Chapter 3

Rellich, Caffarelli–Kohn–Nirenberg, and Sobolev Type Inequalities

This chapter is devoted to other functional inequalities usually associated to the Hardy inequalities. These include Rellich and Caffarell–Kohn–Nirenberg inequalities. We also discuss different aspects of this analysis such as their stability, higher-order inequalities, their weighted and extended versions.

3.1 Rellich inequality

In general, the Rellich type inequalities have the following form

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|_E^\alpha} dx \leq C \int_{\mathbb{R}^n} \frac{|\Delta f(x)|^p}{|x|_E^\beta} dx$$

for certain constants α, β and p. The classical result by Rellich appearing at the 1954 ICM in Amsterdam [Rel56] was the inequality

$$\left\| \frac{f}{|x|_E^2} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)} \|\Delta f\|_{L^2(\mathbb{R}^n)}, \quad n \geq 5. \tag{3.1}$$

To find analogues of (3.1) on the homogeneous groups is an interesting question. The first obstacle to it is that there is neither stratification nor gradation on general homogeneous groups, so there may be no homogeneous left invariant hypoelliptic differential operators on $\mathbb{G}$ at all, to formulate an expression of the form of the right-hand side of (3.1).

However, similar to the results on the Hardy type inequalities before, the inequality (3.1) can be expressed in terms of the radial derivative

$$\partial_r = \frac{x}{|x|_E} \cdot \nabla,$$

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taking the form

$$\left\|\frac{f}{|x|_E^2}\right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)}\left\|\partial_r^2 f+\frac{n-1}{|x|_E}\partial_r f\right\|_{L^2(\mathbb{R}^n)}, \quad n\geq 5. \tag{3.2}$$

It is well known that in the spherical coordinates on $\mathbb{R}^n$ the Laplacian $\Delta_{\mathbb{R}^n}$ decomposes in the radial and spherical parts as

$$\Delta_{\mathbb{R}^n} f=\frac{\partial^2 f}{\partial r^2}+\frac{n-1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} f.$$

So, the operator on the right-hand side in (3.2) is precisely the radial part of the Laplacian on $\mathbb{R}^n$, which can be also expressed as

$$\frac{\partial^2 f}{\partial r^2}+\frac{n-1}{r}\frac{\partial f}{\partial r}=\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial f}{\partial r}\right). \tag{3.3}$$

Although there is no analogue of the Laplacian on general homogeneous groups, using the radial operator $\mathcal{R}$ from Section 1.3, the expression (3.3) on a homogeneous group of homogeneous dimension Q and homogeneous quasi-norm $|\cdot|$ makes perfect sense in the form of

$$\mathcal{R}^2 f+\frac{Q-1}{|x|}\mathcal{R}f=\frac{1}{|x|^{Q-1}}\mathcal{R}\left(|x|^{Q-1}\mathcal{R}f\right). \tag{3.4}$$

One aim of this section is to show that the Rellich inequality in the form (3.2) extends to general homogeneous groups using the radial operator $\mathcal{R}$, taking the form:

$$\left\|\frac{f}{|x|^2}\right\|_{L^2(\mathbb{G})} \leq \frac{4}{Q(Q-4)}\left\|\mathcal{R}^2 f+\frac{Q-1}{|x|}\mathcal{R}f\right\|_{L^2(\mathbb{G})}, \quad Q\geq 5, \tag{3.5}$$

for all complex-valued functions $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$. The operator on the right-hand side of (3.5) is thus an analogue of the radial part of the Laplacian on $\mathbb{R}^n$.

Thus, in the sequel we will be frequently use the *Rellich type operator* appearing in the right-hand side of (3.8) which we may denote by

$$\tilde{\mathcal{R}}f:=\mathcal{R}^2 f+\frac{Q-1}{|x|}\mathcal{R}f. \tag{3.6}$$

Similarly to the Hardy type inequalities from Chapter 2, the expression on the right-hand side of (3.5) appears to be natural since there is no analogue of homogeneous Laplacian or sub-Laplacian on general homogeneous groups to extend (3.1). Moreover, even on a group where such operators exist, this would usually give a refinement of those inequalities since derivatives only in one direction appear.

3.1.1 Rellich type inequalities in L^2

In this section we prove the Rellich type inequality (3.5) and its weighted version. This will be a corollary of the following identity relating different expressions involving the radial derivatives and radially symmetric weights.

Theorem 3.1.1 (Identity leading to Rellich inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 5$. Then for arbitrary homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ and every complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the identity*

$$\left\|\mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f + \frac{Q(Q-4)}{4|x|^2}f\right\|^2_{L^2(\mathbb{G})} + \frac{Q(Q-4)}{2}\left\|\frac{1}{|x|}\mathcal{R}f + \frac{Q-4}{2|x|^2}f\right\|^2_{L^2(\mathbb{G})}$$
$$= \left\|\mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})} - \left(\frac{Q(Q-4)}{4}\right)^2\left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{G})}. \tag{3.7}$$

Since the left-hand side of (3.7) is non-negative, this implies the following Rellich type inequality on $\mathbb{G}$ for $Q \geq 5$:

Corollary 3.1.2 (Rellich type inequality in $L^2(\mathbb{G})$)**.** *For all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left\|\frac{f}{|x|^2}\right\|_{L^2(\mathbb{G})} \leq \frac{4}{Q(Q-4)}\left\|\mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f\right\|_{L^2(\mathbb{G})}, \quad Q \geq 5, \tag{3.8}$$

where the constant $\frac{4}{Q(Q-4)}$ is sharp and it is attained if and only if $f = 0$.

Remark 3.1.3. Let us show that the constant $\frac{4}{Q(Q-4)}$ in (3.8) is sharp and is never attained unless $f = 0$. Indeed, if the equality in (3.8) is attained, it follows that both terms on the left-hand side of (3.7) must be zero. In particular, it means that

$$\frac{1}{|x|}\mathcal{R}f + \frac{Q-4}{2|x|^2}f = 0, \tag{3.9}$$

and hence

$$\mathbb{E}f = -\frac{Q-4}{2}f.$$

In view of Proposition 1.3.1, Part (i), the function f must be positively homogeneous of order $-\frac{Q-4}{2}$ which is impossible since $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ unless $f = 0$, so that the constant is not attained unless $f = 0$.

Furthermore, the first term in (3.7) must be also zero, and by using (3.9) this is equivalent to

$$\mathcal{R}^2 f + \frac{Q-2}{2|x|}\mathcal{R}f = 0$$

which means that $\mathcal{R}f$ is positively homogeneous of order $-\frac{Q-2}{2}$. Thus, by taking an approximation of homogeneous functions f of order $-\frac{Q-4}{2}$, we have that $\mathcal{R}f$ is homogeneous of order $-\frac{Q-2}{2}$, so that the left-hand side of (3.7) converges to zero. Therefore, the constant $\frac{4}{Q(Q-4)}$ in (3.8) is sharp.

Thus, in view of Remark 3.1.3, the statement of Corollary 3.1.2 follows from the identity (3.7), which we will now prove.

Proof of Theorem 3.1.1. Introducing polar coordinates

$$(r, y) = \left(|x|, \frac{x}{|x|}\right) \in (0, \infty) \times \wp$$

on $\mathbb{G}$, using the polar decomposition formula from Proposition 1.2.10, as well as integrating by parts we obtain

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^4} dx &= \int_0^\infty \int_\wp \frac{|f(ry)|^2}{r^4} r^{Q-1} d\sigma(y) dr \\
&= -\frac{2}{Q-4} \mathrm{Re} \int_0^\infty r^{Q-4} \int_\wp f(ry) \overline{\frac{df(ry)}{dr}} d\sigma(y) dr \\
&= \frac{2}{(Q-3)(Q-4)} \mathrm{Re} \int_0^\infty r^{Q-3} \int_\wp \left(\left| \frac{df(ry)}{dr} \right|^2 + f(ry) \overline{\frac{d^2 f(ry)}{dr^2}} \right) d\sigma(y) dr \\
&= \frac{2}{(Q-3)(Q-4)} \left(\left\| \frac{1}{|x|} \mathcal{R} f \right\|^2_{L^2(\mathbb{G})} + \mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\mathcal{R}^2 f(x)} dx \right). \qquad (3.10)
\end{aligned}$$

For the first term, using identity (2.16) with $\alpha = 1$, we have identity (2.17), i.e.,

$$\left\| \frac{1}{|x|} \mathcal{R} f \right\|^2_{L^2(\mathbb{G})} = \left(\frac{Q-4}{2} \right)^2 \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})} + \left\| \frac{1}{|x|} \mathcal{R} f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})}. \qquad (3.11)$$

For the second term a direct calculation shows

$$\begin{aligned}
&\mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\mathcal{R}^2 f(x)} dx \\
&= \mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} dx - (Q-1) \mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^3} \overline{\mathcal{R} f(x)} dx \\
&= \mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} dx \\
&\quad - \frac{Q-1}{2} \int_0^\infty r^{Q-4} \int_\wp \frac{d|f(ry)|^2}{dr} d\sigma(y) dr \\
&= \mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} dx \\
&\quad + \frac{(Q-1)(Q-4)}{2} \int_0^\infty r^{Q-5} \int_\wp |f(ry)|^2 d\sigma(y) dr
\end{aligned}$$

$$= \mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} dx$$
$$+ \frac{(Q-1)(Q-4)}{2} \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}. \tag{3.12}$$

Combining (3.11) and (3.12) with (3.10) we arrive at

$$\left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})} = \frac{2}{(Q-3)(Q-4)} \Bigg(\left(\frac{Q-4}{2} \right)^2 \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}$$
$$+ \left\| \frac{1}{|x|} \mathcal{R} f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})}$$
$$+ \mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} dx + \frac{(Q-1)(Q-4)}{2} \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})} \Bigg).$$

Collecting same terms, this gives

$$0 = \frac{2Q}{(Q-3)4} \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}$$
$$+ \frac{2}{(Q-3)(Q-4)} \Bigg(\mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} dx$$
$$+ \left\| \frac{1}{|x|} \mathcal{R} f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})} \Bigg),$$

that is,

$$\frac{Q(Q-4)}{4} \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}$$
$$= -\mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \overline{\left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} dx - \left\| \frac{1}{|x|} \mathcal{R} f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})}.$$

Multiplying both sides by $\frac{4}{Q(Q-4)}$ and simplifying we obtain

$$\mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \left(\overline{\frac{f(x)}{|x|^2} + \frac{4}{Q(Q-4)} \left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} \right) dx$$
$$= -\frac{4}{Q(Q-4)} \left\| \frac{1}{|x|} \mathcal{R} f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})}. \tag{3.13}$$

