected, in its proof we will use an already established version of a fractional Sobolev inequality on the homogeneous groups.

Theorem 4.4.1 (Fractional Gagliargo–Nirenberg inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$, $Q > sp$ and*

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}.$$

Then there exists $C = C(s,p,Q,a,\alpha) > 0$ such that

$$\|u\|_{L^\tau(\mathbb{G})} \leq C[u]^a_{s,p,|\cdot|} \|u\|^{1-a}_{L^\alpha(\mathbb{G})} \tag{4.40}$$

holds for all $u \in C^1_c(\mathbb{G})$.

Remark 4.4.2. Theorem 4.4.1 was proved in [KRS18a]. In the Abelian case $(\mathbb{R}^N, +)$ with the standard Euclidean distance instead of the quasi-norm, Theorem 4.4.1 covers the fractional Gagliardo–Nirenberg inequality which was proved in [NS18a].

Proof of Theorem 4.4.1*.* By using Hölder's inequality, for every $\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}$ we get

$$\|u\|^\tau_{L^\tau(\mathbb{G})} = \int_{\mathbb{G}} |u|^\tau dx = \int_{\mathbb{G}} |u|^{a\tau}|u|^{(1-a)\tau} dx \leq \|u\|^{a\tau}_{L^{p^*}(\mathbb{G})} \|u\|^{(1-a)\tau}_{L^\alpha(\mathbb{G})}, \tag{4.41}$$

where $p^* = \frac{Qp}{Q-sp}$. From (4.41), by using the fractional Sobolev inequality (Theorem 4.3.1), we obtain

$$\|u\|^\tau_{L^\tau(\mathbb{G})} \leq \|u\|^{a\tau}_{L^{p^*}(\mathbb{G})} \|u\|^{(1-a)\tau}_{L^\alpha(\mathbb{G})} \leq C[u]^{a\tau}_{s,p,|\cdot|} \|u\|^{(1-a)\tau}_{L^\alpha(\mathbb{G})},$$

that is,

$$\|u\|_{L^\tau(\mathbb{G})} \leq C[u]^a_{s,p,|\cdot|} \|u\|^{1-a}_{L^\alpha(\mathbb{G})}, \tag{4.42}$$

where C is a positive constant independent of u. Theorem 4.4.1 is proved. $\square$

4.5 Fractional Caffarelli–Kohn–Nirenberg inequalities

In this section we discuss the weighted fractional Caffarelli–Kohn–Nirenberg inequalities on homogeneous groups. First, let us define a weighted version of fractional Sobolev spaces from Definition 4.1.2.

Definition 4.5.1 (Weighted fractional Sobolev spaces)**.** We define the *weighted fractional Sobolev space* on a homogeneous group $\mathbb{G}$ with homogeneous dimension Q and homogeneous quasi-norm $|\cdot|$ by

$$\begin{aligned} W^{s,p,\beta}(\mathbb{G}) = \Bigg\{ & u \in L^p(\mathbb{G}) : u \text{ is measurable}, \\ & [u]_{s,p,\beta,|\cdot|} := \left(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|x|^{\beta_1 p}|y|^{\beta_2 p}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{1}{p}} < +\infty \Bigg\}, \end{aligned} \tag{4.43}$$

where $\beta_1,\ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$. We note that the space $W^{s,p,\beta}(\mathbb{G})$ depends on β_1 and β_2. At the same time, it is independent of a particular choice of a quasi-norm due to their equivalence, see Proposition 1.2.3.

For a Haar measurable set $\Omega \subset \mathbb{G}$, $p \geq 1$, $s \in (0,1)$ and $\beta_1,\ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$, we define the weighted fractional Sobolev space on Ω by

$$\begin{aligned} W^{s,p,\beta}(\Omega) = \Bigg\{ & u \in L^p(\Omega) : u \text{ is measurable}, \\ & [u]_{s,p,\beta,|\cdot|,\Omega} := \left(\int_{\Omega} \int_{\Omega} \frac{|x|^{\beta_1 p}|y|^{\beta_2 p}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{1}{p}} < +\infty \Bigg\}. \end{aligned} \tag{4.44}$$

Theorem 4.5.2 (Fractional Caffarelli–Kohn–Nirenberg inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$, β_1, β_2, β, μ, $\gamma \in \mathbb{R}$, $\beta_1 + \beta_2 = \beta$ and*

$$\frac{1}{\tau} + \frac{\gamma}{Q} = a\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{4.45}$$

In addition, assume that, $0 \leq \beta - \sigma$ with $\gamma = a\sigma + (1-a)\mu$. We also assume

$$\beta - \sigma \leq s \ \text{ only if } \ \frac{1}{\tau} + \frac{\gamma}{Q} = \frac{1}{p} + \frac{\beta - s}{Q}. \tag{4.46}$$

Then when $\frac{1}{\tau} + \frac{\gamma}{Q} > 0$ we have

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,\beta,|\cdot|} \||x|^{\mu} u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{4.47}$$

for all $u \in C^1_c(\mathbb{G})$, and when $\frac{1}{\tau} + \frac{\gamma}{Q} < 0$ we have

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,\beta,|\cdot|} \||x|^{\mu} u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{4.48}$$

for all $u \in C^1_c(\mathbb{G} \setminus \{0\})$.

Remark 4.5.3.

1. The critical case $\frac{1}{\tau} + \frac{\gamma}{Q} = 0$ will be considered in Theorem 4.5.5.
2. In the Abelian (Euclidean) case $(\mathbb{R}^N, +)$ with the usual Euclidean distance instead of the quasi-norm in Theorem 4.5.2, we get the (Abelian) fractional Caffarelli–Kohn–Nirenberg inequality (see, e.g., [NS18a], Theorem 1.1). In (4.48) by setting $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, and $\gamma = -s$, one has an analogue of the fractional Hardy inequality on homogeneous groups.
3. In the Abelian (Euclidean) case $(\mathbb{R}^N, +)$ again with the usual Euclidian distance instead of the quasi-norm and by taking in (4.48) the values $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, and $\gamma = -s$, we get the fractional Hardy inequality (see [FS74, Theorem 1.1]).
4. The results of this section were obtained in [KRS18a] and we follow the presentation there in our proof.

To prove the fractional weighted Caffarelli–Kohn–Nirenberg inequality on $\mathbb{G}$ we will use Theorem 4.4.1 in the proof of the following lemma.

Lemma 4.5.4. *Let $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$ and*

$$\frac{1}{\tau} \geq a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}.$$

Let $\lambda > 0$ and $0 < r < R$ and set

$$\Omega = \{x \in \mathbb{G} : \ \lambda r < |x| < \lambda R\}.$$

Then, for all $u \in C^1(\overline{\Omega})$, we have

$$\left(\fint_{\Omega} |u - u_\Omega|^\tau dx\right)^{\frac{1}{\tau}} \leq C_{r,R} \lambda^{\frac{a(sp-Q)}{p}} [u]^a_{s,p,|\cdot|,\Omega} \left(\fint_{\Omega} |u|^\alpha dx\right)^{\frac{1-a}{\alpha}}, \tag{4.49}$$

where $C_{r,R}$ is a positive constant independent of u and λ.

Proof of Lemma 4.5.4. Without loss of generality, we assume that $0 < s' \leq s$ and $\tau' \geq \tau$ are such that

$$\frac{1}{\tau'} = a\left(\frac{1}{p} - \frac{s'}{Q}\right) + \frac{1-a}{\alpha},$$

and $\lambda = 1$ with

$$\Omega_1 := \{x \in \mathbb{G} : \ r < |x| < R\}.$$

By using Theorem 4.4.1, Jensen's inequality and $[u]_{s',p,|\cdot|,\Omega} \leq C[u]_{s,p,|\cdot|,\Omega}$, we compute

$$\left(\fint_{\Omega_1} |u - u_{\Omega_1}|^\tau dx\right)^{1/\tau} = \frac{1}{|\Omega_1|^{\frac{1}{\tau}}} \|u - u_{\Omega_1}\|_\tau \leq C_{r,R} \|u - u_{\Omega_1}\|_{L^{\tau'}(\Omega_1)}$$

$$
\begin{aligned}
&\le C_{r,R}[u-u_{\Omega_1}]^a_{s',p,|\cdot|,\Omega_1}\|u\|^{1-a}_{L^\alpha(\Omega_1)}\\
&\le C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{|u(x)-u_{\Omega_1}-u(y)+u_{\Omega_1}|^p}{|y^{-1}x|^{Q+s'p}}dxdy\right)^{a/p}\|u\|^{1-a}_{L^\alpha(\Omega_1)}\\
&\le C_{r,R}[u]^a_{s,p,|\cdot|,\Omega_1}\|u\|^{1-a}_{L^\alpha(\Omega_1)}\le C_{r,R}[u]^a_{s,p,|\cdot|,\Omega_1}\left(\fint_{\Omega_1}|u|^\alpha dx\right)^{(1-a)/\alpha},
\end{aligned}
$$

where $C_{r,R}>0$. Let us apply the above inequality to $u(\lambda x)$ instead of $u(x)$. This yields

$$
\begin{aligned}
&\left(\fint_{\Omega_1}\left|u(\lambda x)-\fint_{\Omega_1}u(\lambda x)dx\right|^\tau dx\right)^{1/\tau}\\
&\quad\le C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{|u(\lambda x)-u(\lambda y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{a/p}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}|u(\lambda x)|^\alpha dx\right)^{(1-a)/\alpha}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\left(\fint_{\Omega}\left|u(x)-\fint_{\Omega}u(x)dx\right|^\tau dx\right)^{\frac{1}{\tau}}=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(x)-\frac{1}{|\Omega|}\int_{\Omega}u(x)dx\right|^\tau dx\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(\lambda y)-\frac{1}{|\Omega|}\int_{\Omega}u(\lambda y)d(\lambda y)\right|^\tau d(\lambda y)\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^Q}{\lambda^Q}\left|u(\lambda y)-\frac{\lambda^Q}{\lambda^Q|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^\tau dy\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\left|u(\lambda y)-\frac{1}{|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^\tau dy\right)^{\frac{1}{\tau}}\\
&\le C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{|u(\lambda x)-u(\lambda y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}|u(\lambda x)|^\alpha dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{\lambda^{2Q}\lambda^{Q+sp}|u(\lambda x)-u(\lambda y)|^p}{\lambda^{2Q}\lambda^{Q+sp}|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^Q}{\lambda^Q}|u(\lambda x)|^\alpha dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(\lambda x)-u(\lambda y)|^p}{|(\lambda y)^{-1}\lambda x|^{Q+sp}}d(\lambda x)d(\lambda y)\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(\lambda x)|^\alpha d(\lambda x)\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^\alpha dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\lambda^{\frac{a(sp-Q)}{p}}[u]^a_{s,p,|\cdot|,\Omega}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^\alpha dx\right)^{\frac{1-a}{\alpha}}.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 4.5.2. First let us consider the case (4.46), that is, $\beta - \sigma \leq s$ and $\frac{1}{\tau} + \frac{\gamma}{Q} = \frac{1}{p} + \frac{\beta - s}{Q}$. By using Lemma 4.5.4 with $\lambda = 2^k$, $r = 1$, $R = 2$ and $\Omega = A_k$, we get

$$\left(\fint_{A_k} |u - u_{A_k}|^\tau dx \right)^{1/\tau} \leq C 2^{\frac{ak(sp-Q)}{p}} [u]^a_{s,p,|\cdot|,A_k} \left(\fint_{A_k} |u|^\alpha dx \right)^{(1-a)/\alpha}. \quad (4.50)$$

Here and below A_k is the quasi-annulus defined by

$$A_k := \{x \in \mathbb{G} : \ 2^k \leq |x| < 2^{k+1}\},$$

for $k \in \mathbb{Z}$. Now by using (4.50) we obtain

$$\begin{aligned}
\int_{A_k} |u|^\tau dx &= \int_{A_k} |u - u_{A_k} + u_{A_k}|^\tau dx \\
&\leq C \left(\int_{A_k} |u_{A_k}|^\tau dx + \int_{A_k} |u - u_{A_k}|^\tau dx \right) \\
&= C \left(\int_{A_k} |u_{A_k}|^\tau dx + \frac{|A_k|}{|A_k|} \int_{A_k} |u - u_{A_k}|^\tau dx \right) \\
&= C \left(|A_k||u_{A_k}|^\tau + |A_k| \fint_{A_k} |u - u_{A_k}|^\tau dx \right) \qquad (4.51) \\
&\leq C \left(|A_k||u_{A_k}|^\tau + 2^{\frac{ak(sp-Q)\tau}{p}} |A_k| [u]^{a\tau}_{s,p,|\cdot|,A_k} \left(\frac{1}{|A_k|} \int_{A_k} |u|^\alpha dx \right)^{\frac{(1-a)\tau}{\alpha}} \right) \\
&\leq C \left(2^{Qk} |u_{A_k}|^\tau + 2^{\frac{ak(sp-Q)\tau}{p}} 2^{kQ} 2^{-\frac{Q(1-a)\tau k}{\alpha}} [u]^{a\tau}_{s,p,|\cdot|,A_k} \|u\|^{(1-a)\tau}_{L^\alpha(A_k)} \right).
\end{aligned}$$