On the other hand, we also have

$$2\mathrm{Re} \int_{\mathbb{G}} \frac{f(x)}{|x|^2} \left(\overline{\frac{f(x)}{|x|^2} + \frac{4}{Q(Q-4)} \left(\mathcal{R}^2 f(x) + \frac{Q-1}{|x|} \mathcal{R} f(x) \right)} \right) dx$$

$$= \left\| \frac{f}{|x|^2} + \frac{4}{Q(Q-4)} \left(\mathcal{R}^2 f + \frac{Q-1}{|x|} \mathcal{R} f \right) \right\|^2_{L^2(\mathbb{G})} + \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}$$
$$- \left\| \frac{4}{Q(Q-4)} \left(\mathcal{R}^2 f + \frac{Q-1}{|x|} \mathcal{R} f \right) \right\|^2_{L^2(\mathbb{G})}. \tag{3.14}$$

From (3.13) and (3.14) we obtain

$$- \frac{8}{Q(Q-4)} \left\| \frac{1}{|x|} \mathcal{R} f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})}$$
$$= \left(\frac{4}{Q(Q-4)} \right)^2 \left\| \frac{Q(Q-4)}{4} \frac{f}{|x|^2} + \mathcal{R}^2 f + \frac{Q-1}{|x|} \mathcal{R} f \right\|^2_{L^2(\mathbb{G})} + \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}$$
$$- \left(\frac{4}{Q(Q-4)} \right)^2 \left\| \mathcal{R}^2 f + \frac{Q-1}{|x|} \mathcal{R} f \right\|^2_{L^2(\mathbb{G})},$$

thus,

$$\left\| \mathcal{R}^2 f + \frac{Q-1}{|x|} \mathcal{R} f + \frac{Q(Q-4)}{4|x|^2} f \right\|^2_{L^2(\mathbb{G})} + \frac{Q(Q-4)}{2} \left\| \frac{1}{|x|} \mathcal{R} f + \frac{Q-4}{2|x|^2} f \right\|^2_{L^2(\mathbb{G})}$$
$$= \left\| \mathcal{R}^2 f + \frac{Q-1}{|x|} \mathcal{R} f \right\|^2_{L^2(\mathbb{G})} - \left(\frac{Q(Q-4)}{4} \right)^2 \left\| \frac{f}{|x|^2} \right\|^2_{L^2(\mathbb{G})}.$$

This gives identity (3.7). □

In the Euclidean setting of $\mathbb{R}^n$ the equalities of the type of Theorem 3.1.1 were analysed in [MOW17b]. Theorem 3.1.1 was obtained in [RS17b] and it can be extended to all $\alpha \in \mathbb{R}$. We present these results next, following [Ngu17], also correcting relevant statements. We will be always using the notation

$$\tilde{\mathcal{R}} f := \mathcal{R}^2 f + \frac{Q-1}{|x|} \mathcal{R} f. \tag{3.15}$$

Theorem 3.1.4 (Weighted L^2-Rellich inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 5$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then we have the following properties:*

(1) *For any $\alpha \in \mathbb{R}$, for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the identity*

$$\left\| \frac{\tilde{\mathcal{R}} f}{|x|^\alpha} \right\|^2_{L^2(\mathbb{G})} = C_\alpha^2 \left\| \frac{f}{|x|^{\alpha+2}} \right\|^2_{L^2(\mathbb{G})} + \left\| \frac{\tilde{\mathcal{R}} f}{|x|^\alpha} + C_\alpha \frac{f}{|x|^{\alpha+2}} \right\|^2_{L^2(\mathbb{G})}$$
$$+ 2C_\alpha \left\| \frac{\mathcal{R} f}{|x|^{1+\alpha}} + \frac{Q-4-2\alpha}{2|x|^{\alpha+2}} f \right\|^2_{L^2(\mathbb{G})}, \tag{3.16}$$

where

$$C_\alpha = \frac{(Q+2\alpha)(Q-4-2\alpha)}{4}. \tag{3.17}$$

(2) *For any $\alpha \in (-Q/2, (Q-4)/2)$, we have the following weighted Rellich inequality for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$,*

$$C_\alpha \left\| \frac{f}{|x|^{\alpha+2}} \right\|_{L^2(\mathbb{G})} \leq \left\| \frac{\tilde{\mathcal{R}} f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}, \tag{3.18}$$

where the constant $C_\alpha > 0$ as in (3.17) is sharp and it is attained if and only if $f = 0$.

Proof of Theorem 3.1.4. Part (2) is an immediate consequence of Part (1), so we prove Part (1) now.

We can assume that $C_\alpha \neq 0$ since otherwise (3.16) trivially holds. Moreover, we can assume for the proof below that $Q-3-2\alpha \neq 0$. If this is zero, the statement (3.16) would follow by continuity from the same identity for other α's.

As in the proof of Theorem 3.1.1, we calculate

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^{4+2\alpha}} dx &= \int_0^\infty r^{Q-5-2\alpha} \int_\wp |f(ry)|^2 d\sigma(y) dr \\
&= \frac{1}{Q-4-2\alpha} \int_0^\infty (r^{Q-4-2\alpha})' \int_\wp |f(ry)|^2 d\sigma(y) dr \\
&= -\frac{2}{Q-4-2\alpha} \mathrm{Re} \int_0^\infty r^{Q-4-2\alpha} \int_\wp f(ry) \overline{\mathcal{R} f(ry)} d\sigma(y) dr \\
&= -\frac{2}{(Q-4-2\alpha)(Q-3-2\alpha)} \mathrm{Re} \int_0^\infty (r^{Q-3-2\alpha})' \int_\wp f(ry) \overline{\mathcal{R} f(ry)} d\sigma(y) dr \\
&= \frac{2}{(Q-4-2\alpha)(Q-3-2\alpha)} \mathrm{Re} \int_0^\infty r^{Q-3-2\alpha} \\
&\quad \times \int_\wp \left(|\mathcal{R} f(ry)|^2 + f(ry) \overline{\mathcal{R}^2 f(ry)} \right) d\sigma(y) dr \\
&= \frac{2}{(Q-4-2\alpha)(Q-3-2\alpha)} \mathrm{Re} \int_{\mathbb{G}} \left(\frac{|\mathcal{R} f(x)|^2}{|x|^{2+2\alpha}} + \frac{f(x) \overline{\mathcal{R}^2 f(x)}}{|x|^{2+2\alpha}} \right) dx. \quad (3.19)
\end{aligned}$$

By Theorem 3.1.1 we have

$$\int_{\mathbb{G}} \frac{|\mathcal{R} f|^2}{|x|^{2+2\alpha}} dx = \frac{(Q-4-2\alpha)^2}{4} \int_{\mathbb{G}} \frac{|f|^2}{|x|^{4+2\alpha}} dx + \int_{\mathbb{G}} \left| \frac{\mathcal{R} f}{|x|^{1+\alpha}} + \frac{Q-4-2\alpha}{2|x|^{2+\alpha}} f \right|^2 dx. \tag{3.20}$$

By integration by parts we obtain

$$\mathrm{Re} \int_{\mathbb{G}} \frac{f \overline{\mathcal{R}^2 f}}{|x|^{2+2\alpha}} dx = \mathrm{Re} \int_{\mathbb{G}} \frac{f \overline{\tilde{\mathcal{R}} f}}{|x|^{2+2\alpha}} dx - (Q-1) \mathrm{Re} \int_{\mathbb{G}} \frac{f \overline{\mathcal{R} f}}{|x|^{3+2\alpha}} dx \tag{3.21}$$

$$= \mathrm{Re} \int_{\mathbb{G}} \frac{f\overline{\tilde{\mathcal{R}}f}}{|x|^{2+2\alpha}}dx + \frac{(Q-1)(Q-4-2\alpha)}{2} \int_{\mathbb{G}} \frac{|f|^2}{|x|^{4+2\alpha}}dx.$$

Plugging (3.20) and (3.21) into (3.19) we get

$$\int_{\mathbb{G}} \frac{|f|^2}{|x|^{4+2\alpha}}dx = -\frac{1}{C_\alpha}\mathrm{Re}\int_{\mathbb{G}} \frac{f\overline{\tilde{\mathcal{R}}f}}{|x|^{2+2\alpha}}dx - \frac{1}{C_\alpha}\int_{\mathbb{G}} \left| \frac{\mathcal{R}f}{|x|^{1+\alpha}} + \frac{Q-4-2\alpha}{2|x|^{2+\alpha}}f \right|^2 dx$$
$$= \frac{1}{2C_\alpha^2}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}f|^2}{|x|^{2\alpha}}dx + \frac{1}{2}\int_{\mathbb{G}} \frac{|f|^2}{|x|^{2\alpha+4}}dx - \frac{1}{2C_\alpha^2}\int_{\mathbb{G}} \left| \frac{\tilde{\mathcal{R}}f}{|x|^\alpha} + C_\alpha \frac{f}{|x|^{\alpha+2}} \right|^2 dx$$
$$- \frac{1}{C_\alpha}\int_{\mathbb{G}} \left| \frac{\mathcal{R}f}{|x|^{\alpha+1}} + \frac{Q-4-2\alpha}{2|x|^{\alpha+2}}f \right|^2 dx,$$

which gives equality (3.16).

The inequality (3.18) is a straightforward consequence of (3.16) since $C_\alpha > 0$ for $\alpha \in (-Q/2, (Q-4)/2)$. Let us now show that this constant is sharp. For this, we consider the test function $g(r) := r^{-(Q-4-2\alpha)/2}$ which can be approximated by smooth functions. For example, let $\eta \in C_0^\infty(\mathbb{R})$ be such that $\eta = 1$ on $(-1,1)$ and $\eta = 0$ on $\mathbb{R}\backslash(-2,2)$. For any $\epsilon > 0$, define $f_\epsilon(r) := (1-\eta(r/\epsilon))r^{-\frac{Q-4-2\alpha}{2}}\eta(\epsilon r)$, then

$$\lim_{\epsilon\to 0} f_\epsilon(r) = g(r).$$

We have

$$\frac{\left\| \frac{\tilde{\mathcal{R}}g}{|x|^\alpha} \right\|^2_{L^2(\mathbb{G})}}{\left\| \frac{g}{|x|^{\alpha+2}} \right\|^2_{L^2(\mathbb{G})}} = \frac{\int_{\mathbb{G}} \frac{1}{|x|^{2\alpha}} \left| \mathcal{R}^2 g(|x|) + \frac{Q-1}{|x|}\mathcal{R}g(|x|) \right|^2 dx}{\int_{\mathbb{G}} \frac{|g(|x|)|^2}{|x|^{4+2\alpha}}dx} = C_\alpha^2,$$

which shows the sharpness of C_α. If for some function f, there is equality in (3.18), then from (3.16) we must have

$$\mathcal{R}f + \frac{Q-4-2\alpha}{2|x|}f = 0.$$

In terms of the Euler operator this can be equivalently expressed by

$$\mathbb{E}f = -\frac{Q-4-2\alpha}{2}f.$$

By Proposition 1.3.1 this implies that f is positively homogeneous of order $-(Q-4-2\alpha)/2$, that is, there exists function $h : \wp \to \mathbb{C}$ such that

$$f(x) = |x|^{-(Q-4-2\alpha)/2}h(x/|x|).$$

Since $f(x)/|x|^{2+\alpha}$ is in $L^2(\mathbb{G})$, we must have $h = 0$ on $\wp$ and, consequently, $f = 0$ on $\wp$. $\square$

3.1.2 Rellich type inequalities in L^p

In this section we describe the L^p-versions of the L^2-properties presented in Section 3.1.1, where we have followed the proofs in [RS17b]. In this section we follow [Ngu17]. We start with the identity analogous to that in Theorem 3.1.1.

Theorem 3.1.5 (Identity leading to L^p-Rellich inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $1 < p < Q$ and $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\begin{aligned}\int_{\mathbb{G}} \frac{1}{|x|^{p\alpha}} \left|\mathcal{R}f + \frac{Q-1}{|x|}f\right|^p dx = \frac{|Q+p'\alpha|^p}{(p')^p} \int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(1+\alpha)}}dx \\ + p\int_{\mathbb{G}} \frac{1}{|x|^{p\alpha}} R_p\left(\frac{Q+p'\alpha}{p'}\frac{f}{|x|}, \mathcal{R}f + \frac{Q-1}{|x|}f\right)dx,\end{aligned} \tag{3.22}$$

where $p' = p/(p-1)$ and R_p is as in (2.22).

Proof of Theorem 3.1.5. First, a direct calculation shows

$$\begin{aligned}&\int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(1+\alpha)}}dx = \int_0^\infty r^{Q-p(1+\alpha)-1}\int_\wp |f(ry)|^p d\sigma dr \\ &= \frac{1}{Q-p(1+\alpha)}\int_0^\infty (r^{Q-p(1+\alpha)})'\int_\wp |f(ry)|^p d\sigma dr \\ &= -\frac{p}{Q-p(1+\alpha)}\mathrm{Re}\int_0^\infty r^{Q-p(1+\alpha)}\int_\wp |f(ry)|^{p-2}f(ry)\overline{\mathcal{R}f(ry)}d\sigma(y)dr \\ &= -\frac{p}{Q-p(1+\alpha)}\mathrm{Re}\int_{\mathbb{G}} \frac{|f|^{p-2}f\overline{\mathcal{R}f}}{|x|^{p(1+\alpha)-1}}dx \\ &= -\frac{p}{Q-p(1+\alpha)}\left(\mathrm{Re}\int_{\mathbb{G}} \frac{|f|^{p-2}f}{|x|^{(p-1)(1+\alpha)}}\frac{\overline{\mathcal{R}f+\frac{Q-1}{|x|}f}}{|x|^\alpha} - (Q-1)\int_{\mathbb{G}}\frac{|f|^p}{|x|^{p(1+\alpha)}}dx\right).\end{aligned}$$

This implies further equalities,

$$\begin{aligned}\int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(1+\alpha)}}dx &= \frac{p'}{Q+p'\alpha}\mathrm{Re}\int_{\mathbb{G}} \frac{|f|^{p-2}f}{|x|^{(p-1)(1+\alpha)}}\frac{\overline{\mathcal{R}f+\frac{Q-1}{|x|}f}}{|x|^\alpha}dx \\ &= \frac{p-1}{p}\int_{\mathbb{G}}\frac{|f|^p}{|x|^{p(1+\alpha)}}dx + \frac{1}{p}\frac{(p')^p}{|Q+p'\alpha|^p}\left|\mathcal{R}f+\frac{Q-1}{|x|}f\right|^2 dx \\ &\quad - \int_{\mathbb{G}} R_p\left(\frac{f}{|x|^{1+\alpha}}, \frac{p'}{Q+p'\alpha}\frac{\mathcal{R}f+\frac{Q-1}{|x|}f}{|x|^\alpha}\right)dx,\end{aligned}$$

which proves (3.22). $\square$

Since the remainder term on the right-hand side of (3.22) is non-negative, this implies the following L^p-version of the Rellich type inequality on $\mathbb{G}$:

Corollary 3.1.6 (Rellich type inequality in $L^p(\mathbb{G})$). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $1 < p < Q$ and $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left\| \frac{1}{|x|^\alpha} \left(\mathcal{R}f + \frac{Q-1}{|x|} f \right) \right\|_{L^p(\mathbb{G})} \geq \frac{|Q+p'\alpha|}{p'} \left\| \frac{f}{|x|^{1+\alpha}} \right\|_{L^p(\mathbb{G})},$$

where the constant $\frac{4}{Q(Q-4)}$ is sharp and it is attained if and only if $f = 0$.

As a consequence of Theorem 3.1.5 we have

Corollary 3.1.7. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$ and $1 < p < Q/2$. Then for any complex-valued $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, we have*

$$\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}f|^p}{|x|^{p\alpha}} dx = \frac{|Q+p'\alpha|^p}{(p')^p} \int_{\mathbb{G}} \frac{|\mathcal{R}f|^p}{|x|^{p(1+\alpha)}} dx + p \int_{\mathbb{G}} \frac{1}{|x|^{p\alpha}} R_p \left(\frac{Q+p'\alpha}{p'} \frac{\mathcal{R}f}{|x|}, \tilde{\mathcal{R}}f \right) dx \tag{3.23}$$

for any $\alpha \in \mathbb{R}$. Here R_p is as in (2.22)*. As a consequence, we obtain the following weighted L^p-Rellich type inequality*

$$\frac{|Q+p'\alpha|}{p'} \left\| \frac{\mathcal{R}f}{|x|^{(1+\alpha)}} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{\tilde{\mathcal{R}}f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \tag{3.24}$$

for any $\alpha \in \mathbb{R}$ and any complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Moreover, the inequality (3.24) *is sharp and equality holds if and only if $f = 0$.*

Proof of Corollary 3.1.7. The equality (3.23) is exactly (3.22) with f being replaced by $\mathcal{R}f$. The inequality (3.24) follows immediately from (3.23) by dropping the non-negative remainder term on the right-hand side of (3.23). The sharpness of (3.24) is proved by using approximations of the function $r^{-(Q'p(1+\alpha))/p}$. If the equality occurs in (3.24) for some function f, then by (3.23) we must have

$$\tilde{\mathcal{R}}f = \frac{Q+p'\alpha}{p'|x|} \mathcal{R}f,$$

which is equivalent to

$$\mathcal{R}^2 f + \frac{Q-p(1+\alpha)}{p|x|} \mathcal{R}f = 0.$$

This can be also expressed as

$$\mathbb{E}(\mathcal{R}f) = -\frac{Q-p(1+\alpha)}{p} \mathcal{R}f.$$

Hence $\mathcal{R}f$ is positively homogeneous of degree $-(Q'p(1+\alpha))/p$, which forces $\mathcal{R}f = 0$ since $\mathcal{R}f/|x|^{1+\alpha}$ is in $L^p(\mathbb{G})$. Thus, we get $f = 0$. $\square$

Theorem 3.1.8 (Another weighted L^p-Rellich inequality). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $1 < p < Q/2$. Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then we have the following properties.*

(1) *For all $\alpha \in \mathbb{R}$ and for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the identity*

$$\begin{aligned}\left\|\frac{\tilde{\mathcal{R}}f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}^p &= |C_{p,\alpha}|^p\int_{\mathbb{G}}\frac{|f|^p}{|x|^{p(2+\alpha)}}dx + p\int_{\mathbb{G}}\frac{1}{|x|^{p\alpha}}R_p\left(C_{p,\alpha}\frac{f}{|x|^2}, -\tilde{\mathcal{R}}f\right)dx\\ &+ p|C_{p,\alpha}|^{p-2}C_{p,\alpha}(p-1)\int_{\mathbb{G}}\frac{|f|^{p-2}}{|x|^{p(2+\alpha)-2}}\left|\mathcal{R}|f| + \frac{Q-p(2+\alpha)}{p|x|}|f|\right|^2 dx\\ &+ p|C_{p,\alpha}|^{p-2}C_{p,\alpha}\int_{\mathbb{G}}\frac{|f|^{p-4}(\operatorname{Im}(f\overline{\mathcal{R}f}))^2}{|x|^{p(2+\alpha)-2}}dx. \end{aligned}\tag{3.25}$$

Here R_p is as in (2.22) *and*

$$C_{p,\alpha} = \frac{(Q-2p-p\alpha)(Q+p'\alpha)}{pp'}, \quad \frac{1}{p}+\frac{1}{p'} = 1. \tag{3.26}$$

(2) *For any $\alpha \in (-(p-1)Q/p, (Q-2p)/p)$ and all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$C_{p,\alpha}\left\|\frac{f}{|x|^{2+\alpha}}\right\|_{L^p(\mathbb{G})} \le \left\|\frac{\tilde{\mathcal{R}}f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}. \tag{3.27}$$

Moreover, the constant $C_{p,\alpha} > 0$ as in (3.26) *is sharp and equality in* (3.27) *holds if and only if $f = 0$.*

Proof of Theorem 3.1.8. Inequality (3.27) in Part (2) follows from Part (1) in view of the positivity of the constant $C_{p,\alpha} > 0$ under the corresponding conditions on α, so we prove Part (1) now. For the argument below we may assume that $C_{p,\alpha} \neq 0$, and that the constant on the left-hand side in the following estimate is non-zero. Then a direct calculation using Proposition 1.2.10 and integration by parts gives