Then, from (4.51) we get

$$\begin{aligned}
&\int_{A_k} |x|^{\gamma\tau} |u|^\tau dx \leq 2^{(k+1)\gamma\tau} \int_{A_k} |u|^\tau dx \\
&\leq C 2^{(Q+\gamma\tau)k} |u_{A_k}|^\tau + C 2^{\gamma\tau k} 2^{kQ} 2^{\frac{ak(sp-Q)\tau}{p}} 2^{-\frac{Q(1-a)\tau k}{\alpha}} [u]^{a\tau}_{s,p,|\cdot|,A_k} \|u\|^{(1-a)\tau}_{L^\alpha(A_k)} \\
&= C 2^{(Q+\gamma\tau)k} |u_{A_k}|^\tau \\
&\quad + C 2^{\left(\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha}\right)k} \left(\int_{A_k} \int_{A_k} \frac{2^{kp\beta_1} 2^{kp\beta_2} |u(x) - u(y)|^p}{2^{kp\beta} |y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{a\tau}{p}} \\
&\quad\quad \times \left(\int_{A_k} \frac{2^{k\alpha\mu}}{2^{k\alpha\mu}} |u(x)|^\alpha dx \right)^{\frac{(1-a)\tau}{\alpha}} \\
&\leq C 2^{(Q+\gamma\tau)k} |u_{A_k}|^\tau \\
&\quad + C 2^{\left(\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha} - a\beta\tau - \mu\tau(1-a)\right)k} \\
&\quad\quad \times \left(\int_{A_k} \int_{A_k} \frac{|x|^{p\beta_1} |y|^{p\beta_2} |u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{a\tau}{p}}
\end{aligned}$$

$$\times \left(\int_{A_k} |x|^{\alpha\mu}|u(x)|^{\alpha}dx\right)^{\frac{(1-a)\tau}{\alpha}}$$
$$\leq C2^{(Q+\gamma\tau)k}|u_{A_k}|^{\tau} \tag{4.52}$$
$$+ C2^{\left(\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}-a\beta\tau-\mu\tau(1-a)\right)k}[u]^{a\tau}_{s,p,\beta,|\cdot|,A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_k)}.$$

Here by (4.45), we have

$$\begin{aligned}
&\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}-a\beta\tau-\mu\tau(1-a)\\
&\quad= Q\tau\left(\frac{\gamma}{Q}+\frac{1}{\tau}+\frac{a(sp-Q)}{Qp}-\frac{(1-a)}{\alpha}-\frac{a\beta}{Q}-\frac{\mu(1-a)}{Q}\right)\\
&\quad= Q\tau\left(a\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right)\right)\\
&\qquad+Q\tau\left(\frac{a(sp-Q)}{Qp}-\frac{(1-a)}{\alpha}-\frac{a\beta}{Q}-\frac{\mu(1-a)}{Q}\right)=0.
\end{aligned} \tag{4.53}$$

Thus, we obtain

$$\int_{A_k}|x|^{\gamma\tau}|u|^{\tau}dx \leq C2^{(\gamma\tau+Q)k}|u_{A_k}|^{\tau}+C[u]^{a\tau}_{s,p,\beta,|\cdot|,A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_k)}, \tag{4.54}$$

and by summing over k from m to n, we get

$$\begin{aligned}
\int_{\cup_{k=m}^{n}A_k}|x|^{\gamma\tau}|u|^{\tau}dx &= \int_{\{2^m<|x|<2^{n+1}\}}|x|^{\gamma\tau}|u|^{\tau}dx \\
&\leq C\sum_{k=m}^{n}2^{(\gamma\tau+Q)k}|u_{A_k}|^{\tau}+C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,|\cdot|,A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_k)},
\end{aligned} \tag{4.55}$$

where $k,m,n\in\mathbb{Z}$ and $m\leq n-2$.

To prove (4.47) let us choose n such that

$$\operatorname{supp} u \subset B_{2^n}, \tag{4.56}$$

where B_{2^n} is a quasi-ball of $\mathbb{G}$ with the radius 2^n.

Let us consider the following integral

$$\begin{aligned}
&⨍_{A_{k+1}\cup A_k}\left|u-⨍_{A_{k+1}\cup A_k}u\right|^{\tau}dx = \frac{1}{|A_{k+1}|+|A_k|}\int_{A_{k+1}\cup A_k}\left|u-⨍_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\quad= \frac{1}{|A_{k+1}|+|A_k|}\left(\int_{A_{k+1}}\left|u-⨍_{A_{k+1}\cup A_k}u\right|^{\tau}dx+\int_{A_k}\left|u-⨍_{A_{k+1}\cup A_k}u\right|^{\tau}dx\right).
\end{aligned}$$

On the other hand, a direct calculation gives

$$
\begin{aligned}
&\fint_{A_{k+1}\cup A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\quad=\frac{1}{|A_{k+1}|+|A_k|}\left(\int_{A_{k+1}}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx+\int_{A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\right)\\
&\quad\geq\frac{1}{|A_{k+1}|+|A_k|}\int_{A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\quad\geq\frac{1}{|A_{k+1}|+|A_k|}\left|\int_{A_k}\left(u-\fint_{A_{k+1}\cup A_k}u\right)dx\right|^{\tau}\\
&\quad=\frac{1}{|A_{k+1}|+|A_k|}\left|\int_{A_k}udx-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_k}udx-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_{k+1}}udx\right|^{\tau}\\
&\quad=\frac{1}{|A_{k+1}|+|A_k|}\left|\frac{|A_{k+1}|}{|A_{k+1}|+|A_k|}\int_{A_k}udx-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_{k+1}}udx\right|^{\tau}\\
&\quad=\frac{1}{(|A_{k+1}|+|A_k|)^{\tau+1}}\left||A_{k+1}|\int_{A_k}udx-|A_k|\int_{A_{k+1}}udx\right|^{\tau}\\
&\quad=\frac{|A_{k+1}|^{\tau}|A_k|^{\tau}}{(|A_{k+1}|+|A_k|)^{\tau+1}}\left|\frac{1}{|A_k|}\int_{A_k}udx-\frac{1}{|A_{k+1}|}\int_{A_{k+1}}udx\right|^{\tau}\\
&\quad=\frac{|A_{k+1}|^{\tau}|A_k|^{\tau}}{(|A_{k+1}|+|A_k|)^{\tau+1}}|u_{A_{k+1}}-u_{A_k}|^{\tau}\\
&\quad\geq C|u_{A_{k+1}}-u_{A_k}|^{\tau}.
\end{aligned}\tag{4.57}
$$

From (4.57) and Lemma 4.5.4, we obtain

$$
\begin{aligned}
|u_{A_{k+1}}-u_{A_k}|^{\tau}&\leq C\fint_{A_{k+1}\cup A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\leq C2^{\frac{ak(sp-Q)}{p}}[u]^{\tau a}_{s,p,|\cdot|,A_{k+1}\cup A_k}\left(\fint_{A_{k+1}\cup A_k}|u|^{\alpha}dx\right)^{\frac{(1-a)\tau}{\alpha}}.
\end{aligned}\tag{4.58}
$$

By using this fact, taking $\tau=1$ we have

$$
\begin{aligned}
|u_{A_k}|&\leq|u_{A_{k+1}}-u_{A_k}|+|u_{A_{k+1}}|\\
&\leq|u_{A_{k+1}}|+C2^{\frac{ak(sp-Q)}{p}}[u]^{a}_{s,p,|\cdot|,A_{k+1}\cup A_k}\left(\fint_{A_{k+1}\cup A_k}|u|^{\alpha}dx\right)^{\frac{(1-a)}{\alpha}}.
\end{aligned}\tag{4.59}
$$

On the other hand (see, e.g., [NS18b, Lemma 2.2]), there exists a positive constant C depending $\xi > 1$ and $\eta > 1$ such that $1 < \zeta < \xi$,

$$(|a| + |b|)^\eta \leq \zeta |a|^\eta + \frac{C}{(\zeta - 1)^{\eta - 1}} |b|^\eta, \quad \forall\, a, b \in \mathbb{R}. \tag{4.60}$$

Thus, by using with $\eta = \tau$, $\zeta = 2^{\gamma\tau + Q} c$, where $c = \frac{2}{1 + 2^{\gamma\tau + Q}} < 1$, since $\gamma\tau + Q > 0$, from the previous inequality we have

$$2^{(\gamma\tau+Q)k} |u_{A_k}|^\tau \leq c 2^{(k+1)(\gamma\tau+Q)} |u_{A_{k+1}}|^\tau + C[u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u \|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}.$$

By summing over k from m to n and by using (4.56) we have

$$\begin{aligned} \sum_{k=m}^{n} 2^{(\gamma\tau+Q)k} |u_{A_k}|^\tau \leq & \sum_{k=m}^{n} c 2^{(k+1)(\gamma\tau+Q)} |u_{A_{k+1}}|^\tau \\ & + C \sum_{k=m}^{n} [u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u \|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \end{aligned} \tag{4.61}$$

By using (4.61), we compute

$$\begin{aligned} (1-c) \sum_{k=m}^{n} 2^{(\gamma\tau+Q)k} |u_{A_k}|^\tau & \leq 2^{(\gamma\tau+Q)m} |u_{A_m}|^\tau + (1-c) \sum_{k=m+1}^{n} 2^{(\gamma\tau+Q)k} |u_{A_k}|^\tau \\ & \leq C \sum_{k=m}^{n} [u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u \|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \end{aligned}$$

This yields

$$\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k} |u_{A_k}|^\tau \leq C \sum_{k=m}^{n} [u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u \|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{4.62}$$

From (4.55) and (4.62), we have

$$\int_{\{2^m < |x| < 2^{n+1}\}} |x|^{\gamma\tau} |u|^\tau dx \leq C \sum_{k=m}^{n} [u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u \|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{4.63}$$

Let $s, t \geq 0$ be such that $s + t \geq 1$. Then for any $x_k, y_k \geq 0$, we have

$$\sum_{k=m}^{n} x_k^s y_k^t \leq \left(\sum_{k=m}^{n} x_k \right)^s \left(\sum_{k=m}^{n} y_k \right)^t. \tag{4.64}$$

By using this inequality in (4.63) with $s = \frac{\tau a}{p}$, $t = \frac{(1-a)\tau}{\alpha}$, $\frac{a}{p} + \frac{1-a}{\alpha} \geq \frac{1}{\tau}$ and $s \geq \beta - \sigma$, we obtain

$$\int_{\{|x| > 2^m\}} |x|^{\gamma\tau} |u|^\tau dx \leq C[u]^{a\tau}_{s,p,\beta,|\cdot|,\cup_{k=m}^\infty A_k} \| |x|^\mu u \|^{(1-a)\tau}_{L^\alpha(\cup_{k=m}^\infty A_k)}.$$

Inequality (4.47) is proved.

Let us prove (4.48). The strategy of the proof is similar to the previous case. Choose m such that

$$\operatorname{supp} u \cap B_{2^m} = \emptyset. \tag{4.65}$$

From Lemma 4.5.4 we have

$$|u_{A_{k+1}} - u_{A_k}|^\tau \le C 2^{\frac{a\tau k(sp-Q)}{p}} [u]^{\tau a}_{s,p,|\cdot|,A_{k+1}\cup A_k} \left(\fint_{A_{k+1}\cup A_k} |u|^\alpha dx \right)^{\frac{(1-a)\tau}{\alpha}}.$$

By (4.60) and choosing $c = \frac{1+2^{\gamma\tau+Q}}{2} < 1$, since $\gamma\tau + Q < 0$, we have

$$2^{(\gamma\tau+Q)(k+1)} |u_{A_{k+1}}|^\tau \le c 2^{k(\gamma\tau+Q)} |u_{A_k}|^\tau + C [u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)},$$

and by summing over k from m to n and by using (4.65) we obtain

$$\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k} |u_{A_k}|^\tau \le C \sum_{k=m-1}^{n-1} [u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{4.66}$$

From (4.55) and (4.66), we establish that

$$\int_{\{2^m<|x|<2^{n+1}\}} |x|^{\gamma\tau} |u|^\tau dx \le C \sum_{k=m-1}^{n-1} [u]^{\tau a}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \| |x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}.$$

Now by using (4.64) we get

$$\int_{\{|x|<2^{n+1}\}} |x|^{\gamma\tau} |u|^\tau dx \le C [u]^{\tau a}_{s,p,\beta,|\cdot|,\cup_{k=-\infty}^{n} A_k} \| |x|^\mu u\|^{(1-a)\tau}_{L^\alpha(\cup_{k=-\infty}^{n} A_k)}.$$

The proof of the case $s \ge \beta - \sigma$ is complete.