$$\begin{aligned}&\frac{(Q-p(2+\alpha))(Q-p(2+\alpha)+1)}{p}\int_{\mathbb{G}}\frac{|f(x)|^p}{|x|^{p(2+\alpha)}}dx\\ &= \frac{(Q-p(2+\alpha))(Q-p(2+\alpha)+1)}{p}\int_0^\infty r^{Q-p(2+\alpha)-1}\int_{\wp}|f(ry)|^p d\sigma(y)dr\\ &= \frac{(Q-p(2+\alpha)+1)}{p}\int_0^\infty (r^{Q-p(2+\alpha)})'\int_{\wp}|f(ry)|^p d\sigma(y)dr\\ &= -(Q-p(2+\alpha)+1)\operatorname{Re}\int_0^\infty r^{Q-p(2+\alpha)}\int_{\wp}|f(ry)|^{p-2}f(ry)\overline{\mathcal{R}f(ry)}d\sigma(y)dr\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{Re}\int_0^\infty (r^{Q-p(2+\alpha)+1})'\int_\wp |f(ry)|^{p-2}f(ry)\overline{\mathcal{R}f(ry)}d\sigma(y)dr\\
&= \mathrm{Re}\int_0^\infty r^{Q-p(2+\alpha)+1}\int_\wp (p-2)|f(ry)|^{p-4}(\mathrm{Re}(f(ry)\overline{\mathcal{R}f(ry)}))^2\\
&\quad + |f(ry)|^{p-2}|\mathcal{R}f(ry)|^2 + |f(ry)|^{p-2}f(ry)\overline{\mathcal{R}^2f(ry)}d\sigma(y)dr\\
&= \mathrm{Re}\int_{\mathbb{G}} \frac{1}{|x|^{p(2+\alpha)-2}}\left((p-2)|f|^{p-4}(\mathrm{Re}(f\overline{\mathcal{R}f}))^2 + |f|^{p-2}|\mathcal{R}f|^2 + |f|^{p-2}f\overline{\mathcal{R}^2f}\right)dx\\
&= \mathrm{Re}\int_{\mathbb{G}} \frac{|f|^{p-2}f}{|x|^{(p-1)(2+\alpha)}}\frac{\overline{\tilde{\mathcal{R}}f}}{|x|^\alpha}dx - (Q-1)\mathrm{Re}\int_{\mathbb{G}}\frac{|f|^{p-2}f\overline{\mathcal{R}f}}{|x|^{p(2+\alpha)-1}}dx\\
&\quad + (p-1)\int_{\mathbb{G}}\frac{|f|^{p-4}(\mathrm{Re}(f\overline{\mathcal{R}f}))^2}{|x|^{p(2+\alpha)-2}}dx + \int_{\mathbb{G}}\frac{|f|^{p-4}(\mathrm{Im}(f\overline{\mathcal{R}f}))^2}{|x|^{p(2+\alpha)-2}}dx\\
&= \mathrm{Re}\int_{\mathbb{G}} \frac{|f|^{p-2}f}{|x|^{(p-1)(2+\alpha)}}\frac{\overline{\tilde{\mathcal{R}}f}}{|x|^\alpha}dx + \frac{(Q-1)(Q-p(2+\alpha))}{p}\int_{\mathbb{G}}\frac{|f|^p}{|x|^{p(2+\alpha)}}dx\\
&\quad + \frac{4(p-1)}{p^2}\int_{\mathbb{G}}\frac{(\mathcal{R}(|f|^{p/2}))^2}{|x|^{p(2+\alpha)-2}}dx + \int_{\mathbb{G}}\frac{|f|^{p-4}(\mathrm{Im}(f\overline{\mathcal{R}f}))^2}{|x|^{p(2+\alpha)-2}}dx.
\end{aligned}
\tag{3.28}
$$

By using Theorem 3.1.1 for $|f|^{\frac{p}{2}}$, we obtain

$$
\begin{aligned}
\int_{\mathbb{G}}\frac{(\mathcal{R}(|f|^{p/2}))^2}{|x|^{p(2+\alpha)-2}}dx &= \frac{(Q-p(2+\alpha))^2}{4}\int_{\mathbb{G}}\frac{|f|^p}{|x|^{p(2+\alpha)}}dx\\
&\quad + \int_{\mathbb{G}}\frac{1}{|x|^{p(2+\alpha)-2}}\left|\mathcal{R}|f|^{p/2} + \frac{Q-p(2+\alpha)}{2|x|}|f|^{p/2}\right|^2dx.
\end{aligned}
$$

Plugging this in (3.28) we get

$$
\begin{aligned}
&\int_{\mathbb{G}}\frac{|f(x)|^p}{|x|^{p(2+\alpha)}}dx\\
&= -\frac{1}{C_{p,\alpha}}\mathrm{Re}\int_{\mathbb{G}}\frac{|f|^{p-2}f}{|x|^{(p-1)(2+\alpha)}}\frac{\overline{\tilde{\mathcal{R}}f}}{|x|^\alpha}dx - \frac{1}{C_{p,\alpha}}\int_{\mathbb{G}}\frac{|f|^{p-4}(\mathrm{Im}(f\overline{\mathcal{R}f}))^2}{|x|^{p(2+\alpha)-2}}dx\\
&\quad - \frac{1}{C_{p,\alpha}}\frac{4(p-1)}{p^2}\int_{\mathbb{G}}\frac{1}{|x|^{p(2+\alpha)-2}}\left|\mathcal{R}|f|^{\frac{p}{2}} + \frac{Q-p(2+\alpha)}{2|x|}|f|^{\frac{p}{2}}\right|^2dx\\
&= \frac{p-1}{p}\int_{\mathbb{G}}\frac{|f|^p}{|x|^{p(2+\alpha)}}dx + \frac{1}{p}\frac{1}{|C_{p,\alpha}|^p}\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}f|^p}{|x|^{p\alpha}}dx - \int_{\mathbb{G}}R_p\left(\frac{f}{|x|^{2+\alpha}}, -\frac{1}{C_{p,\alpha}}\frac{\tilde{\mathcal{R}}f}{|x|^\alpha}\right)dx\\
&\quad - \frac{4(p-1)}{p^2C_{p,\alpha}}\int_{\mathbb{G}}\frac{1}{|x|^{p(2+\alpha)-2}}\left|\mathcal{R}|f|^{\frac{p}{2}} + \frac{Q-p(2+\alpha)}{2|x|}|f|^{\frac{p}{2}}\right|^2dx\\
&\quad - \frac{1}{C_{p,\alpha}}\int_{\mathbb{G}}\frac{|f|^{p-4}(\mathrm{Im}(f\overline{\mathcal{R}f}))^2}{|x|^{p(2+\alpha)-2}}dx.
\end{aligned}
\tag{3.29}
$$

The equality (3.25) now follows from (3.29) and the equality

$$\mathcal{R}(|f|^{\frac{p}{2}}) = \frac{p}{2}|f|^{\frac{p}{2}-1}\mathcal{R}(|f|).$$

Part (2). Clearly, inequality (3.27) follows from equality (3.25). As in the proof of Theorem 3.1.4, the sharpness of (3.27) follows by considering appropriate approximations of the function $r^{-(Q-p(2+\alpha))/p}$. Moreover, if we have equality in (3.27) for some function f, then in view of Part (1) we must have

$$\mathcal{R}(|f|) + \frac{Q-p(2+\alpha)}{p|x|}|f| = 0.$$

This means that

$$\mathbb{E}(|f|) = -\frac{Q-p(2+\alpha)}{p}|f|.$$

By Proposition 1.3.1 the function $|f|$ must be positively homogeneous of degree $-(Q-p(2+\alpha))/p$. Since $|f|/|x|^{2+\alpha}$ is in $L^p(\mathbb{G})$ we must have $f=0$. $\square$

3.1.3 Stability of Rellich type inequalities

The method used in the previous section also allows one to obtain the following stability property for Rellich type inequalities. We present such a result following [RS18].

Theorem 3.1.9 (Stability of Rellich type inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$ and let $p \geq 1$. Let $k \geq 2$, $k \in \mathbb{N}$, be such that $kp < Q$. Then for all real-valued radial functions $u \in C_0^\infty(\mathbb{G})$ we have*

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}u|^p}{|x|^{(k-2)p}}dx - K_{k,p}^p \int_{\mathbb{G}} \frac{|u|^p}{|x|^{kp}}dx \\ \geq C \sup_{R>0} \int_{\mathbb{G}} \frac{\left||u(x)|^{\frac{p-2}{2}}u(x) - R^{\frac{Q-kp}{2}}|u(R)|^{\frac{p-2}{2}}u(R)|x|^{-\frac{Q-kp}{2}}\right|^2}{|x|^{kp}\left|\log\frac{R}{|x|}\right|^2}dx,\end{aligned} \tag{3.30}$$

where

$$\tilde{\mathcal{R}}f = \mathcal{R}^2 f + \frac{Q-1}{|x|}\mathcal{R}f$$

and

$$K_{k,p} = \frac{(Q-kp)[(k-2)p+(p-1)Q]}{p^2}. \tag{3.31}$$

Proof of Theorem 3.1.9. For $k \geq 2$, $k \in \mathbb{N}$, and $kp < Q$, as in the assumptions of the theorem, let us denote

$$v(r) := r^{\frac{Q-kp}{p}} u(r), \quad \text{where} \quad r \in [0, \infty). \tag{3.32}$$

In particular, we have $v(0) = 0$ and $v(\infty) = 0$. We then calculate as follows:

$$\begin{aligned}
-\tilde{\mathcal{R}}u =& -\mathcal{R}^2 \left(r^{\frac{kp-Q}{p}} v(r) \right) - \frac{Q-1}{r} \mathcal{R} \left(r^{\frac{kp-Q}{p}} v(r) \right) \\
=& -\mathcal{R} \left(\frac{kp-Q}{p} r^{\frac{kp-Q}{p}-1} v(r) + r^{\frac{kp-Q}{p}} \mathcal{R} v(r) \right) \\
&- \frac{Q-1}{r} \frac{kp-Q}{p} r^{\frac{kp-Q}{p}-1} v(r) - \frac{Q-1}{r} r^{\frac{kp-Q}{p}} \mathcal{R} v(r) \\
=& -\frac{kp-Q}{p} \left(\frac{kp-Q}{p} - 1 \right) r^{\frac{kp-Q}{p}-2} v(r) - \frac{kp-Q}{p} r^{\frac{kp-Q}{p}-1} \mathcal{R} v(r) \\
&- \frac{kp-Q}{p} r^{\frac{kp-Q}{p}-1} \mathcal{R} v(r) - r^{\frac{kp-Q}{p}} \mathcal{R}^2 v(r) \\
&- \frac{Q-1}{r} \frac{kp-Q}{p} r^{\frac{kp-Q}{p}-1} v(r) - \frac{Q-1}{r} r^{\frac{kp-Q}{p}} \mathcal{R} v(r) \\
=& -r^{\frac{kp-Q}{p}-2} \left(\frac{(kp-Q)(kp-Q-p)}{p^2} + \frac{(Q-1)(kp-Q)}{p} \right) v(r) \\
&- r^{\frac{kp-Q}{p}-2} r^2 \left(\mathcal{R}^2 v(r) + \frac{1}{r} \left(\frac{2(kp-Q)}{p} + (Q-1) \right) \mathcal{R} v(r) \right) \\
=& \, r^{k-2-\frac{Q}{p}} (K_{k,p} v(r) - r^2 \tilde{\mathcal{R}}_k v(r)),
\end{aligned}$$

where $K_{k,p}$ is as in (3.31), and where we denote

$$\tilde{\mathcal{R}}_k f := \mathcal{R}^2 f + \frac{2k + \frac{Q(p-2)}{p} - 1}{r} \mathcal{R} f.$$

By using the first inequality in Lemma 2.4.2 with $a = K_{k,p} v(r)$ and $b = r^2 \tilde{\mathcal{R}}_k v(r)$, and the fact that $\int_0^\infty |v|^{p-2} v v' dr = 0$ in view of $v(0) = 0$ and $v(\infty) = 0$, we obtain

$$\begin{aligned}
J :=& \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}} u|^p}{|x|^{(k-2)p}} dx - K_{k,p}^p \int_{\mathbb{G}} \frac{|u|^p}{|x|^{kp}} dx \\
=& \, |\wp| \int_0^\infty | - \tilde{\mathcal{R}} u(r)|^p r^{Q-1-(k-2)p} dr - K_{k,p}^p |\wp| \int_0^\infty |u(r)|^p r^{Q-kp-1} dr \\
=& \, |\wp| \int_0^\infty \left(|K_{k,p} v(r) - r^2 \tilde{\mathcal{R}}_k v(r)|^p - (K_{k,p} v(r))^p \right) r^{-1} dr \\
\geq& -p |\wp| K_{k,p}^{p-1} \int_0^\infty |v|^{p-2} v \tilde{\mathcal{R}}_k v r dr
\end{aligned}$$

$$
\begin{aligned}
&= -p|\wp|K_{k,p}^{p-1}\int_0^\infty |v|^{p-2}v\left(v''+\frac{2k+\frac{Q(p-2)}{p}-1}{r}v'\right)rdr \\
&= -p|\wp|K_{k,p}^{p-1}\int_0^\infty |v|^{p-2}vv''rdr.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
-\int_0^\infty |v|^{p-2}vv''rdr &= (p-1)\int_0^\infty |v|^{p-2}(v')^2rdr+\int_0^\infty |v|^{p-2}vv'dr \\
&= (p-1)\int_0^\infty |v|^{p-2}(v')^2rdr \\
&= \frac{4(p-1)}{p^2}\int_0^\infty \left(\frac{p-2}{2}\right)^2|v|^{p-2}(v')^2dr \\
&\quad + \frac{4(p-1)}{p^2}\int_0^\infty (p-2)|v|^{p-2}(v')^2+|v|^{p-2}(v')^2rdr \\
&= \frac{4(p-1)}{p^2}\int_0^\infty \left(\left(|v|^{\frac{p-2}{2}}\right)'v+|v|^{\frac{p-2}{2}}v'\right)^2rdr \\
&= \frac{4(p-1)}{p^2}\int_0^\infty |(|v|^{\frac{p-2}{2}}v)'|^2rdr \\
&= \frac{4(p-1)}{|\wp_2|p^2}\int_{\mathbb{G}_2}\left|\mathcal{R}(|v|^{\frac{p-2}{2}}v)\right|^2dx,
\end{aligned}
$$

where $\mathbb{G}_2$ is a homogeneous group of homogeneous degree 2 and $|\wp_2|$ is the measure of the corresponding unit 2-quasi-ball. By using Remark 2.2.9 for $|v|^{\frac{p-2}{2}}v\in C_0^\infty(\mathbb{G}_2\backslash\{0\})$ in $p=Q=2$ case, and combining the above equalities, we obtain

$$
\begin{aligned}
J &\geq C_1\int_{\mathbb{G}_2}\frac{\left||v(x)|^{\frac{p-2}{2}}v(x)-|v(R\frac{x}{|x|})|^{\frac{p-2}{2}}v(R\frac{x}{|x|})\right|^2}{|x|^2\left|\log\frac{R}{|x|}\right|^2}dx \\
&= C_1\int_0^\infty \frac{\left||v(r)|^{\frac{p-2}{2}}v(r)-|v(R)|^{\frac{p-2}{2}}v(R)\right|^2}{r\left|\log\frac{R}{r}\right|^2}dr \\
&= C_1\int_0^\infty \frac{\left||u(r)|^{\frac{p-2}{2}}u(r)-R^{\frac{Q-kp}{2}}|u(R)|^{\frac{p-2}{2}}u(R)r^{-\frac{Q-kp}{2}}\right|^2}{r^{1-Q+kp}\left|\log\frac{R}{r}\right|^2}dr
\end{aligned}
$$

for any $R>0$. That is, we have

$$
J\geq C\sup_{R>0}\int_{\mathbb{G}}\frac{\left||u(x)|^{\frac{p-2}{2}}u(x)-R^{\frac{Q-kp}{2}}|u(R)|^{\frac{p-2}{2}}u(R)|x|^{-\frac{Q-kp}{2}}\right|^2}{|x|^{kp}\left|\log\frac{R}{|x|}\right|^2}dx.
$$

The proof is complete. $\square$

3.1.4 Higher-order Hardy–Rellich inequalities

In this section we show that by iterating the already established weighted Hardy inequalities we get inequalities of higher order. An interesting feature is that we also obtain the exact formula for the remainder which yields the sharpness of the constants as well. That is, when $p = 2$, one can iterate the exact representation formulae of the remainder that we obtained in Theorem 3.1.1. It implies higher-order remainder equalities that can be then also used to argue the sharpness of the constant.

Theorem 3.1.10 (Higher-order Hardy–Rellich identities and inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$. Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ be such that*

$$\prod_{j=0}^{k-1} \left| \frac{Q-2}{2} - (\alpha + j) \right| \neq 0.$$

Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have

$$\left\| \frac{f}{|x|^{k+\alpha}} \right\|_{L^2(\mathbb{G})} \leq \left[\prod_{j=0}^{k-1} \left| \frac{Q-2}{2} - (\alpha + j) \right| \right]^{-1} \left\| \frac{1}{|x|^\alpha} \mathcal{R}^k f \right\|_{L^2(\mathbb{G})}, \tag{3.33}$$

where the constant above is sharp, and is attained if and only if $f = 0$.

Moreover, for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, the following identity holds:

$$\begin{aligned}
\left\| \frac{1}{|x|^\alpha} \mathcal{R}^k f \right\|^2_{L^2(\mathbb{G})} &= \left[\prod_{j=0}^{k-1} \left(\frac{Q-2}{2} - (\alpha + j) \right)^2 \right] \left\| \frac{f}{|x|^{k+\alpha}} \right\|^2_{L^2(\mathbb{G})} \\
&\quad + \sum_{l=1}^{k-1} \left[\prod_{j=0}^{l-1} \left(\frac{Q-2}{2} - (\alpha + j) \right)^2 \right] \\
&\qquad \times \left\| \frac{1}{|x|^{l+\alpha}} \mathcal{R}^{k-l} f + \frac{Q - 2(l+1+\alpha)}{2|x|^{l+1+\alpha}} \mathcal{R}^{k-l-1} f \right\|^2_{L^2(\mathbb{G})} \\
&\quad + \left\| \frac{1}{|x|^\alpha} \mathcal{R}^k f + \frac{Q-2-2\alpha}{2|x|^{1+\alpha}} \mathcal{R}^{k-1} f \right\|^2_{L^2(\mathbb{G})}.
\end{aligned} \tag{3.34}$$

Remark 3.1.11. Let us point out some special cases of inequality (3.33).

1. For $k = 1$ inequality (3.33) gives the weighted L^2-Hardy inequalities from Corollary 2.1.6.
2. In particular, for $k = 1$ and $\alpha = 0$, inequality (3.33) gives the L^2-Hardy inequality, i.e., (2.2) in the case of $p = 2$.
3. For $k = 2$, inequality (3.33) can be thought of as a (weighted) Hardy–Rellich type inequality, while for larger k this corresponds to higher-order (weighted) Rellich inequalities.

Proof of Theorem 3.1.10. For any $\alpha \in \mathbb{R}$ and $Q \geq 3$ let us iterate the identity

$$\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})} = \left(\frac{Q-2}{2}-\alpha\right)^2 \left\|\frac{f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})} + \left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f + \frac{Q-2(\alpha+1)}{2|x|^{\alpha+1}}f\right\|^2_{L^2(\mathbb{G})}, \tag{3.35}$$

given in (2.16), as follows. First, replacing f in (3.35) by $\mathcal{R}f$ we have

$$\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^2 f\right\|^2_{L^2(\mathbb{G})} = \left(\frac{Q-2}{2}-\alpha\right)^2 \left\|\frac{\mathcal{R}f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})} + \left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^2 f + \frac{Q-2(\alpha+1)}{2|x|^{\alpha+1}}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})}. \tag{3.36}$$

Furthermore, replacing α by $\alpha+1$, (3.35) implies that

$$\left\|\frac{1}{|x|^{\alpha+1}}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})} = \left(\frac{Q-2}{2}-(\alpha+1)\right)^2 \left\|\frac{f}{|x|^{\alpha+2}}\right\|^2_{L^2(\mathbb{G})} + \left\|\frac{1}{|x|^{\alpha+1}}\mathcal{R}f + \frac{Q-2(\alpha+2)}{2|x|^{\alpha+2}}f\right\|^2_{L^2(\mathbb{G})}.$$

Combination of this with (3.36) gives

$$\begin{aligned}\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^2 f\right\|^2_{L^2(\mathbb{G})} &= \left(\frac{Q-2}{2}-\alpha\right)^2 \left(\frac{Q-2}{2}-(\alpha+1)\right)^2 \left\|\frac{f}{|x|^{\alpha+2}}\right\|^2_{L^2(\mathbb{G})} \\ &+ \left(\frac{Q-2}{2}-\alpha\right)^2 \left\|\frac{1}{|x|^{\alpha+1}}\mathcal{R}f + \frac{Q-2(\alpha+2)}{2|x|^{\alpha+2}}f\right\|^2_{L^2(\mathbb{G})} \\ &+ \left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^2 f + \frac{Q-2-2\alpha}{2|x|^{\alpha+1}}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

By using this iteration process further, we eventually arrive at the family of identities

$$\begin{aligned}\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^k f\right\|^2_{L^2(\mathbb{G})} &= \left[\prod_{j=0}^{k-1}\left(\frac{Q-2}{2}-(\alpha+j)\right)^2\right] \left\|\frac{f}{|x|^{k+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\ &+ \sum_{l=1}^{k-1}\left[\prod_{j=0}^{l-1}\left(\frac{Q-2}{2}-(\alpha+j)\right)^2\right] \\ &\times \left\|\frac{1}{|x|^{l+\alpha}}\mathcal{R}^{k-l}f + \frac{Q-2(l+1+\alpha)}{2|x|^{l+1+\alpha}}\mathcal{R}^{k-l-1}f\right\|^2_{L^2(\mathbb{G})}\end{aligned}$$

$$+\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^k f+\frac{Q-2-2\alpha}{2|x|^{1+\alpha}}\mathcal{R}^{k-1}f\right\|^2_{L^2(\mathbb{G})},\quad k=1,2,\ldots,$$

which give (3.34).

Now, by dropping positive terms, these identities imply that

$$\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^k f\right\|^2_{L^2(\mathbb{G})}\geq C_{k,Q}\left\|\frac{f}{|x|^{k+\alpha}}\right\|^2_{L^2(\mathbb{G})},$$

where

$$C_{k,Q}=\prod_{j=0}^{k-1}\left(\frac{Q-2}{2}-(\alpha+j)\right)^2.$$

If $C_{k,Q}\neq 0$, this can be written as

$$\left\|\frac{f}{|x|^{k+\alpha}}\right\|^2_{L^2(\mathbb{G})}\leq\frac{1}{C_{k,Q}}\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^k f\right\|^2_{L^2(\mathbb{G})},$$

which gives (3.33).