Let us prove the case of $\beta - \sigma > s$. Without loss of generality, we assume that

$$[u]_{s,p,\beta,|\cdot|} = \|u\|_{L^\alpha(\mathbb{G})} = 1,$$

where

$$\frac{1}{p} + \frac{\beta - s}{Q} \neq \frac{1}{\alpha} + \frac{\mu}{Q}.$$

We also assume that $a_1 > 0,\ 1 > a_2$ and $\tau_1,\ \tau_2 > 0$ with

$$\frac{1}{\tau_2} = \frac{a_2}{p} + \frac{1-a_2}{\alpha},$$

and

$$\begin{aligned} &\text{if } \ \frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} > 0, \quad \text{then } \ \frac{1}{\tau_1} = \frac{a_1}{p} + \frac{1-a_1}{\alpha} - \frac{a_1 s}{Q}, \\ &\text{if } \ \frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} \le 0, \quad \text{then } \ \frac{1}{\tau} > \frac{1}{\tau_1} \ge \frac{a_1}{p} + \frac{1-a_1}{\alpha} - \frac{a_1 s}{Q}. \end{aligned} \tag{4.67}$$

Taking $\gamma_1 = a_1\beta + (1-a_1)\mu$ and $\gamma_2 = a_2(\beta - s) + (1-a_2)\mu$, we obtain

$$\frac{1}{\tau_1} + \frac{\gamma_1}{Q} \geq a_1\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a_1)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right) \tag{4.68}$$

and

$$\frac{1}{\tau_2} + \frac{\gamma_2}{Q} = a_2\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a_2)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{4.69}$$

Let a_1 and a_2 be such that

$$|a - a_1| \text{ and } |a - a_2| \text{ are small enough,} \tag{4.70}$$

$$\begin{aligned} &a_2 < a < a_1, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} > \frac{1}{\alpha} + \frac{\mu}{Q}, \\ &a_1 < a < a_2, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} < \frac{1}{\alpha} + \frac{\mu}{Q}. \end{aligned} \tag{4.71}$$

By using (4.70)–(4.71) in (4.68), (4.69) and (4.45), we establish

$$\frac{1}{\tau_1} + \frac{\gamma_1}{Q} > \frac{1}{\tau} + \frac{\gamma}{Q} > \frac{1}{\tau_2} + \frac{\gamma_2}{Q} > 0.$$

From (4.67) in the case $\frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} > 0$ with $a > 0$, $\beta - \sigma > s$ and (4.70), we get

$$\frac{1}{\tau} - \frac{1}{\tau_1} = (a - a_1)\left(\frac{1}{p} - \frac{s}{Q} - \frac{1}{\alpha}\right) + \frac{a}{Q}(\beta - \sigma) > 0, \tag{4.72}$$

and

$$\frac{1}{\tau} - \frac{1}{\tau_2} = (a - a_2)\left(\frac{1}{p} - \frac{1}{\alpha}\right) + \frac{a}{Q}(\beta - \sigma - s) > 0. \tag{4.73}$$

From (4.67), (4.72) and (4.73), we have

$$\tau_1 > \tau, \quad \tau_2 > \tau.$$

Thus, using this, (4.70) and Hölder's inequality, we obtain

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{G}\setminus B_1)} \leq C\||x|^{\gamma_1} u\|_{L^{\tau_1}(\mathbb{G})},$$

and

$$\||x|^{\gamma} u\|_{L^{\tau}(B_1)} \leq C\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{G})},$$

where B_1 is the unit quasi-ball. By using the previous case, we establish

$$\||x|^{\gamma_1} u\|_{L^{\tau_1}(\mathbb{G})} \leq C[u]^{a_1}_{s,p,\beta,|\cdot|}\||x|^{\mu} u\|^{1-a_1}_{L^{\alpha}(\mathbb{G})} \leq C,$$

and

$$\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{G})} \leq C[u]^{a_2}_{s,p,\beta,|\cdot|}\||x|^{\mu} u\|^{1-a_2}_{L^{\alpha}(\mathbb{G})} \leq C.$$

The proof of Theorem 4.5.2 is complete. □

Now we consider the critical case $\frac{1}{\tau} + \frac{\gamma}{Q} = 0$.

Theorem 4.5.5 (Fractional critical Caffarelli–Kohn–Nirenberg inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 1$, $a \in (0,1]$, β_1, β_2, β, μ, $\gamma \in \mathbb{R}$, $\beta_1 + \beta_2 = \beta$,*

$$\frac{1}{\tau} + \frac{\gamma}{Q} = a\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{4.74}$$

In addition, assume that $0 \leq \beta - \sigma \leq s$ with $\gamma = a\sigma + (1-a)\mu$.

If $\frac{1}{\tau} + \frac{\gamma}{Q} = 0$ and $\operatorname{supp} u \subset B_R = \{x \in \mathbb{G} : |x| < R\}$, then we have

$$\left\| \frac{|x|^{\gamma}}{\ln \frac{2R}{|x|}} u \right\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,\beta,|\cdot|} \||x|^{\mu} u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{4.75}$$

for all $u \in C^1_c(\mathbb{G})$.

Proof of Theorem 4.5.5. The proof is similar to the proof of Theorem 4.5.2. In (4.54), summing over k from m to n and fixing $\varepsilon > 0$, we have

$$\begin{aligned}\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{1+\varepsilon}\left(\frac{2R}{|x|}\right)} |u|^{\tau} dx \leq C \sum_{k=m}^{n} \frac{1}{(n+1-k)^{1+\varepsilon}} |u_{A_k}|^{\tau} \\ + C \sum_{k=m}^{n} [u]^{a\tau}_{s,p,\beta,|\cdot|,A_k} \||x|^{\mu} u\|^{(1-a)\tau}_{L^{\alpha}(A_k)}.\end{aligned} \tag{4.76}$$

From Lemma 4.5.4, we have

$$|u_{A_{k+1}} - u_{A_k}| \leq C 2^{\frac{ak(sp-Q)}{p}} [u]^{a}_{s,p,|\cdot|,A_{k+1}\cup A_k} \left(\fint_{A_{k+1}\cup A_k} |u|^{\alpha} dx \right)^{\frac{1-a}{\alpha}}.$$

By using (4.60) with $\zeta = \frac{(n+1-k)^{\varepsilon}}{(n+\frac{1}{2}-k)^{\varepsilon}}$ we get

$$\begin{aligned}\frac{|u_{A_k}|^{\tau}}{(n+1-k)^{\varepsilon}} \leq \frac{|u_{A_{k+1}}|^{\tau}}{(n+\frac{1}{2}-k)^{\varepsilon}} \\ + C(n+1-k)^{\tau-1-\varepsilon} [u]^{a\tau}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \||x|^{\mu} u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}.\end{aligned} \tag{4.77}$$

For $\varepsilon > 0$ and $n \geq k$, we have

$$\frac{1}{(n-k+1)^{\varepsilon}} - \frac{1}{(n-k+\frac{3}{2})^{\varepsilon}} \sim \frac{1}{(n-k+1)^{1+\varepsilon}}. \tag{4.78}$$

By using this fact, (4.77), (4.78) and $\varepsilon = \tau - 1$, we obtain

$$\sum_{k=m}^{n} \frac{|u_{A_k}|^{\tau}}{(n+1-k)^{\tau}} \leq C \sum_{k=m}^{n} [u]^{a\tau}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \|q^{\mu}(x) u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}. \tag{4.79}$$

From (4.76) and (4.79), we establish

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^\tau \frac{2R}{|x|}} |u|^\tau dx \leq C \sum_{k=m}^{n} [u]^{a\tau}_{s,p,\beta,|\cdot|,A_{k+1}\cup A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}.$$

By using (4.64) with (4.74) and $0 \leq \beta - \sigma \leq s$, where $s = \frac{\tau a}{p}$, $t = \frac{(1-a)\tau}{\alpha}$, we have $s + t \geq 1$ and we arrive at

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^\tau \frac{2R}{|x|}} |u|^\tau dx \leq C \sum_{k=m}^{n} [u]^{a\tau}_{s,p,\beta,|\cdot|,\cup_{k=m}^{\infty} A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(\cup_{k=m}^{\infty} A_k)}.$$

Theorem 4.5.5 is proved. □

4.6 Lyapunov inequalities on homogeneous groups

In this section we give an application of the preceding results to derive a Lyapunov type inequality for the fractional p-sub-Laplacian with homogeneous Dirichlet boundary condition on homogeneous groups. First, we summarize the basic results concerning the classical Lyapunov inequality.

Remark 4.6.1 (Euclidean Lyapunov inequalities)**.**

1. In [Lya07], Lyapunov obtained the following result for the one-dimensional homogeneous Dirichlet boundary value problem. Consider the second-order ordinary differential equation

$$\begin{cases} u''(x) + \omega(x)u(x) = 0, \ x \in (a, b), \\ u(a) = u(b) = 0. \end{cases} \tag{4.80}$$

   Then, if (4.80) has a non-trivial solution u, and $\omega = \omega(x)$ is a real-valued and continuous function on $[a, b]$, then we must have

$$\int_a^b |\omega(x)| dx > \frac{4}{b-a}. \tag{4.81}$$

   Inequality (4.81) is called a (classical) *Lyapunov inequality.*

2. Nowadays, there are many extensions of the Lyapunov inequality (4.81). For example, in [Elb81] the Lyapunov inequality for the one-dimensional Dirichlet p-Laplacian was obtained: if

$$\begin{cases} (|u'(x)|^{p-2}u'(x))' + \omega(x)u(x) = 0, \ x \in (a, b), \ \ 1 < p < \infty, \\ u(a) = u(b) = 0, \end{cases} \tag{4.82}$$

has a non-trivial solution u for $\omega \in L^1(a,b)$, then

$$\int_a^b |\omega(x)|dx > \frac{2^p}{(b-a)^{p-1}}, \quad 1 < p < \infty. \tag{4.83}$$

Obviously, taking $p = 2$ in (4.83), we recover (4.81).

3. In [JKS17], the following Lyapunov inequality was obtained for the multi-dimensional fractional p-Laplacian $(-\Delta_p)^s$, $1 < p < \infty$, $s \in (0,1)$, with a homogeneous Dirichlet boundary condition, that is, for the equation

$$\begin{cases} (-\Delta_p)^s u = \omega(x)|u|^{p-2}u, \ x \in \Omega, \\ u(x) = 0, \ x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.84}$$

where $\Omega \subset \mathbb{R}^N$ is a measurable set, $1 < p < \infty$, and $s \in (0,1)$. More precisely, let $\omega \in L^\theta(\Omega)$ with $N > sp$, $\frac{N}{sp} < \theta < \infty$, be a non-negative weight. Suppose that the problem (4.84) has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then we have

$$\left(\int_\Omega \omega^\theta(x)\, dx \right)^{1/\theta} > \frac{C}{r_\Omega^{sp - \frac{N}{\theta}}}, \tag{4.85}$$

where $C > 0$ is a universal constant and r_Ω is the inner radius of Ω.

The appearance of the inner radius in (4.85) motivates one to define its analogue also in the setting of homogeneous groups.

Definition 4.6.2 (Inner quasi-radius)**.** Let $p > 1$ and $s \in (0,1)$ be such that $Q > sp$. Let $\Omega \subset \mathbb{G}$ be a Haar measurable set. We denote by $r_{\Omega,q}$ the *inner quasi-radius* of Ω, that is,

$$r_\Omega = r_{\Omega,|\cdot|} := \sup\{|x| : \ x \in \Omega\}. \tag{4.86}$$

Clearly, the exact values depend on the choice of a homogeneous quasi-norm $|\cdot|$. As before, if the quasi-norm is fixed, we can omit it from the notation.

4.6.1 Lyapunov type inequality for fractional *p*-sub-Laplacians

We now turn our attention to the Lyapunov inequalities for the fractional p-sub-Laplacian $(-\Delta_p)^s$ from Definition 4.1.1. Let us consider the boundary value problem

$$\begin{cases} (-\Delta_p)^s u(x) = \omega(x)|u(x)|^{p-2}u(x), \ x \in \Omega, \\ u(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \end{cases} \tag{4.87}$$

where $\omega \in L^\infty(\Omega)$. A function $u \in W_0^{s,p}(\Omega)$ is called a *weak solution of the problem* (4.87) if we have

$$\begin{aligned} &\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|y^{-1}x|^{Q+sp}} dxdy \\ &= \int_\Omega \omega(x)|u(x)|^{p-2}u(x)v(x)dx \quad \text{for all } v \in W_0^{s,p}(\Omega). \end{aligned}$$

Theorem 4.6.3 (Lyapunov inequality for fractional p-sub-Laplacian)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $\Omega \subset \mathbb{G}$ be a Haar measurable set. Let $\omega \in L^{\theta}(\Omega)$ be a non-negative function with $\frac{Q}{sp} < \theta < \infty$, and with $Q > ps$. Suppose that the boundary value problem (4.87) has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then we have*

$$\|\omega\|_{L^{\theta}(\Omega)} \geq C \Big/ r_{\Omega,|\cdot|}^{sp-Q/\theta}, \tag{4.88}$$

for some $C = C(Q, p, s, |\cdot|) > 0$.