Now it remains to show the sharpness of the constant and the equality in (3.33). To do this, we rewrite the equality

$$\frac{\mathcal{R}^{k-l}f}{|x|^{l+\alpha}}+\frac{Q-2(l+1+\alpha)}{2|x|^{l+1+\alpha}}\mathcal{R}^{k-l-1}f=0$$

as

$$|x|\mathcal{R}(\mathcal{R}^{k-l-1}f)+\frac{Q-2(l+1+\alpha)}{2}(\mathcal{R}^{k-l-1}f)=0,$$

and by Proposition 1.3.1, Part (i), this means that $\mathcal{R}^{k-l-1}f$ is positively homogeneous of degree $-\frac{Q}{2}+l+1+\alpha$. So, all the remainder terms vanish if f is positively homogeneous of degree $k-\frac{Q}{2}+\alpha$. As this can be approximated by functions in $C_0^{\infty}(\mathbb{G}\backslash\{0\})$, the constant $C_{k,Q}$ is sharp.

If this constant was attained, it would be on functions f which are positively homogeneous of degree $k-\frac{Q}{2}+\alpha$, in which case $\frac{f}{|x|^{k+\alpha}}$ would be positively homogeneous of degree $-\frac{Q}{2}$. These are in L^2 if and only if they are zero. $\square$

Theorem 3.1.4 and Theorem 3.1.10 were proved in [RS17b]. They can be extended further to derivatives of higher order. Such results have been recently obtained in [Ngu17] and in the rest of this subsection we describe such extensions.

Theorem 3.1.12 (Further higher-order Hardy–Rellich identities and inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. We denote*

$$\tilde{\mathcal{R}}f:=\mathcal{R}^2 f+\frac{Q-1}{|x|}\mathcal{R}f,\quad C_{\beta}:=\frac{(Q+2\beta)(Q-4-2\beta)}{4},\tag{3.37}$$

for $\beta \in \mathbb{R}$, for the constants appearing below. Let $k \in \mathbb{N}$ be a positive integer and let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the following identities:

(1) *If $Q \geq 4k+1$, then we have*

$$\begin{aligned}
&\left(\prod_{i=0}^{k-1} C_{2i+\alpha}\right)^2 \left\|\frac{f}{|x|^{2k+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\
&= \left\|\frac{\tilde{\mathcal{R}}^k f}{|x|^\alpha}\right\|^2_{L^2(\mathbb{G})} - \left\|\frac{1}{|x|^\alpha}\left|\tilde{\mathcal{R}}^k f + C_\alpha \frac{\tilde{\mathcal{R}}^{k-1} f}{|x|^2}\right|\right\|^2_{L^2(\mathbb{G})} \\
&\quad - \sum_{j=1}^{k-1}\left(\prod_{i=0}^{k-1} C_{2i+\alpha}\right)^2 \left\|\frac{1}{|x|^{2j+\alpha}}\left|\tilde{\mathcal{R}}^{k-j} f + C_{2j+\alpha}\frac{\tilde{\mathcal{R}}^{k-j-1} f}{|x|^2}\right|\right\|^2_{L^2(\mathbb{G})} \\
&\quad - 2C_\alpha \left\|\frac{1}{|x|^{1+\alpha}}\left|\mathcal{R}(\tilde{\mathcal{R}}^{k-1} f) + \frac{Q-4-2\alpha}{2|x|}\tilde{\mathcal{R}}^{k-1} f\right|\right\|^2_{L^2(\mathbb{G})} \\
&\quad - 2\sum_{j=1}^{k-1}\left(\prod_{i=0}^{k-1} C_{2i+\alpha}\right)^2 C_{2j+\alpha} \qquad\qquad (3.38)\\
&\qquad \times \left\|\frac{1}{|x|^{1+\alpha+2j}}\left|\mathcal{R}(\tilde{\mathcal{R}}^{k-j-1} f) + \frac{Q-4-2\alpha-4j}{2|x|}\tilde{\mathcal{R}}^{k-j-1} f\right|\right\|^2_{L^2(\mathbb{G})}.
\end{aligned}$$

(2) *If $Q \geq 4k+3$, then we have*

$$\begin{aligned}
&\left(\frac{Q-2-2\alpha}{2}\prod_{i=0}^{k-1} C_{2i+1+\alpha}\right)^2 \left\|\frac{f}{|x|^{2k+1+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\
&= \left\|\frac{\mathcal{R}(\tilde{\mathcal{R}}^k f)}{|x|^\alpha}\right\|^2_{L^2(\mathbb{G})} - \left\|\frac{1}{|x|^\alpha}\left|\mathcal{R}(\tilde{\mathcal{R}}^k f) + \frac{Q-2-2\alpha}{2|x|}\tilde{\mathcal{R}}^k f\right|\right\|^2_{L^2(\mathbb{G})} \\
&\quad - \frac{(Q-2-2\alpha)^2}{4}\left\|\frac{1}{|x|^\alpha}\left|\tilde{\mathcal{R}}^k f + C_{1+\alpha}\frac{\tilde{\mathcal{R}}^{k-1} f}{|x|^2}\right|\right\|^2_{L^2(\mathbb{G})} \\
&\quad - \frac{(Q-2-2\alpha)^2}{4}\sum_{j=1}^{k-1}\left(\prod_{i=0}^{k-1} C_{2i+\alpha}\right)^2 \\
&\qquad \times \left\|\frac{1}{|x|^{1+2j+\alpha}}\left|\tilde{\mathcal{R}}^{k-j} f + C_{2j+1+\alpha}\frac{\tilde{\mathcal{R}}^{k-j-1} f}{|x|^2}\right|\right\|^2_{L^2(\mathbb{G})} \\
&\quad - \frac{(Q-2-2\alpha)^2}{4} C_{\alpha+1}\left\|\frac{1}{|x|^{2+\alpha}}\left|\mathcal{R}(\tilde{\mathcal{R}}^{k-1} f) + \frac{Q-6-2\alpha}{2|x|}\tilde{\mathcal{R}}^{k-1} f\right|\right\|^2_{L^2(\mathbb{G})}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(Q-2-2\alpha)^2}{4}\sum_{j=1}^{k-1}\left(\prod_{i=0}^{k-1}C_{2i+1+\alpha}\right)^2 C_{2j+1+\alpha} \\
&\quad\times\left\|\frac{1}{|x|^{2+\alpha+2j}}\left|\mathcal{R}(\tilde{\mathcal{R}}^{k-j-1}f)+\frac{Q-6-2\alpha-4j}{2|x|}\tilde{\mathcal{R}}^{k-j-1}f\right|\right\|^2_{L^2(\mathbb{G})}.
\end{aligned}
\tag{3.39}
$$

As consequences, for all complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have the following estimates:*

(3) *If* $Q \geq 4k+1$ *and* $\alpha \in (-Q/2, (Q-4k)/2)$*, then we have*

$$
\left(\prod_{i=0}^{k-1}C_{2i+\alpha}\right)\left\|\frac{f}{|x|^{2k+\alpha}}\right\|_{L^2(\mathbb{G})} \leq \left\|\frac{\tilde{\mathcal{R}}^k f}{|x|^\alpha}\right\|_{L^2(\mathbb{G})}. \tag{3.40}
$$

(4) *If* $Q \geq 4k+3$ *and* $\alpha \in (-(Q+2)/2, (Q-4k-2)/2)$*, then we have*

$$
\left(\frac{Q-2-2\alpha}{2}\prod_{i=0}^{k-1}C_{2i+1+\alpha}\right)\left\|\frac{f}{|x|^{2k+1+\alpha}}\right\|_{L^2(\mathbb{G})} \leq \left\|\frac{\mathcal{R}(\tilde{\mathcal{R}}^k f)}{|x|^\alpha}\right\|_{L^2(\mathbb{G})}. \tag{3.41}
$$

Moreover, these inequalities in Parts (3) *and* (4) *are sharp and the equalities hold if only if* $f = 0$*.*

Proof of Theorem 3.1.12*.* The equality (3.38) follows by induction using Theorem 3.1.4, which gives the case of $k = 1$. Furthermore, we have

$$
\begin{aligned}
\int_{\mathbb{G}}\frac{|\mathcal{R}(\tilde{\mathcal{R}}^k f)|^2}{|x|^{2\alpha}}dx &= \frac{(Q-2-2\alpha)^2}{4}\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^k f|^2}{|x|^{2+2\alpha}}dx \\
&\quad + \int_{\mathbb{G}}\frac{1}{|x|^{2\alpha}}\left|\mathcal{R}(\tilde{\mathcal{R}}^k f)+\frac{Q-2-2\alpha}{2|x|}\tilde{\mathcal{R}}^k f\right|^2 dx.
\end{aligned}
\tag{3.42}
$$

Thus, (3.39) follows from (3.38) and (3.42). The inequalities (3.40) and (3.41) follow directly from these equalities since $C_{\alpha+2i} > 0, C_{\alpha+2i+1} > 0$ for any $i = 0, \ldots, k-1$ corresponding to each case. To check the sharpness of constants, we use the arguments similar to those in the proof of Theorem 3.1.4, considering approximations of the function $r^{-(Q-4k-2\alpha)/2}$. Indeed, one has

$$
\int_{\mathbb{G}}\frac{|f_\epsilon|^2}{|x|^{4k+2\alpha}}dx = (-\ln\epsilon)|\wp| + O(1),
$$

and

$$
\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^k f|^2}{|x|^{2\alpha}}dx = \left(\prod_{i=0}^{k-1}C_{2i+\alpha}\right)^2 (-\ln\epsilon)|\wp| + O(1).
$$

This proves the sharpness of (3.40). For (3.41), we use the approximation of the function $r^{-(Q-2-4k-2\alpha)/2}$ and similar calculations.

Finally, suppose that for some function f we have an equality in (3.40). Then by (3.38), we must have

$$\mathcal{R}f + \frac{Q-4k-2\alpha}{2|x|}f = 0.$$

This is equivalent to

$$\mathbb{E}f = -\frac{Q-4k-2\alpha}{2}f.$$

By Proposition 1.3.1, Part (i), it follows that f is positively homogeneous of order $-(Q-2\alpha-4k)/2$. Since $f/|x|^{\alpha+2k} \in L^2(\mathbb{G})$, this forces to have $f=0$. The sharpness of (3.41) is proved in a similar way. $\square$

Theorem 3.1.12 can be used to derive further weighted Rellich type inequalities. First we start with one iteration, continuing using the notation $\tilde{\mathcal{R}}$ as in (3.37).

Theorem 3.1.13 (Iterated weighted Rellich inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 5$, and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then for any $\alpha \in \mathbb{R}$ and for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the identity*

$$\left\|\frac{\tilde{\mathcal{R}}f}{|x|^{\alpha}}\right\|^2_{L^2(\mathbb{G})} = \frac{(Q+2\alpha)^2}{4}\left\|\frac{\mathcal{R}f}{|x|^{1+\alpha}}\right\|^2_{L^2(\mathbb{G})} + \left\|\frac{1}{|x|^{\alpha}}\left|\mathcal{R}^2 f + \frac{Q-2-2\alpha}{2|x|}\mathcal{R}f\right|\right\|^2_{L^2(\mathbb{G})}. \tag{3.43}$$

As a consequence, for any $\alpha \in \mathbb{R}$ we get the inequality

$$\frac{|Q+2\alpha|}{2}\left\|\frac{\mathcal{R}f}{|x|^{1+\alpha}}\right\|_{L^2(\mathbb{G})} \leq \left\|\frac{\tilde{\mathcal{R}}f}{|x|^{\alpha}}\right\|_{L^2(\mathbb{G})}, \tag{3.44}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. For $Q+2\alpha \neq 0$ the inequality (3.44) is sharp and the equality holds if and only if $f=0$.

Proof of Theorem 3.1.13. By definition (3.37) we have

$$\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}f|^2}{|x|^{2\alpha}}dx = \int_{\mathbb{G}}\frac{|\mathcal{R}(\mathcal{R}f)|^2}{|x|^{2\alpha}}dx + 2(Q-1)\mathrm{Re}\int_{\mathbb{G}}\frac{\mathcal{R}(\mathcal{R}f)\overline{\mathcal{R}f}}{|x|^{2\alpha+1}}dx + (Q-1)^2\int_{\mathbb{G}}\frac{|\mathcal{R}f|^2}{|x|^{2+2\alpha}}dx.$$

Moreover, we have

$$2\mathrm{Re}\int_{\mathbb{G}}\frac{\mathcal{R}(\mathcal{R}f)\overline{\mathcal{R}f}}{|x|^{2\alpha+1}}dx = -(Q-2-2\alpha)\int_{\mathbb{G}}\frac{|\mathcal{R}f|^2}{|x|^{2+2\alpha}}dx.$$

On the other hand, by Theorem 3.1.1 we have the identity

$$\int_{\mathbb{G}} \frac{|\mathcal{R}(\mathcal{R}f)|^2}{|x|^{2\alpha}} dx = \frac{(Q-2-2\alpha)^2}{4} \int_{\mathbb{G}} \frac{|\mathcal{R}f|^2}{|x|^{2+2\alpha}} dx + \int_{\mathbb{G}} \frac{1}{|x|^{2\alpha}} \left| \mathcal{R}^2 f + \frac{Q-2-2\alpha}{2|x|} \mathcal{R}f \right|^2 dx.$$

Combining these inequalities we obtain (3.43).

Identity (3.43) then implies inequality (3.44). The sharpness of inequality (3.44) can be verified by using approximations of the function $r^{-(Q-2\alpha-4k)/2}$. If the equality in (3.44) holds for some function f, then we must have

$$\mathcal{R}^2 f + \frac{Q-2-2\alpha}{2|x|} \mathcal{R}f = 0.$$

This is equivalent to

$$\mathbb{E}(\mathcal{R}f) = -\frac{Q-2-2\alpha}{2} \mathcal{R}f$$

which implies, by Proposition 1.3.1, Part (i), that $\mathcal{R}f$ is positively homogeneous of degree $-(Q-2\alpha-2k)/2$. Since $\mathcal{R}f/|x|^{1+\alpha}$ is in $L^2(\mathbb{G})$, this implies that $\mathcal{R}f = 0$. Consequently, we must also have $f = 0$. $\square$

Theorem 3.1.13 implies further identities and inequalities.

Theorem 3.1.14 (Further higher-order Rellich type identities and inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $k, l \in \mathbb{N}$ be such that $Q \geq 4k+1$ and $k \geq l+1$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\begin{aligned}
\left\| \frac{\tilde{\mathcal{R}}^k f}{|x|^\alpha} \right\|^2_{L^2(\mathbb{G})} &= \frac{4}{(Q-2\alpha)^2} \left(\prod_{i=0}^{k-l-1} \frac{Q^2 - 4(2i+\alpha)^2}{4} \right)^2 \left\| \frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|^{2(k-l)-1+\alpha}} \right\|^2_{L^2(\mathbb{G})} \\
&+ \left(\prod_{i=0}^{k-l-2} C_{2i+\alpha} \right)^2 \left\| \frac{\left| \mathcal{R}^2 \tilde{\mathcal{R}}^l f + \frac{Q+2-4(k-l)-2\alpha}{2|x|} \mathcal{R}\tilde{\mathcal{R}}^l f \right|}{|x|^{2(k-l-1)+\alpha}} \right\|^2_{L^2(\mathbb{G})} \\
&+ \left\| \frac{\left| \tilde{\mathcal{R}}^k f + C_\alpha \frac{\tilde{\mathcal{R}}^{k-1} f}{|x|^2} \right|}{|x|^\alpha} \right\|^2_{L^2(\mathbb{G})} + \sum_{j=1}^{k-l-2} \left(\prod_{i=0}^{j-1} C_{2i+\alpha} \right)^2 \left\| \frac{\left| \tilde{\mathcal{R}}^{k-j} f + C_{2j+\alpha} \frac{\tilde{\mathcal{R}}^{k-j-1} f}{|x|^2} \right|}{|x|^{2j+\alpha}} \right\|^2_{L^2(\mathbb{G})} \\
&+ 2C_\alpha \left\| \frac{\left| \mathcal{R}(\tilde{\mathcal{R}}^{k-1} f) + \frac{Q-4-2\alpha}{2|x|} \tilde{\mathcal{R}}^{k-1} f \right|}{|x|^{1+\alpha}} \right\|^2_{L^2(\mathbb{G})}
\end{aligned}$$

$$+ 2\sum_{j=1}^{k-l-2}\left(\prod_{i=0}^{j-1} C_{2i+\alpha}\right)^2 C_{2j+\alpha}\left\|\frac{\left|\mathcal{R}(\tilde{\mathcal{R}}^{k-j-1}f)+\frac{Q-4-2\alpha-4j}{2|x|}\tilde{\mathcal{R}}^{k-j-1}f\right|}{|x|^{1+2j+\alpha}}\right\|^2_{L^2(\mathbb{G})} \tag{3.45}$$

and

$$\begin{aligned}
\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^\alpha}\right\|^2_{L^2(\mathbb{G})} &= \left(\prod_{i=0}^{k-l-1}\frac{Q^2-4(1+2i+\alpha)^2}{4}\right)^2\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|^{2(k-l)+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\
&+ \frac{(Q-2-2\alpha)^2}{4}\left(\prod_{i=0}^{k-l-2} C_{2i+1+\alpha}\right)^2\left\|\frac{\left|\mathcal{R}^2\tilde{\mathcal{R}}^l f+\frac{Q+2-4(k-l)-2\alpha}{2|x|}\mathcal{R}\tilde{\mathcal{R}}^l f\right|}{|x|^{2(k-l-1)+1+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\
&+ \frac{(Q-2-2\alpha)^2}{4}\left\|\frac{\left|\tilde{\mathcal{R}}^k f + C_\alpha\frac{\tilde{\mathcal{R}}^{k-1}f}{|x|^2}\right|}{|x|^\alpha}\right\|^2_{L^2(\mathbb{G})} \\
&+ \frac{(Q-2-2\alpha)^2}{4}\sum_{j=1}^{k-l-2}\left(\prod_{i=0}^{j-1} C_{2i+1+\alpha}\right)^2\left\|\frac{\left|\tilde{\mathcal{R}}^{k-j}f + C_{2j+1+\alpha}\frac{\tilde{\mathcal{R}}^{k-j-1}f}{|x|^2}\right|}{|x|^{2j+\alpha+1}}\right\|^2_{L^2(\mathbb{G})} \\
&+ 2C_{1+\alpha}\frac{(Q-2-2\alpha)^2}{4}\left\|\frac{\left|\mathcal{R}(\tilde{\mathcal{R}}^{k-1}f)+\frac{Q-6-2\alpha}{2|x|}\tilde{\mathcal{R}}^{k-1}f\right|}{|x|^{2+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\
&+ 2\frac{(Q-2-2\alpha)^2}{4}\sum_{j=1}^{k-l-2}\left(\prod_{i=0}^{j-1} C_{2i+1+\alpha}\right)^2 C_{2j+1+\alpha} \\
&\quad\times\left\|\frac{\left|\mathcal{R}(\tilde{\mathcal{R}}^{k-j-1}f)+\frac{Q-6-2\alpha-4j}{2|x|}\tilde{\mathcal{R}}^{k-j-1}f\right|}{|x|^{1+2j+\alpha}}\right\|^2_{L^2(\mathbb{G})} \\
&+ \left\|\frac{\left|\mathcal{R}\tilde{\mathcal{R}}^k f + \frac{Q-2-2\alpha}{2}\frac{\tilde{\mathcal{R}}^k f}{|x|}\right|}{|x|^\alpha}\right\|^2_{L^2(\mathbb{G})}.
\end{aligned} \tag{3.46}$$

As consequences, we have the following weighted Rellich type inequalities for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$:

(1) *For any $\alpha \in (-Q/2, (Q-4(k-l-1))/2)$ if $k \geq l+2$ and $\alpha \in \mathbb{R}$ if $k = l+1$, we have*

$$\frac{4}{(Q-2\alpha)^2}\left(\prod_{i=0}^{k-l-1}\frac{Q^2-4(2i+\alpha)^2}{4}\right)^2\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|^{2(k-l)-1+\alpha}}\right\|^2_{L^2(\mathbb{G})} \leq \left\|\frac{\tilde{\mathcal{R}}^k f}{|x|^\alpha}\right\|^2_{L^2(\mathbb{G})}. \tag{3.47}$$

(2) *For any* $\alpha \in (-(Q+2)/2, (Q-4(k-l)+2)/2)$ *if* $k \geq l+2$ *and* $\alpha \in \mathbb{R}$ *if* $k = l+1$, *we have*

$$\left(\prod_{i=0}^{k-l-1} \frac{Q^2-4(1+2i+\alpha)^2}{4}\right)^2 \left\| \frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|^{2(k-l)+\alpha}} \right\|^2_{L^2(\mathbb{G})} \leq \left\| \frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^{\alpha}} \right\|^2_{L^2(\mathbb{G})}. \tag{3.48}$$

Moreover, inequalities (3.47) *and* (3.48) *are sharp and there is an equality in each of them if and only if* $f = 0$.

Proof of Theorem 3.1.14. Let $g := \tilde{\mathcal{R}}^{k+1} f$. Applying (3.38) to $\tilde{\mathcal{R}}^{k-l-1} g$ and then Theorem 3.1.13 to $\tilde{\mathcal{R}}\tilde{\mathcal{R}}^l f$, one obtains equality (3.45). Note that

$$\frac{Q+4(k-l-1)+2\alpha}{2} \prod_{i=0}^{k-l-2} C_{2i+\alpha} = \frac{2}{Q-2\alpha} \prod_{i=0}^{k-l-1} \frac{Q^2-4(2i+\alpha)^2}{4}.$$

By applying Theorem 3.1.1 to $\mathcal{R}\tilde{\mathcal{R}}^k f$, and then using (3.45) for $\tilde{\mathcal{R}}^k f$ with weights $|x|^{2(1+\alpha)}$, together with the equality

$$\frac{Q-2-2\alpha}{2} \frac{(Q+4(k-l-1)+2+2\alpha)^2}{4} \prod_{i=0}^{k-l-2} C_{2i+\alpha} = \prod_{i=0}^{k-l-1} \frac{Q^2-4(1+\alpha+2i)^2}{4},$$

we also obtain (3.46).