Proof of Theorem 4.6.3. We fix a homogeneous quasi-norm $|\cdot|$ thus also eliminating it from the notation. Let $\alpha := \frac{\theta-\theta/sp}{\theta-1} \in (0,1)$ and let p^* be the Sobolev conjugate exponent as in Theorem 4.3.1. Let us define

$$\beta := \alpha p + (1-\alpha)p^*.$$

Let $\beta = p\theta'$ with $1/\theta + 1/\theta' = 1$. Then we have

$$\int_{\Omega} \frac{|u(x)|^{\beta}}{r_{\Omega}^{\alpha sp}} dx \leq \int_{\Omega} \frac{|u(x)|^{\beta}}{|x|^{\alpha sp}} dx. \tag{4.89}$$

On the other hand, the Hölder inequality with exponents $\nu = \alpha^{-1}$ and its conjugate $1/\nu + 1/\nu' = 1$ implies

$$\begin{aligned}\int_{\Omega} \frac{|u(x)|^{\beta}}{|x|^{\alpha sp}} dx &\leq \int_{\Omega} \frac{|u(x)|^{\alpha p} |u(x)|^{(1-\alpha)p^*}}{|x|^{\alpha sp}} dx \\ &\leq \left(\int_{\Omega} \frac{|u(x)|^{p}}{|x|^{sp} dx} \right)^{\alpha} \left(\int_{\Omega} |u(x)|^{p^*} dx \right)^{1-\alpha}.\end{aligned}$$

Further, by using Theorem 4.3.1 and Theorem 4.2.1, we get

$$\begin{aligned}\int_{\Omega} \frac{|u(x)|^{\beta}}{|x|^{\alpha sp}} dx &\leq C_1^{\alpha} \left(\int_{\Omega} \int_{\Omega} \frac{|u(x)-u(y)|^{p}}{|y^{-1}x|^{Q+sp}} dx dy \right)^{\alpha/p} \; C_2^{(1-\alpha)p^*/p} [u]_{s,p}^{(1-\alpha)p^*/p} \\ &\leq C_1^{\alpha} [u]_{s,p}^{\alpha} C_2^{(1-\alpha)p^*/p} [u]_{s,p}^{(1-\alpha)p^*/p} \\ &= C \left([u]_{s,p}^{p} \right)^{(\alpha p + (1-\alpha)p^*)/p} = C \left(\int_{\Omega} \omega(x) |u(x)|^{p} dx \right)^{\theta'} \\ &\leq C \left(\int_{\Omega} \omega^{\theta}(x) dx \right)^{\theta'/\theta} \int_{\Omega} |u(x)|^{p\theta'} dx = C \|\omega\|_{L^{\theta}(\Omega)}^{\theta'} \int_{\Omega} |u(x)|^{\beta} dx.\end{aligned}$$

That is, we obtain

$$\int_{\Omega} \frac{|u(x)|^{\beta}}{|x|^{\alpha sp}} dx \leq C \|\omega\|_{L^{\theta}(\Omega)}^{\theta'} \int_{\Omega} |u(x)|^{\beta} dx.$$

Thus, from (4.89) we have

$$\frac{1}{r_{\Omega}^{\alpha sp}} \int_{\Omega} |u(x)|^{\beta} dx \leq \int_{\Omega} \frac{|u(x)|^{\beta}}{|x|^{\alpha sp}} dx \leq C \|\omega\|_{L^{\theta}(\Omega)}^{\theta'} \int_{\Omega} |u(x)|^{\beta} dx.$$

Finally, we arrive at

$$\frac{C}{r_{\Omega}^{sp-Q/\theta}} \leq \|\omega\|_{L^{\theta}(\Omega)}.$$

Theorem 4.6.3 is proved. □

As an application of the Lyapunov inequality, let us consider the spectral problem for the (nonlinear) fractional p-sub-Laplacian $(-\Delta_p)^s$, $1 < p < \infty$, $s \in (0,1)$, with the Dirichlet boundary condition:

$$\begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2} u, \ x \in \Omega, \\ u(x) = 0, \ x \in \mathbb{G} \setminus \Omega. \end{cases} \tag{4.90}$$

We define the corresponding *Rayleigh quotient* by

$$\lambda_1 := \inf_{u \in W_0^{s,p}(\Omega),\ u \neq 0} \frac{[u]_{s,p}^p}{\|u\|_{L^p(\mathbb{G})}^p}. \tag{4.91}$$

Clearly, its precise value may depend on the choice of the homogeneous quasi-norm $|\cdot|$. As a consequence of Theorem 4.6.3 we have

Theorem 4.6.4 (First eigenvalue for the fractional p-sub-Laplacian)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let λ_1 be the first eigenvalue of problem (4.90) given by (4.91). Let $Q > sp$, $s \in (0,1)$ and $1 < p < \infty$. Then we have*

$$\lambda_1 \geq \sup_{\frac{Q}{sp} < \theta < \infty} \frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,|\cdot|}^{sp-Q/\theta}}, \tag{4.92}$$

where C is a positive constant given in Theorem 4.6.3 and $|\Omega|$ is the Haar measure of Ω.

Proof of Theorem 4.6.4. In Theorem 4.6.3, taking $\omega = \lambda \in L^{\theta}(\Omega)$ and using Lyapunov type inequality (4.88), we get that

$$\|\omega\|_{L^{\theta}(\Omega)} = \|\lambda\|_{L^{\theta}(\Omega)} = \left(\int_{\Omega} \lambda^{\theta} dx\right)^{1/\theta} \geq \frac{C}{r_{\Omega,|\cdot|}^{sp-Q/\theta}}.$$

For every $\theta > \frac{Q}{sp}$, we have

$$\lambda_1 \geq \frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,|\cdot|}^{sp-Q/\theta}}.$$

Thus, we obtain

$$\lambda_1 \geq \sup_{\frac{Q}{sp} < \theta < \infty} \frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,|\cdot|}^{sp-Q/\theta}},$$

for all $\frac{Q}{sp} < \theta < \infty$. Theorem 4.6.4 is proved. □

4.6.2 Lyapunov type inequality for systems

In the previous section we have presented the Lyapunov type inequality for the fractional p-sub-Laplacian with the homogeneous Dirichlet condition. Now we discuss the Lyapunov type inequality for the fractional p-sub-Laplacian system for the homogeneous Dirichlet problem.

Let $\Omega \subset \mathbb{G}$ be a Haar measurable set, let $\omega_i \in L^1(\Omega)$, $\omega_i \geq 0$, $s_i \in (0,1)$, $p_i \in (1,\infty)$. As before in Definition 4.1.1, for each p_i we denote by $(-\Delta_{p_i})^{s_i}$ the fractional p-sub-Laplacian on $\mathbb{G}$ defined by

$$(-\Delta_{p_i})^{s_i} u_i(x) = 2 \lim_{\delta \searrow 0} \int_{\mathbb{G}\setminus B(x,\delta)} \frac{|u_i(x)-u_i(y)|^{p_i-2}(u_i(x)-u_i(y))}{|y^{-1}x|^{Q+s_ip_i}} dy, \quad x \in \mathbb{G},$$
$$i = 1,\ldots,n.$$

Here $B(x,\delta)$ is a quasi-ball with respect to a fixed quasi-norm $|\cdot|$, with radius δ, centred at $x \in \mathbb{G}$. Let α_i be positive parameters such that

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = 1. \tag{4.93}$$

Then, we consider the following system of the fractional p-sub-Laplacians:

$$\begin{cases} (-\Delta_{p_1})^{s_1} u_1(x) = \omega_1(x)|u_1(x)|^{\alpha_1-2}u_1(x)|u_2(x)|^{\alpha_2}\cdots|u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ (-\Delta_{p_2})^{s_2} u_2(x) = \omega_2(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2-2}u_2(x)\cdots|u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ \qquad\cdots \\ (-\Delta_{p_n})^{s_n} u_n(x) = \omega_n(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2}\cdots|u_n(x)|^{\alpha_n-2}u_n(x), \ x \in \Omega, \end{cases} \tag{4.94}$$

with homogeneous Dirichlet conditions

$$u_i(x) = 0, \quad x \in \mathbb{G}\setminus\Omega, \quad i = 1,\ldots,n. \tag{4.95}$$

We denote by r_Ω the inner quasi-radius of Ω, that is,

$$r_\Omega = r_{\Omega,|\cdot|} := \max\{|x| : \ x \in \Omega\}.$$

Definition 4.6.5 (Weak solutions of the p-sub-Laplacian system)**.** We will say that $(u_1,\ldots,u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$ is a *weak solution* of (4.94)–(4.95) if for all $(v_1,\ldots,v_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$, we have

$$\begin{aligned} &\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u_i(x)-u_i(y)|^{p_i-2}(u_i(x)-u_i(y))(v_i(x)-v_i(y))}{|y^{-1}x|^{Q+s_ip_i}} dxdy \\ &= \int_\Omega \omega_i(x)\left(\prod_{j=1}^{i-1}|u_j(x)|^{\alpha_j}\right)\left(\prod_{j=i+1}^{n}|u_j(x)|^{\alpha_j}\right)|u_i(x)|^{\alpha_i-2}u_i(x)v_i(x)dx, \end{aligned} \tag{4.96}$$

for every $i = 1,\ldots,n$.

Now we present the following analogue of the Lyapunov type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$.

Theorem 4.6.6 (Lyapunov inequality for fractional p-sub-Laplacian system)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $s_i \in (0,1)$ and $p_i \in (1,\infty)$ be such that $Q > s_i p_i$ with $i = 1,\ldots,n$. Let $\omega_i \in L^\theta(\Omega)$ be a non-negative weight and assume that*

$$1 < \max_{i=1,\ldots,n} \left\{ \frac{Q}{s_i p_i} \right\} < \theta < \infty.$$

If (4.94)–(4.95) *admits a non-trivial weak solution, then we have*

$$\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^{n} s_j\alpha_j}, \tag{4.97}$$

where C is a positive constant.

Proof of Theorem 4.6.6. Set

$$\xi_i := \gamma_i p_i + (1-\gamma_i)p_i^*, \quad i = 1,\ldots,n,$$

and

$$\gamma_i := \frac{\theta - \frac{Q}{s_i p_i}}{\theta - 1}, \tag{4.98}$$

where $p_i^* = \frac{Q}{Q - s_i p_i}$ is the Sobolev conjugate exponent as in Theorem 4.3.1. Notice that for all $i = 1,\ldots,n$ we have $\gamma_i \in (0,1)$ and $\xi_i = p_i\theta'$, where $\theta' = \frac{\theta}{\theta-1}$. Then for every i we have

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_i s_i p_i}} dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx,$$

and by using the Hölder inequality with $\nu_i = \frac{1}{\gamma_i}$ and $\frac{1}{\nu_i} + \frac{1}{\nu_i'} = 1$, we get

$$\begin{aligned} \int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx &= \int_\Omega \frac{|u_i(x)|^{\gamma_i p_i} |u_i(x)|^{(1-\gamma_i)p_i^*}}{|x|^{\gamma_i s_i p_i}} dx \\ &\leq \left(\int_\Omega \frac{|u_i(x)|^{p_i}}{|x|^{s_i p_i}} dx \right)^{\gamma_i} \left(\int_\Omega |u_i(x)|^{p_i^*} dx \right)^{1-\gamma_i}. \end{aligned} \tag{4.99}$$

On the other hand, from Theorem 4.3.1, we obtain

$$\left(\int_\Omega |u_i(x)|^{p_i^*} dx \right)^{1-\gamma_i} \leq C[u_i]_{s_i,p_i,|\cdot|}^{p_i^*(1-\gamma_i)},$$

and from Theorem 4.2.1, we have

$$\left(\int_\Omega \frac{|u_i(x)|^{p_i}}{|x|^{s_i p_i}} dx \right)^{\gamma_i} \leq C[u_i]_{s_i,p_i,|\cdot|}^{p_i\gamma_i}.$$

Thus, from (4.99) and by setting $u_i(x) = v_i(x)$ in (4.96), we get

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx \leq C([u_i]^{p_i}_{s_i,p_i,|\cdot|,\Omega})^{\frac{\xi_i}{p_i}} \leq C([u_i]^{p_i}_{s_i,p_i,|\cdot|})^{\frac{\xi_i}{p_i}}$$
$$= C\left(\int_\Omega \omega_i(x) \prod_{j=1}^n |u_j|^{\alpha_j} dx\right)^{\frac{\xi_i}{p_i}} = C\left(\int_\Omega \omega_i(x) \prod_{j=1}^n |u_j|^{\alpha_j} dx\right)^{\theta'},$$

for every $i = 1, \ldots, n$. Therefore, by using the Hölder inequality with exponents θ and θ', we obtain

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx \leq C\|\omega_i\|^{\frac{\theta}{\theta-1}}_{L^\theta(\Omega)} \int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j \theta'} dx.$$

Again by using the Hölder inequality and (4.93), we get

$$\int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j \theta'} dx \leq \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j} dx\right)^{\frac{\alpha_j}{p_j}}.$$

It yields that

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx \leq C\|\omega_i\|^{\frac{\theta}{\theta-1}}_{L^\theta(\Omega)} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j} dx\right)^{\frac{\alpha_j}{p_j}}.$$

That is, we have

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_\Omega^{\gamma_i s_i p_i}} dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx \leq C\|\omega_i\|^{\frac{\theta}{\theta-1}}_{L^\theta(\Omega)} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j} dx\right)^{\frac{\alpha_j}{p_j}}.$$

Thus, for every $e_i > 0$ we have

$$\left(\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_\Omega^{\gamma_i s_i p_i}} dx\right)^{e_i} = \frac{1}{r_\Omega^{e_i \gamma_i s_i p_i}} \left(\int_\Omega |u_i(x)|^{\xi_i} dx\right)^{e_i}$$
$$\leq C\|\omega_i\|^{\frac{e_i\theta}{\theta-1}}_{L^\theta(\Omega)} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j} dx\right)^{\frac{e_i \alpha_j}{p_j}},$$

so that

$$\frac{1}{r_\Omega^{\sum_{j=1}^n \gamma_j s_j p_j e_j}} \prod_{i=1}^n \left(\int_\Omega |u_i(x)|^{\theta' p_i} dx\right)^{e_i}$$
$$\leq C\left(\prod_{i=1}^n \|\omega_i\|^{\frac{e_i\theta}{\theta-1}}_{L^\theta(\Omega)}\right) \left(\prod_{i=1}^n \left(\int_\Omega |u_i(x)|^{\theta' p_i} dx\right)^{\frac{\alpha_i \sum_{j=1}^n e_j}{p_i}}\right).$$

This implies

$$\frac{1}{r_\Omega^{\sum_{j=1}^n \gamma_j s_j p_j e_j}} \le C\left(\prod_{i=1}^n \|\omega_i\|_{L^\theta(\Omega)}^{\frac{e_i\theta}{\theta-1}}\right)\left(\prod_{i=1}^n \left(|u_i(x)|^{\theta' p_i}dx\right)^{\frac{\alpha_i\sum_{j=1}^n e_j}{p_i}-e_i}\right), \quad (4.100)$$

where C is a positive constant. Then, we choose e_i, $i=1,\ldots,n$, such that $\frac{\alpha_i\sum_{j=1}^n e_j}{p_i}-e_i=0$. Consequently, from (4.93) we have the solution of this system

$$e_i=\frac{\alpha_i}{p_i},\ i=1,\ldots,n. \quad (4.101)$$

Combining (4.100), (4.98) and (4.101) we arrive at

$$\prod_{i=1}^n \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \ge C r_\Omega^{Q-\theta\sum_{j=1}^n s_j\alpha_j}. \quad (4.102)$$

Theorem 4.6.6 is proved. □

In order to discuss an application of the Lyapunov type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$, we consider the spectral problem for the system of fractional p-sub-Laplacians:

$$\begin{cases} (-\Delta_{p_1,q})^{s_1}u_1(x)=\lambda_1\alpha_1\varphi(x)|u_1(x)|^{\alpha_1-2}u_1(x)|u_2(x)|^{\alpha_2}\cdots|u_n(x)|^{\alpha_n},\ x\in\Omega,\\ (-\Delta_{p_2,q})^{s_2}u_2(x)=\lambda_2\alpha_2\varphi(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2-2}u_2(x)\cdots|u_n(x)|^{\alpha_n},\ x\in\Omega,\\ \cdots\\ (-\Delta_{p_n,q})^{s_n}u_n(x)=\lambda_n\alpha_n\varphi(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2}\cdots|u_n(x)|^{\alpha_n-2}u_n(x),\ x\in\Omega, \end{cases} \quad (4.103)$$

with

$$u_i(x)=0,\quad x\in\mathbb{G}\setminus\Omega,\ \ i=1,\ldots,n, \quad (4.104)$$

where $\Omega\subset\mathbb{G}$ is a Haar measurable set, $\varphi\in L^1(\Omega)$, $\varphi\ge 0$ and $s_i\in(0,1)$, $p_i\in(1,\infty)$, $i=1,\ldots,n$.

Definition 4.6.7 (Eigenvalues of p-sub-Laplacian system (4.103)–(4.104))**.** We say that $\lambda=(\lambda_1,\ldots,\lambda_n)$ is an eigenvalue if the problem (4.103)–(4.104) admits at least one non-trivial weak solution $(u_1,\ldots,u_n)\in\prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$.

Theorem 4.6.8 (Eigenvalue estimates for p-sub-Laplacian system)**.** *Let $s_i\in(0,1)$ and $p_i\in(1,\infty)$ be such that $Q>s_ip_i$, for all $i=1,\ldots,n$, and*

$$1<\max_{i=1,\ldots,n}\left\{\frac{Q}{s_ip_i}\right\}<\theta<\infty.$$

Let $\varphi\in L^\theta(\Omega)$ with $\|\varphi\|_{L^\theta(\Omega)}\neq 0$. Then we have

$$\lambda_k\ge\frac{C}{\alpha_k}\left(\frac{1}{\prod_{i=1,i\neq k}^n\lambda_i^{\frac{\alpha_i}{p_i}}}\right)^{\frac{p_k}{\alpha_k}}\left(\frac{1}{r_\Omega^{\theta\sum_{i=1}^n\alpha_is_i-Q}\prod_{i=1,i\neq k}^n\alpha_i^{\frac{\theta\alpha_i}{p_i}}\int_\Omega\varphi^\theta(x)dx}\right)^{\frac{p_k}{\theta\alpha_k}},$$

where C is a positive constant and $k=1,\ldots,n$.

Proof of Theorem 4.6.8. In Theorem 4.6.6 by setting $\omega_k = \lambda_k \alpha_k \varphi(x)$, $k = 1, \ldots, n$, we get

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}} \lambda_k^{\frac{\theta\alpha_k}{p_k}} \prod_{i=1, i\neq k}^{n} (\alpha_i \lambda_i)^{\frac{\theta\alpha_i}{p_i}} \prod_{i=1}^{n} \|\varphi\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^{n} s_j\alpha_j}.$$

So from (4.93) we have

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}} \lambda_k^{\frac{\theta\alpha_k}{p_k}} \prod_{i=1, i\neq k}^{n} (\alpha_i \lambda_i)^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)dx \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^{n} s_j\alpha_j}.$$

This yields

$$\lambda_k^{\frac{\theta\alpha_k}{p_k}} \geq \frac{C}{\alpha_k^{\frac{\theta\alpha_k}{p_k}} r_{\Omega,q}^{\theta\sum_{j=1}^{n} s_j\alpha_j - Q} \prod_{i=1, i\neq k}^{n} (\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)dx}, \; k = 1, \ldots, n.$$

Therefore, we have

$$\lambda_k \geq \frac{C}{\alpha_k} \left(\frac{1}{\prod_{i=1, i\neq k}^{n} \lambda_i^{\frac{\alpha_i}{p_i}}} \right)^{\frac{p_k}{\alpha_k}} \left(\frac{1}{r_\Omega^{\theta\sum_{i=1}^{n} \alpha_i s_i - Q} \prod_{i=1, i\neq k}^{n} \alpha_i^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)dx} \right)^{\frac{p_k}{\theta\alpha_k}},$$
$$k = 1, \ldots, n,$$

completing the proof. □

4.6.3 Lyapunov type inequality for Riesz potentials

In this section we discuss the Lyapunov type inequality for the Riesz potential operators on homogeneous groups. As an application, we discuss a two-sided estimate for the first eigenvalue of the Riesz potential.

Definition 4.6.9 (Riesz potentials)**.** Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a quasi-norm $|\cdot|$. The *Riesz potential* on a Haar measurable set $\Omega \subset \mathbb{G}$ is the operator given by the formula

$$\mathfrak{R}u(x) = \int_\Omega \frac{u(y)}{|y^{-1}x|^{Q-2s}} dy, \quad 0 < 2s < Q. \tag{4.105}$$

The (*weighted*) *Riesz potential* is defined by

$$\mathfrak{R}(\omega u)(x) = \int_\Omega \frac{\omega(y)u(y)}{|y^{-1}x|^{Q-2s}} dy, \quad 0 < 2s < Q. \tag{4.106}$$

A Lyapunov type inequality for the weighted Riesz potentials can be formulated as follows.

Theorem 4.6.10 (Lyapunov type inequality for weighted Riesz potentials)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Let $\Omega \subset \mathbb{G}$ be a Haar measurable set, let $Q \geq 2 > 2s > 0$ and $1 < p < 2$. Assume that $\omega \in L^{\frac{p}{2-p}}(\Omega)$, $\frac{1}{|y^{-1}x|^{Q-2s}} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)$ and*

$$C_0 = \left\| \frac{1}{|y^{-1}x|^{Q-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)} < \infty.$$

Let $u \in L^{\frac{p}{p-1}}(\Omega)$, $u \neq 0$, satisfy

$$\mathfrak{R}(\omega u)(x) = \int_\Omega \frac{\omega(y)u(y)}{|y^{-1}x|^{Q-2s}} dy = u(x), \text{ for a.e. } x \in \Omega. \tag{4.107}$$

Then

$$\|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \geq \frac{1}{C_0}. \tag{4.108}$$

Proof of Theorem 4.6.10. In (4.107), by using Hölder's inequality for $p, \theta > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{\theta} + \frac{1}{\theta'} = 1$, we get

$$\begin{aligned}
|u(x)| &= \left| \int_\Omega \frac{\omega(y)u(y)}{|y^{-1}x|^{Q-2s}} dy \right| \\
&\leq \left(\int_\Omega |\omega(y)u(y)|^p dy \right)^{\frac{1}{p}} \left(\int_\Omega \left| \frac{1}{|y^{-1}x|^{Q-2s}} \right|^{p'} dy \right)^{\frac{1}{p'}} \\
&\leq \left(\int_\Omega |\omega(y)|^{p\theta} dy \right)^{\frac{1}{p\theta}} \left(\int_\Omega |u(y)|^{\theta' p} dy \right)^{\frac{1}{\theta' p}} \left(\int_\Omega \left| \frac{1}{|y^{-1}x|^{Q-2s}} \right|^{p'} dy \right)^{\frac{1}{p'}} \\
&= \|\omega\|_{L^{p\theta}(\Omega)} \|u\|_{L^{p\theta'}(\Omega)} \left(\int_\Omega \left| \frac{1}{|y^{-1}x|^{Q-2s}} \right|^{p'} dy \right)^{\frac{1}{p'}}.
\end{aligned}$$

Let p' be such that $p' = p\theta'$, so that $\theta = \frac{1}{2-p}$. Then we have

$$|u(x)| \leq \|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \|u\|_{L^{\frac{p}{p-1}}(\Omega)} \left(\int_\Omega \left| \frac{1}{|y^{-1}x|^{Q-2s}} \right|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}}. \tag{4.109}$$

From (4.109) we calculate

$$\begin{aligned}
\|u\|_{L^{\frac{p}{p-1}}(\Omega)} &\leq \|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \|u\|_{L^{\frac{p}{p-1}}(\Omega)} \left(\int_\Omega \int_\Omega \left| \frac{1}{|y^{-1}x|^{Q-2s}} \right|^{\frac{p}{p-1}} dxdy \right)^{\frac{p-1}{p}} \\
&= C_0 \|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \|u\|_{L^{\frac{p}{p-1}}(\Omega)}.
\end{aligned}$$

Finally, since $u \neq 0$, this implies

$$\|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \geq \frac{1}{C_0},$$

completing the proof. $\square$

Let us now consider the following spectral problem for the Riesz potential:

$$\mathfrak{R}u(x) = \int_\Omega \frac{u(y)}{|y^{-1}x|^{Q-2s}}dy = \lambda u(x), \quad x \in \Omega, \quad 0 < 2s < Q. \tag{4.110}$$

We recall the Rayleigh quotient for the Riesz potential:

$$\lambda_1(\Omega) = \sup_{u\neq 0} \frac{\int_\Omega \int_\Omega \frac{u(x)u(y)}{|y^{-1}x|^{Q-2s}}dxdy}{\|u\|^2_{L^2(\Omega)}}, \tag{4.111}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Riesz potential. A direct consequence of Theorem 4.6.10 is the following estimate for this first eigenvalue.

Theorem 4.6.11 (First eigenvalue of the Riesz potential)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $\Omega \subset \mathbb{G}$ be a Haar measurable set, let $Q \geq 2 > 2s > 0$ and $1 < p < 2$ with $\frac{1}{|y^{-1}x|^{Q-2s}} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)$ and set*

$$C_0 := \left\| \frac{1}{|y^{-1}x|^{Q-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)}.$$

Then for the spectral problem (4.110), we have

$$\lambda_1(\Omega) \leq C_0 |\Omega|^{\frac{2-p}{p}}. \tag{4.112}$$

Proof of Theorem 4.6.11. By using (4.111), Theorem 4.6.10 and $\omega = \frac{1}{\lambda_1(\Omega)}$, we obtain

$$\lambda_1(\Omega) \leq C_0 |\Omega|^{\frac{2-p}{p}}. \tag{4.113}$$

Theorem 4.6.11 is proved. $\square$

Euclidean case. Let us now record several applications of the above constructions in the case of the Abelian group $(\mathbb{R}^N, +)$. With the Euclidean distance $|\cdot|_E$, the Riesz potential is given by

$$\mathfrak{R}u(x) = \int_\Omega \frac{u(y)}{|x-y|_E^{N-2s}}dy, \quad 0 < 2s < N, \quad \Omega \subset \mathbb{R}^N, \tag{4.114}$$

and the weighted Riesz potential is

$$\mathfrak{R}(\omega u)(x) = \int_\Omega \frac{\omega(y)u(y)}{|x-y|_E^{N-2s}}dy, \quad 0 < 2s < N, \quad \Omega \subset \mathbb{R}^N. \tag{4.115}$$

Then, in Theorem 4.6.10, setting $\mathbb{G} = (\mathbb{R}^N, +)$ and taking the standard Euclidean distance instead of the quasi-norm, we obtain

Theorem 4.6.12 (Euclidean Lyapunov inequality for the Riesz potential)**.** *Let $\Omega \subset \mathbb{R}^N$ be a measurable set with $|\Omega| < \infty$, $1 < p < 2$ and let $N \geq 2 > 2s > 0$. Let $\omega \in L^{\frac{p}{2-p}}(\Omega)$, $\frac{1}{|x-y|_E^{N-2s}} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)$ and set*

$$S := \left\| \frac{1}{|x-y|_E^{N-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)}.$$

In addition, assume that $u \in L^{\frac{p}{p-1}}(\Omega)$, $u \neq 0$, satisfies

$$\mathfrak{R}(\omega u)(x) = u(x), \quad x \in \Omega.$$

Then we have

$$\|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \geq \frac{1}{S}.$$

Now let us consider the spectral problem for the Euclidean Riesz potential from (4.114):

$$\mathfrak{R}u(x) = \int_\Omega \frac{u(y)}{|x-y|_E^{N-2s}} dy = \lambda u(x), \quad 0 < 2s < N. \tag{4.116}$$

Theorem 4.6.13 (Isoperimetric inequality for the Euclidean Riesz potential)**.** *Let $\Omega \subset \mathbb{R}^N$ be a set with $|\Omega| < \infty$, $1 < p < 2$ and $N \geq 2 > 2s > 0$ and $1 < p < 2$. Assume that $\omega \in L^{\frac{p}{2-p}}(\Omega)$, $\frac{1}{|x-y|_E^{N-2s}} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)$ and set*

$$S := \left\| \frac{1}{|x-y|_E^{N-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)}.$$

Then, for the spectral problem (4.116), we have,

$$\lambda_1(\Omega) \leq \lambda_1(B) \leq S|B|^{\frac{2-p}{p}},$$

where $B \subset \mathbb{R}^N$ is an open ball, $\lambda_1(\Omega)$ is the first eigenvalue of the spectral problem (4.116), for all sets Ω with $|\Omega| = |B|$.

Proof of Theorem 4.6.13. The proof of $\lambda_1(B) \leq S|B|^{\frac{2-p}{p}}$ is the same as the proof of Theorem 4.6.11. From [RRS16] we have

$$\lambda_1(B) \geq \lambda_1(\Omega),$$

which completes the proof of Theorem 4.6.13. $\square$

4.7 Hardy inequalities for fractional sub-Laplacians on stratified groups

In this section we discuss the Hardy inequalities involving fractional powers of the sub-Laplacian from a different point of view. First, we observe another way of writing the well-known one-dimensional L^p-Hardy inequality

$$\int_0^\infty \left|\frac{f(x)}{x}\right|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \left|\frac{df(x)}{dx}\right|^p dx, \quad p>1.$$

Consequently, one can replace $f(x)$ by $xf(x)$ and restate this inequality in terms of the boundedness of the operator

$$Tf(x) = \frac{d(xf(x))}{dx},$$

or of its dual operator

$$T^* f(x) = x\frac{df(x)}{dx}.$$

In this section, we discuss this point of view and its extension to the subelliptic setting. For fractional powers of the sub-Laplacian on stratified groups this has been analysed in [CCR15], with subsequent extensions to more general hypoelliptic operators and more general graded groups in [RY18a]. We discuss this matter in the spirit of the former.

4.7.1 Riesz kernels on stratified Lie groups

In this section we briefly recapture some properties of the Riesz kernels of the sub-Laplacians on stratified Lie groups. Historically, these have been consistently developed in [Fol75]. For a detailed unifying description containing also higher-order hypoelliptic operators on general graded groups, their fractional powers, and the corresponding Riesz and Bessel kernels we refer to the exposition in [FR16, Section 4.3] where one can find proofs of most of the properties described in this section.

The analysis of second-order hypoelliptic operators, including the sub-Laplacian, is significantly simpler than that of higher-order operators. This difference is based on the Hunt theorem [Hun56] which asserts that the semigroup $\left\{e^{-t\mathcal{L}}\right\}_{t>0}$ generated by the sub-Laplacian $\mathcal{L}$ (or more precisely, by its unique self-adjoint extension) consists of convolutions with probability measures. Moreover, the hypoellipticity of $\mathcal{L}$ implies that these measures are absolutely continuous with respect to the Haar measure and have densities in the Schwartz space [FS82], which are strictly positive by the Bony maximum principle [Bon69]. Therefore, for all $t \in \mathbb{R}^+$, there exists the heat kernel of $\mathcal{L}$, that is, a function $p_t \in \mathcal{S}(\mathbb{G})$ such that

$$e^{-t\mathcal{L}} f(x) = f * p_t(x) = \int_{\mathbb{G}} f(xy^{-1}) p_t(y) dy$$

for all $x \in \mathbb{G}$ and all $f \in L^2(\mathbb{G})$. Since $\mathcal{L}$ is homogeneous of order two with respect to the dilations D_r of a stratified group $\mathbb{G}$, we also have

$$p_t(x) = t^{-Q/2} P(D_{t^{-1/2}} x)$$

for all $x \in \mathbb{G}$, where $P = p_1$, which this is a strictly positive function in the Schwartz class $\mathcal{S}(\mathbb{G})$ and

$$\int_{\mathbb{G}} P(x) dx = 1.$$

Let us define the following fractional integral kernel F_α on $\mathbb{G}\backslash\{0\}$ by

$$F_\alpha(x) := \int_0^\infty t^{\alpha/2} p_t(x) \frac{dt}{t} = \int_0^\infty t^{(\alpha-Q)/2} P(D_{t^{-1/2}} x) \frac{dt}{t}, \quad \operatorname{Re}\alpha < Q, \quad (4.117)$$

which converges absolutely and uniformly on compact subsets of $\mathbb{G}\backslash\{0\}$ to a smooth function, homogeneous of degree $\alpha - Q$. Since P is positive F_α is positive when α is real. Thus, the function $|\cdot|_\alpha$, given by the formula

$$|x|_\alpha := \begin{cases} (F_0(x))^{-1/(Q-\alpha)} & \text{if } x \in \mathbb{G}\backslash\{0\}, \\ 0 & \text{if } x = 0, \end{cases} \quad (4.118)$$

is non-negative and homogeneous of degree 1, and vanishes only at the origin. So $|\cdot|_\alpha$ is a homogeneous norm, and for further discussions it will be convenient to use the norm $|\cdot|_0$, which is a limit of the norms $|\cdot|_\alpha$.

The fractional integral $F_\alpha(x)$ in (4.117) is holomorphic with respect to α in $\mathcal{H}_Q := \{\alpha \in \mathbb{C} : \operatorname{Re}\alpha < Q\}$, and its derivatives

$$F_\alpha^{(k)}(x) = \frac{1}{2^k} \int_0^\infty t^{\alpha/2} (\log t)^k p_t(x) \frac{dt}{t}$$

are the absolutely convergent integrals. The functions F_α are smooth and homogeneous of degree $\alpha - Q$ and locally integrable on $\mathbb{G}$ when $0 < \operatorname{Re}\alpha < Q$. Thus, the associated distributions are defined by

$$\begin{aligned} \langle F_\alpha, \phi \rangle &:= \int_{\mathbb{G}} F_\alpha(x)\phi(x) dx = \int_{\mathbb{G}} \int_0^\infty \phi(x) t^{(\alpha-Q)/2-1} P(D_{t^{-1/2}} x) dt dx \\ &= 2 \int_0^\infty \int_{\mathbb{G}} s^{\alpha-1} \phi(D_s x) P(x) dx ds \end{aligned} \quad (4.119)$$

for all $\phi \in C_0^\infty(\mathbb{G})$. Now it is clear that

$$I_\alpha := \Gamma(\alpha/2)^{-1} F_\alpha$$

is the *Riesz kernel of order* α, i.e., the convolution kernel of $\mathcal{L}^{-\alpha/2}$, that is,

$$\mathcal{L}^{-\alpha/2} f = f * I_\alpha, \quad (4.120)$$

with

$$I_\alpha|_{\alpha=0} = \delta_0. \quad (4.121)$$

The family of Riesz kernels can be analytically continued as distributions to the half-plane $\mathcal{H}_Q$ by the identity

$$I_\alpha = \mathcal{L}(I_{\alpha+2}).$$

The analytic continuation to the strip $\{\alpha \in \mathbb{C} : -1 < \operatorname{Re}\alpha < Q\}$ will be enough for our purpose. An explicit expression is obtained by rewriting (4.119) in the form

$$\begin{aligned}\langle I_\alpha, \phi\rangle = &\frac{2}{\Gamma\left(\frac{\alpha}{2}\right)}\int_{\mathbb{G}}\int_0^1 s^{\alpha-1}(\phi(D_s x) - \phi(0))dsP(x)dx + \frac{2\phi(0)}{\alpha\Gamma(\frac{\alpha}{2})}\\ &+ \frac{2}{\Gamma(\frac{\alpha}{2})}\int_{\mathbb{G}}\int_1^\infty s^{\alpha-1}\phi(D_s x)dsP(x)dx.\end{aligned} \tag{4.122}$$

Since $\Gamma'(1) = -\gamma$ and hence

$$\frac{1}{\Gamma(t)} = \frac{t}{\Gamma(t+1)} = t - \gamma t^2 + O(t^3) \text{ as } t \to 0, \tag{4.123}$$

and

$$s^t = 1 + \log s \int_0^t s^u du,$$

the equality (4.122) implies that for α near 0, for $\phi \in C_0^\infty(\mathbb{G})$ we have

$$c\langle I_\alpha, \phi\rangle = \phi(0) + \langle \Lambda, \phi\rangle\alpha + O(\alpha^2), \tag{4.124}$$

where Λ is the distribution defined by

$$\begin{aligned}\langle \Lambda, \phi\rangle := &\int_{\mathbb{G}}\int_0^1 \frac{1}{s}(\phi(D_s x) - \phi(0))dsP(x)dx + \frac{\gamma\phi(0)}{2}\\ &+ \int_{\mathbb{G}}\int_1^\infty \frac{1}{s}\phi(D_s x)dsP(x)dx.\end{aligned} \tag{4.125}$$

Thus, (4.124) is the Taylor expansion of I_α around 0. By the analytic continuation of (4.122) one obtains the following representation of the distribution Λ in terms of the homogeneous norm $|\cdot|_0$ defined in (4.118): There exists $a > 0$ such that

$$\langle \Lambda, \phi\rangle = \frac{1}{2}\left(\int_{\{x\in\mathbb{G}:|x|_0<a\}}(\phi(x) - \phi(0))|x|_0^{-Q}dx + \int_{\{x\in\mathbb{G}:|x|_0>a\}}\phi(x)|x|_0^{-Q}dx\right), \tag{4.126}$$

for all $\phi \in C_0^\infty$.

Let $0 < \operatorname{Re}\alpha$, $0 < \operatorname{Re}\beta$ and $\operatorname{Re}(\alpha+\beta) < Q$. Then the convolution of the Riesz potentials of orders α and β is defined pointwise by absolutely convergent integrals as well as

$$I_\alpha * I_\beta = I_{\alpha+\beta} \tag{4.127}$$

is satisfied pointwise and in the sense of distributions. This identity is equivalent to the functional equality $\mathcal{L}^{-\alpha/2}\mathcal{L}^{-\beta/2} = \mathcal{L}^{-(\alpha+\beta)/2}$ and can be obtained from (4.117) by using the properties of the heat kernel. This fact can be also extended by using the analytical continuation. That is,

$$(\phi * I_\alpha) * I_\beta = \phi * I_{\alpha+\beta}$$

holds if $\operatorname{Re}\alpha > -1$, $\operatorname{Re}\beta > -1$ and $\operatorname{Re}(\alpha+\beta) < Q$. These distributions will be useful to present a family of Hardy type inequalities in the following sections. We refer to [FR16, Section 4.3] for the detailed discussion of these and other properties of the Riesz kernels and fractional powers of left invariant hypoelliptic operators.

4.7.2 Hardy inequalities for fractional powers of sub-Laplacians

As in this whole section, let $\mathcal{L}$ be the sub-Laplacian on the stratified Lie group $\mathbb{G}$ of homogeneous dimension Q. Let us define the operator T_α on $C_0^\infty(\mathbb{G})$ by the formula

$$T_\alpha f := |\cdot|^{-\alpha}\mathcal{L}^{-\alpha/2} f = |\cdot|^{-\alpha}(f * I_\alpha), \quad -1 < \operatorname{Re}\alpha < Q.$$

Consequently, the operator

$$T_\alpha^* g = (|\cdot|^{-\overline{\alpha}} g) * I_{\overline{\alpha}}$$

satisfies

$$\langle f, T_\alpha^* g\rangle = \langle T_\alpha f, g\rangle \tag{4.128}$$

for all $f, g \in C_0^\infty(\mathbb{G})$. It turns out that the operator T_α is bounded on $L^p(\mathbb{G})$ when $1 < p < \infty$ and $0 < \alpha < Q/p$ which, in turn, can be formulated as a version of a Hardy inequality:

Theorem 4.7.1 (Hardy inequality for fractional powers of sub-Laplacian)**.** *Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q. Let $1 < p < \infty$ and $0 < \alpha < Q/p$. Then the operator T_α extends uniquely to a bounded operator on $L^p(\mathbb{G})$.*

Applying this to $\mathcal{L}^{\alpha/2} f$ rather than f we get that for all $f \in C_0^\infty(\mathbb{G})$, we have

$$\left\|\frac{f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})} \le \|T_\alpha\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})} \|\mathcal{L}^{\alpha/2} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty. \tag{4.129}$$

Remark 4.7.2 (Hardy–Sobolev inequalities)**.**

1. Theorem 4.7.1 was established in [CCR15]. It is easy to see that combined with the Sobolev inequality, it implies the following *Hardy–Sobolev inequality*:

   Let $0 \le b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant $C > 0$ such that we have

$$\left\|\frac{f}{|x|^{\frac{b}{q}}}\right\|_{L^q(\mathbb{G})} \le C\|(-\mathcal{L})^{\frac{a}{2}} f\|_{L^p(\mathbb{G})}. \tag{4.130}$$

Such an inequality can be interpreted as a weighted Sobolev embedding of the homogeneous Sobolev space $\dot{L}^p_a(\mathbb{G})$ over L^p of order a, based on the sub-Laplacian $\mathcal{L}$. The theory of such spaces has been extensively developed by Folland [Fol75]. We refer to [RY18a] for the inequality (4.130).

2. The asymptotic behaviour of $\|T_\alpha\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})}$ with respect to α will be given in Theorem 4.7.3; in its proof we will follow the original proof in [CCR15], as well as for Theorem 4.7.4.
3. (Critical global Hardy inequality for $a = Q/p$) Let $1 < p < r < \infty$ and $p < q < (r-1)p'$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(p, q, r, Q) > 0$ such that we have

$$\left\| \frac{f}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \le C(\|f\|_{L^p(\mathbb{G})} + \|(-\mathcal{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbb{G})}). \quad (4.131)$$

4. (Critical local Hardy inequality for $a = Q/p$) Let $1 < p < \infty$ and $\beta \in [0, Q)$. Let $r > 0$ be given and let x_0 be any point of $\mathbb{G}$. Then for any $p \le q < \infty$ there exists a positive constant $C = C(p, Q, \beta, r, q)$ such that we have

$$\left\| \frac{f}{|x|^{\frac{\beta}{q}}} \right\|_{L^q(B(x_0,r))} \le Cq^{1-1/p}(\|f\|_{L^p(B(x_0,r))} + \|(-\mathcal{L})^{\frac{Q}{2p}} f\|_{L^p(B(x_0,r))}), \quad (4.132)$$

and such that

$$\lim_{q\to\infty} \sup C(p, Q, \beta, r, q) < \infty.$$

Inequalities (4.131) and (4.132) have been obtained in [RY18a], to which we refer for their proofs as well as for the expressions for the asymptotically sharp constants in these inequalities.

5. (Trudinger–Moser inequality on stratified groups) Let $Q \ge 3$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then there exists a constant $\alpha_Q > 0$ such that we have

$$\sup_{\|f\|_{L^Q_1(\mathbb{G})} \le 1} \int_{\mathbb{G}} \frac{1}{|x|^\beta} \left(\exp(\alpha |f(x)|^{Q'}) - \sum_{k=0}^{Q-2} \frac{\alpha^k |f(x)|^{kQ'}}{k!} \right) dx < \infty \quad (4.133)$$

for any $\beta \in [0, Q)$ and $\alpha \in (0, \alpha_Q(1 - \beta/Q))$, where $Q' = Q/(Q-1)$. When $\alpha > \alpha_Q(1 - \beta/Q)$, the integral in (4.133) is still finite for any $f \in L^Q_1(\mathbb{G})$, but the supremum is infinite. The space $L^Q_1(\mathbb{G})$ is the Sobolev space with the norm $\|f\|_{L^Q_1(\mathbb{G})} = \|f\|_{L^Q(\mathbb{G})} + \|\nabla_H f\|_{L^Q(\mathbb{G})}$.

The inequality (4.133) was obtained in [RY18a], also with a rather explicit expression for the constant α_Q. We refer to the above paper also for an extensive history of this subject.

6. (Hardy–Sobolev inequalities on graded groups) In fact, Hardy–Sobolev inequalities (4.130), (4.131) and (4.132), Trudinger–Moser inequalities (4.133) and their local versions, have been obtained in [RY18a] for general homogeneous left invariant hypoelliptic differential operators (the so-called Rockland operators) on **general graded Lie groups**. The methods of the proof, however, are rather different, and somewhat more involved, than the proofs for the sub-Laplacians presented in this section. Therefore, here we do not present them in full generality: these results, their proofs, and expressions for (asymptotically) best constants can be found in [RY18a].

Proof of Theorem 4.7.1. As usual here we denote the conjugate number to p by p'. Since the integral kernel of the operator T_α is positive, its L^p-boundedness follows from the following *Schur test* (see [FR75]):

Suppose that there exist a positive function g and constants $A_{\alpha,p}$ and $B_{\alpha,p}$ such that we have

$$T_\alpha(g^{p'})(x) \le A_{\alpha,p} g^{p'}(x) \text{ and } T^*_\alpha(g^p)(x) \le B_{\alpha,p} u^p(x) \tag{4.134}$$

for almost all $x \in \mathbb{G}$. Then the estimate

$$\|T_\alpha f\|_{L^p(\mathbb{G})} \le A^{1/p'}_{\alpha,p} B^{1/p}_{\alpha,p} \|f\|_{L^p(\mathbb{G})} \tag{4.135}$$

holds for all $f \in L^p(\mathbb{G})$.

In order to produce g as in (4.134), let us consider the family of functions $g_\gamma := |\cdot|^{\gamma-Q}$, $\gamma > 0$, and consider the convolutions $g_\gamma^{p'} * I_\alpha$ and $(|\cdot|^{-\alpha} g_\gamma^p) * I_\alpha$ related to the computations of $T_\alpha(g_\gamma^{p'})$ and $T^*_\alpha(g_\gamma^p)$. Let

$$\beta' = Q + (\gamma - Q)p' \text{ and } \beta = Q - \alpha + (\gamma - Q)p, \tag{4.136}$$

so that $g_\gamma^{p'}$ and $|\cdot|^{-\alpha} g_\gamma^p$ have the same homogeneity as $I_{\beta'}$ and I_β, respectively. As in the case of (4.127) these convolution integrals converge absolutely in $\mathbb{G}\backslash\{0\}$ if and only if $0 < \beta < Q - \alpha$ and $0 < \beta' < Q - \alpha$, i.e., for γ such that

$$\max\left(\frac{Q}{p}, \frac{\alpha}{p} + \frac{Q}{p'}\right) < \gamma < Q - \frac{\alpha}{p'}. \tag{4.137}$$

Thus, if this condition is satisfied, then both $T_\alpha(g_\gamma^{p'})$ and $T^*_\alpha(g_\gamma^p)$ are positive functions, continuous away from the origin, and of the same homogeneity as $I_{\beta'}$ and I_β, respectively. The pointwise estimates (4.134) follow directly from the homogeneity. Finally, for γ the condition (4.137) is nontrivial if and only if $0 < \alpha < Q/p$, completing the proof of Theorem 4.7.1. □

By Theorem 4.7.1, the operator T_α is L^p-bounded and the constant in the inequality (4.129) depends on the choice of a homogeneous quasi-norm. While the inequality itself holds true for any homogeneous quasi-norm (due to their equivalence), in the case of particular homogeneous norm $|\cdot|_0$ from the family (4.118) one can give the following estimate for it.

Theorem 4.7.3 (Estimate for L^p-operator norm)**.** *Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q. Let $1 < p < \infty$ and $0 < \alpha < Q/p$. For the particular homogeneous norm $|\cdot|_0$, the operator norm $\|T_\alpha\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})}$ satisfies*

$$\|T_\alpha\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})} \leq 1 + C\alpha + O(\alpha^2).$$

For the rest of this subsection, we assume that $|\cdot| = |\cdot|_0$ and that the operator T_α is defined using $|\cdot|_0$.

Proof of Theorem 4.7.3. The proof of Theorem 4.7.1, and especially (4.135), show that, if

$$0 < \alpha < \frac{Q}{p} \quad \text{and} \quad \frac{\alpha}{p} + \frac{Q}{p'} < \gamma < Q - \frac{\alpha}{p'}, \tag{4.138}$$

then

$$\|T_\alpha\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})} \leq A^{1/p'}_{\alpha,\gamma,p} B^{1/p}_{\alpha,\gamma,p}, \tag{4.139}$$

where

$$\begin{aligned} A_{\alpha,\gamma,p} &:= \sup_{|y|=1} \left(|\cdot|_0^{(\gamma-Q)p'} * I_\alpha \right)(y) \quad = \sup_{|y|=1} \left(|\cdot|_0^{\beta'-Q} * I_\alpha \right)(y), \\ B_{\alpha,\gamma,p} &:= \sup_{|y|=1} \left(|\cdot|_0^{(\gamma-Q)p-\alpha} * I_\alpha \right)(y) = \sup_{|y|=1} \left(|\cdot|_0^{\beta-Q} * I_\alpha \right)(y). \end{aligned} \tag{4.140}$$

First let us show that there is a constant C_p such that

$$\|T_\alpha f\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})} \leq 1 + C_p\alpha + O(\alpha^2), \quad 1 < p < \infty, \tag{4.141}$$

holds for all sufficiently small positive α. Assume that $Q/p' < \gamma < Q$. Then (4.138) is valid for α in a neighborhood of 0^+. Moreover, with β and β' as in (4.136), the constants $A_{\alpha,\gamma,p}$ and $B_{\alpha,\gamma,p}$ in (4.140) are bounded by $1 + L_{\beta'}\alpha + O(\alpha^2)$ and $1 + L_\beta\alpha + O(\alpha^2)$, respectively. By (4.121) we obtain

$$\|T_\alpha\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})} \leq 1 + \left(\frac{L_{\beta'}}{p'} + \frac{L_\beta}{p} \right)\alpha + O(\alpha^2),$$

which confirms (4.141). Combining this with Theorem 4.7.1 completes the proof of Theorem 4.7.3. $\square$

Theorem 4.7.4 (A logarithmic version of uncertainty principle)**.** *Let $\mathbb{G}$ be a stratified Lie group and let $1 < p < \infty$. Then we have*

$$\int_{\mathbb{G}} (\log|x|)|f(x)|^p dx + \int_{\mathbb{G}} \mathrm{Re}((\log \mathcal{L}^{1/2} f)(x)\overline{f(x)})|f(x)|^{p-2} dx \geq -C_p\|f\|^p_{L^p(\mathbb{G})}.$$

Proof of Theorem 4.7.4. Now let us recall the distribution Λ introduced in (4.125). We also define the operator

$$T_0' f := \left(\frac{d}{d\alpha} T_\alpha f \right)\Bigg|_{\alpha=0} = -\log \mathcal{L}^{1/2} f - (\log|\cdot|_0) f = -f * \Lambda - (\log|\cdot|_0) f. \tag{4.142}$$

Setting $\phi = |y \cdot |_0^{\beta-Q}$ in (4.126), the integrals converge absolutely for any $|y| = 1$, and so (4.124) also extends to

$$| \cdot |_0^{\beta-Q} * I_\alpha(y) = 1 + \alpha | \cdot |_0^{\beta-Q} * \Lambda(y) + O(\alpha^2);$$

the term $O(\alpha^2)$ is uniform in $|y| = 1$. Taking the supremum over y yields the logarithmic uncertainty inequality for T_0', that is, this shows that if α is small and $0 < \alpha < Q - \beta$, then

$$\sup_{|y|=1} \left(| \cdot |_0^{\beta-Q} * I_\alpha \right)(y) \leq 1 + L_\beta \alpha + O(\alpha^2), \quad 0 < \beta < Q,$$

where

$$L_\beta = \sup_{|y|=1} \left(| \cdot |_0^{\beta-Q} * \Lambda(y) \right). \tag{4.143}$$

Let $1 < p < \infty$. Taking $f \in C_0^\infty(\mathbb{G})$ with $\|f\|_{L^p(\mathbb{G})} = 1$ and restricting ourselves to positive values of α for which (4.141) holds and $1 + C_p\alpha > 0$, we consider the function

$$\Phi_\epsilon(\alpha) := (1 + (C_p + \epsilon)\alpha)^p - \|T_\alpha f\|^p_{L^p(\mathbb{G})},$$

where $\epsilon \geq 0$. The expression $\|T_\alpha f\|^p_{L^p(\mathbb{G})}$ can be differentiated in α at 0, and

$$\frac{d}{d\alpha} \left(\|T_\alpha f\|^p_{L^p(\mathbb{G})} \right) \Big|_{\alpha=0} = p \int_{\mathbb{G}} \operatorname{Re} \left(T_0' f(x) \overline{f(x)} \right) |f(x)|^{p-2} dx.$$

Hence Φ_ϵ is differentiable at 0. Now $\Phi_\epsilon(0) = 0$ and $\Phi_\epsilon(\alpha) \geq 0$ for all sufficiently small α, by (4.141), and so $\Phi_\epsilon'(0) \geq 0$. Now we let ϵ tend to 0, and deduce that $\Phi_\epsilon(0) \geq 0$. Thus, the statement of Theorem 4.7.4 follows from (4.142). ☐

4.7.3 Landau–Kolmogorov inequalities on stratified groups

In this section we discuss the stratified groups version of the Landau–Kolmogorov inequality, and some of its consequences. We start with a related simpler version of such an inequality.

Proposition 4.7.5 (Two weighted inequalities)**.**

(1) *Let $\mathbb{G}$ be a homogeneous group. If $1 \leq p, q, r \leq \infty$, and*

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}, \quad 0 \leq \theta \leq 1,$$

then

$$\| | \cdot |^\alpha f\|_{L^p(\mathbb{G})} \leq \| | \cdot |^{\alpha/\theta} f\|^\theta_{L^q(\mathbb{G})} \|f\|^{1-\theta}_{L^r(\mathbb{G})}, \quad \alpha \geq 0,$$

holds for all $f \in C_0^\infty(\mathbb{G})$.

(2) *Let $\mathbb{G}$ be a stratified group and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. Let $\beta > 0$, $\gamma > 0$, $p > 1$, $q \geq 1$, $r > 1$ be such that*

$$\gamma < \frac{Q}{r} \quad \textit{and} \quad \frac{\beta+\gamma}{p} = \frac{\gamma}{q} + \frac{\beta}{r}.$$

Then

$$\|f\|_{L^p(\mathbb{G})} \leq C \| |\cdot|^\beta f\|^{\gamma/(\beta+\gamma)}_{L^q(\mathbb{G})} \|\mathcal{L}^{\gamma/2} f\|^{\beta/(\beta+\gamma)}_{L^r(\mathbb{G})} \tag{4.144}$$

holds for all $f \in C_0^\infty(\mathbb{G})$.

Proof. Part (1). Writing

$$(|\cdot|^\alpha |f|)^p = \left(|\cdot|^\alpha |f|^\theta\right)^p \left(|f|^{1-\theta}\right)^p,$$

then applying the Hölder inequality (with index s) we have

$$\left(\int_{\mathbb{G}} (|x|^\alpha |f(x)|^p) dx\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{G}} \left(|x|^\alpha |f(x)|^\theta\right)^{ps} dx\right)^{\frac{1}{ps}} \left(\int_{\mathbb{G}} (|f(x)|^{1-\theta})^{ps'} dx\right)^{\frac{1}{ps'}}.$$

If the index s is chosen so that $q = \theta ps$, then $r = (1-\theta)ps'$, finishing the proof.

Part (2). By using the Hölder inequality and (4.129) with (4.141) we have

$$\begin{aligned} \|f\|_{L^p(\mathbb{G})} &\leq \| |\cdot|^\beta f\|^{\gamma/(\beta+\gamma)}_{L^q(\mathbb{G})} \| |\cdot|^{-\gamma} f\|^{\beta/(\beta+\gamma)}_{L^r(\mathbb{G})} \\ &\leq C \| |\cdot|^\beta f\|^{\gamma/(\beta+\gamma)}_{L^q(\mathbb{G})} \|\mathcal{L}^{\gamma/2} f\|^{\beta/(\beta+\gamma)}_{L^r(\mathbb{G})}, \end{aligned}$$

yielding (4.144). ∎

Now we present the following Landau–Kolmogorov type inequality.

Theorem 4.7.6 (Landau–Kolmogorov type inequality)**.** *Let $\mathbb{G}$ be a stratified group and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. If $1 < p, q, r < \infty$, and*

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}, \quad 0 \leq \theta \leq 1,$$

then we have

$$\|\mathcal{L}^{\alpha/2} f\|_{L^p(\mathbb{G})} \leq C \|\mathcal{L}^{\alpha/2\theta} f\|^\theta_{L^q(\mathbb{G})} \|f\|^{1-\theta}_{L^r(\mathbb{G})}, \quad \alpha \geq 0, \tag{4.145}$$

for all $f \in C_0^\infty(\mathbb{G})$.

Proof of Theorem 4.7.6. To prove this theorem we will use the well-known complex interpolation methods (see, e.g., [BL76] or [Ste56]). First, we have to see that the operator $\mathcal{L}^{iy}$, where $y \in \mathbb{R}$, is bounded on $L^s(\mathbb{G})$ with $1 < s < \infty$, and that

$$\|\mathcal{L}^{iy/2}\|_{L^s \to L^s} \leq C(s)\phi(y) \;\; \text{with} \;\; \phi(y) = e^{\gamma|y|}.$$

In the case of the stratified groups this follows from the Mihlin–Hörmander multiplier theorem (see, e.g., [Chr91], [MM90]).

Let $g \in L^{p'}(\mathbb{G})$ be a compactly supported simple function of norm one. For z in the strip $S := \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq \frac{\alpha}{\theta}\}$, let us define

$$g_z(x) := \|g\|^{az+b}_{L^{p'}(\mathbb{G})} g(x)|g(x)|^{cx+d}$$

for all $x \in \mathbb{G}$, where

$$a = \frac{\theta p'}{\alpha}\left(\frac{1}{q} - \frac{1}{p}\right),\ b = -\frac{p'}{r'},\ c = \frac{\theta p'}{\alpha}\left(\frac{1}{q} - \frac{1}{p}\right),\ d = \frac{p'}{r'} - 1.$$

For $\operatorname{Re} z = \eta$ and

$$\frac{1}{s} = \frac{1}{z} - \frac{\theta\eta}{\alpha}\left(\frac{1}{q} - \frac{1}{p}\right),$$

it follows that

$$\|g_z\|^{s'}_{L^{s'}(\mathbb{G})} = \int_{\mathbb{G}} |g_z(x)|^{s'} dx = \|g\|^{s'(a\eta+b)+p'}_{L^{p'}(\mathbb{G})} = 1.$$

Further, let us fix $f \in C_0^\infty(\mathbb{G})$ and set

$$h(z) := e^{z^2} \int_{\mathbb{G}} \mathcal{L}^{z/2} f(x) g_z(x) dx.$$

By the assumptions on f, for each $z \in S$, the function $\mathcal{L}^{z/2} f$ on $\mathbb{G}$ is smooth, while g_z is a simple function with compact support, and so $h(z)$ is well defined. In addition, if $z = \eta + iy$, then

$$\begin{aligned} |h(z)| &\leq e^{\eta^2 - y^2} \|\mathcal{L}^{z/2} f\|_{L^s(\mathbb{G})} \|g_z\|_{L^{s'}(\mathbb{G})} \\ &= e^{\eta^2 - y^2} \|\mathcal{L}^{iy/2} \mathcal{L}^{\eta/2} f\|_{L^s(\mathbb{G})} \leq C e^{-y^2} \phi(y) \|\mathcal{L}^{\eta/2} f\|_{L^s(\mathbb{G})}, \end{aligned}$$

hence

$$|h(\eta + iy)| \leq C \|\mathcal{L}^{\eta/2} f\|_{L^s(\mathbb{G})}.$$

Moreover, if $\operatorname{Re} z = 0$, then $|h(z)| \leq C\|f\|_{L^r(\mathbb{G})}$, while if $\operatorname{Re} z = \alpha/\theta$, then $|h(z)| \leq C\|\mathcal{L}^{\alpha/2\theta} f\|_{L^q(\mathbb{G})}$. On the other hand, the Phragmen–Lindelölf theorem implies that

$$|h(\alpha)| \leq C\|\mathcal{L}^{\alpha/2\theta} f\|^{\theta}_{L^q(\mathbb{G})} \|f\|^{1-\theta}_{L^r(\mathbb{G})}.$$

Since

$$h(\alpha) = \int_{\mathbb{G}} \mathcal{L}^{\alpha/2} f(x) g_\alpha(x) dx,$$

and g_α is an arbitrary simple function on $\mathbb{G}$ with compact support and $L^{p'}(\mathbb{G})$-norm equal to 1, this completes the proof. $\square$

Let us present the following consequence of the Landau–Kolmogorov inequality.

Corollary 4.7.7 (Consequence of the Landau–Kolmogorov inequality)**.** *Let* $\beta > 0$*,* $\delta > 0$*,* $p > 1$*,* $s \geq 1$*,* $r > 1$ *be such that*

$$\frac{\beta+\delta}{p} = \frac{\delta}{s} + \frac{\beta}{r}.$$

Then we have

$$\|f\|_{L^p(\mathbb{G})} \leq C\||\cdot|^\beta f\|_{L^s(\mathbb{G})}^{\delta/(\beta+\delta)} \|\mathcal{L}^{\delta/2} f\|_{L^r(\mathbb{G})}^{\beta/(\beta+\delta)}$$

for all $f \in C_0^\infty(\mathbb{G})$*.*

Proof of Corollary 4.7.7*.* Obviously, if $\delta < Q/r$, there is nothing to prove. Otherwise, we use Proposition 4.7.5, Part (2), and the version of the Landau–Kolmogorov inequality (4.145). Let $0 < \theta < Q/(r\delta)$ and $\gamma = \theta\delta$. Then Proposition 4.7.5, Part (2), gives that

$$\|f\|_{L^p(\mathbb{G})} \leq C\||\cdot|^\beta f\|_{L^q(\mathbb{G})}^{\gamma/(\beta+\gamma)} \|\mathcal{L}^{\gamma/2} f\|_{L^r(\mathbb{G})}^{\beta/(\beta+\gamma)}$$

for all $f \in C_0^\infty(\mathbb{G})$. By Theorem 4.7.6, with r, s and p instead of p, q, and r, we have

$$\|\mathcal{L}^{\gamma/2} f\|_{L^r(\mathbb{G})} \leq C\|\mathcal{L}^{\delta/2} f\|_{L^s(\mathbb{G})}^{\theta} \|f\|_{L^p(\mathbb{G})}^{1-\theta},$$

that is,

$$\|f\|_{L^p(\mathbb{G})} \leq C\||\cdot|_0^\beta f\|_{L^q(\mathbb{G})}^{\gamma/(\beta+\gamma)} \|\mathcal{L}^{\gamma/2} f\|_{L^s(\mathbb{G})}^{\theta\beta/(\beta+\gamma)} \|f\|_{L^p(\mathbb{G})}^{(1-\theta)\beta/(\beta+\gamma)}.$$

This completes the proof. ☐