Consequently, inequalities (3.47) and (3.48) follow from respective identities (3.45) and (3.46) by dropping non-negative remainder terms on the right-hand side. The sharpness of (3.47) and (3.48) follows by considering, respectively, as in the previous theorems, approximations of the function $r^{-(Q-2\alpha-4k)/2}$ and of the function $r^{-(Q-2\alpha-2-4k)/2}$.

If for some function f we have the equality in (3.47), then (3.45) implies that we have

$$\mathcal{R}(\tilde{\mathcal{R}}^{k-1} f) + \frac{Q-4-2\alpha}{2|x|} \tilde{\mathcal{R}}^{k-1} f = 0.$$

This means that

$$\mathbb{E}(\tilde{\mathcal{R}}^{k-1} f) = -\frac{Q-4-2\alpha}{2} \tilde{\mathcal{R}}^{k-1} f.$$

By Proposition 1.3.1, Part (i), it follows that $\tilde{\mathcal{R}}^{k-1} f$ is positively homogeneous of order $-(Q-4-2\alpha)/2$. Since $\tilde{\mathcal{R}}^{k-1} f/|x|^{2\alpha} \in L^2(\mathbb{G})$, we then must have $\tilde{\mathcal{R}}^{k-1} f = 0$, which in turn implies $f = 0$. The same arguments also work for (3.48). $\square$

Using the established theorems, as a corollary one gets the following weighted L^p-Hardy–Rellich type identities which will, in turn, imply the corresponding inequalities.

Theorem 3.1.15 (Higher-order weighted L^p-Hardy–Rellich type identities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q, with a homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$. Let $k \in \mathbb{N}$ be a positive integer, let $1 < p < Q/k$, and let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the identities:*

(1) *If $k = 2l$, $l \geq 2$, then*

$$
\begin{aligned}
&\left\|\frac{\tilde{\mathcal{R}}^l f}{|x|^\alpha}\right\|^p_{L^p(\mathbb{G})} \\
&= \left|\prod_{i=0}^{l-1} C_{p,2i+\alpha}\right|^p \int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(k+\alpha)}} dx + p\int_{\mathbb{G}} \frac{1}{|x|^{p\alpha}} R_p\left(C_{p,\alpha}\frac{\tilde{\mathcal{R}}^{l-1}f}{|x|^2}, -\tilde{\mathcal{R}}^l f\right) dx \\
&+ p\sum_{j=1}^{l-1}\left|\prod_{i=0}^{j-1} C_{p,2i+\alpha}\right|^p \int_{\mathbb{G}} \frac{1}{|x|^{p(2j+\alpha)}} R_p\left(C_{p,2j+\alpha}\frac{\tilde{\mathcal{R}}^{l-j-1}f}{|x|^2}, -\tilde{\mathcal{R}}^{l-j} f\right) dx \\
&+ B_{p,\alpha}(p-1)\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{l-1}f|^{p-2}}{|x|^{p(2j+\alpha)-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{l-1}f| + \frac{Q-p(2+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{l-1}f|\right|^2 dx \\
&+ B_{p,\alpha}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{l-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{l-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{l-1}f}))^2}{|x|^{p(2+\alpha)-2}} dx \qquad (3.49)\\
&+ \sum_{j=1}^{l-1}\left|\prod_{i=0}^{j-1} C_{p,2i+\alpha}\right|^p B_{p,\alpha+2j}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{l-j-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{l-j-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{l-j-1}f}))^2}{|x|^{p(2(j+1)+\alpha)-2}} dx \\
&+ (p-1)\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{l-j-1}f|^{p-2}}{|x|^{p(2(j+1)+\alpha)-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{l-j-1}f| + \frac{Q-p(2(j+1)+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{l-j-1}f|\right|^2 dx,
\end{aligned}
$$

where R_p is as in (2.22), *i.e.,*

$$R_p(\xi,\eta) := \frac{1}{p}|\eta|^p + \frac{p-1}{p}|\xi|^p - \mathrm{Re}(|\xi|^{p-2}\xi\overline{\eta}),$$

as well as

$$B_{p,\alpha+2j} = p|C_{p,\alpha+2j}|^{p-2}C_{p,\alpha+2j},$$

and

$$C_{p,\alpha} = \frac{(Q-2p-p\alpha)(Q+p'\alpha)}{pp'}, \quad p' = \frac{p}{p-1}.$$

(2) *If $k = 2l+1$, $l \geq 1$, then*

$$
\begin{aligned}
&\left\|\frac{\mathcal{R}(\tilde{\mathcal{R}}^l f)}{|x|^\alpha}\right\|^p_{L^p(\mathbb{G})} = \left|\frac{Q-p(1+\alpha)}{p}\right|^p \left|\prod_{i=0}^{l-1} C_{p,2i+1+\alpha}\right|^p \int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(k+\alpha)}} dx \\
&+ p\int_{\mathbb{G}} \frac{1}{|x|^{p\alpha}} R_p\left(-\frac{Q-p(1+\alpha)}{p}\frac{\tilde{\mathcal{R}}^l f}{|x|}, \mathcal{R}\tilde{\mathcal{R}}^l f\right) dx \\
&+ A_{p,\alpha}\int_{\mathbb{G}} \frac{1}{|x|^{p(1+\alpha)}} R_p\left(C_{p,1+\alpha}\frac{\tilde{\mathcal{R}}^{l-1}f}{|x|^2}, -\tilde{\mathcal{R}}^l f\right) dx
\end{aligned}
$$

$$+ A_{p,\alpha}\sum_{j=1}^{l-1}\left|\prod_{i=0}^{j-1}C_{p,2i+1+\alpha}\right|^{p}\int_{\mathbb{G}}\frac{1}{|x|^{p(2j+1+\alpha)}}R_p\left(C_{p,2j+1+\alpha}\frac{\tilde{\mathcal{R}}^{l-j-1}f}{|x|^2},-\tilde{\mathcal{R}}^{l-j}f\right)dx$$
$$+ A_{p,\alpha}\frac{B_{p,\alpha+1}}{p}\left[(p-1)\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{l-1}f|^{p-2}}{|x|^{p(3+\alpha)-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{l-1}f|+\frac{Q-p(3+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{l-1}f|\right|^{2}dx\right]$$
$$+ A_{p,\alpha}\frac{B_{p,\alpha+1}}{p}\left[\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{l-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{l-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{l-1}f}))^2}{|x|^{p(3+\alpha)-2}}dx\right] \tag{3.50}$$
$$+ A_{p,\alpha}\sum_{j=1}^{l-1}\left|\prod_{i=0}^{j-1}C_{p,2i+1+\alpha}\right|^{p}\frac{B_{p,\alpha+2j+1}}{p}\left[\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{l-j-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{l-j-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{l-j-1}f}))^2}{|x|^{p(2j+3+\alpha)-2}}dx\right.$$
$$\left.+ (p-1)\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{l-j-1}f|^{p-2}}{|x|^{p(2j+3+\alpha)-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{l-j-1}f|+\frac{Q-p(2j+3+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{l-j-1}f|\right|^{2}dx\right],$$

where $A_{p,\alpha}=p\left|\frac{Q-p(1+\alpha)}{p}\right|^{p}$.

Proof of Theorem 3.1.15. The equality (3.49) follows from (3.25). The equality (3.50) is consequence of (3.49) and (3.32). Note that in (3.50), if $k=2l+1, l\geq 1$, then the terms concerning the sum from 1 to $l-1$ do not appear if $l=1$. □

By dropping the non-negative remainder terms in (3.49) and (3.50), we obtain the following higher-order weighted L^p-Hardy–Rellich type inequalities.

Corollary 3.1.16 (Higher-order weighted L^p-Hardy–Rellich type inequalities)**.** *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* Q *and let* $|\cdot|$ *be a homogeneous quasi-norm on* $\mathbb{G}$. *Then for any* $\alpha\in\mathbb{R}$ *and all complex-valued functions* $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have:*

(1) *if* $1<p<Q/2k$ *and* $\alpha\in(-Q(p-1))/p,(Q-2pk)/p)$, *then*

$$\left(\prod_{i=0}^{k-1}C_{p,2i+\alpha}\right)\left\|\frac{f}{|x|^{(2k+\alpha)}}\right\|_{L^p(\mathbb{G})}\leq\left\|\frac{\tilde{\mathcal{R}}^k f}{|x|^{\alpha}}\right\|_{L^p(\mathbb{G})}. \tag{3.51}$$

(2) *if* $1<p<Q/(2k+1)$ *and* $\alpha\in(-(Q+p')/p',(Q-p(2k+1))/p)$, *then*

$$\frac{Q-p(1+\alpha)}{p}\left(\prod_{i=0}^{k-1}C_{p,2i+1+\alpha}\right)\left\|\frac{f}{|x|^{(2k+1+\alpha)}}\right\|_{L^p(\mathbb{G})}\leq\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^{\alpha}}\right\|_{L^p(\mathbb{G})}. \tag{3.52}$$

In the above inequalities

$$C_{p,\alpha}=\frac{(Q-2p-p\alpha)(Q+p'\alpha)}{pp'},\quad p'=\frac{p}{p-1}.$$

Inequalities (3.51) *and* (3.52) *are sharp and equalities hold in each of them if and only if* $f=0$.

Proof of Corollary 3.1.16. Inequalities (3.51) and (3.52) are immediate consequences of equalities (3.49) and (3.50), respectively, since $C_{p,2i+\alpha+1} \geq 0$ for $0 \leq i \leq k-1$ under the respective assumptions on indices. The sharpness of constants in (3.51) and (3.52) can be checked by approximating the function $r^{-(Q-p(2k+\alpha))/p}$ and the function $r^{-(Q-p(2k+1+\alpha))/p}$, respectively, by smooth compactly supported functions. Moreover, if for some function f an equality holds in any of these inequalities, then it follows from Theorem 3.1.15 that $|f|$ is positively homogeneous of degree $-(Q-p(2k+\alpha))/p$ and $-(Q-p(2k+1+\alpha))/p$, respectively, which implies that $f = 0$ in view of the condition of L^p integrability. □

As usual, Hardy inequalities imply the corresponding uncertainty principles.

Corollary 3.1.17 (Higher-order Hardy–Rellich uncertainty principles)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $k \in \mathbb{N}$ be a positive integer and let $p > 1$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have:*

(1) *if $k = 2l$, $l \geq 1$, and $\alpha \in (-Q(p-1)/p), (Q-2pl)/p)$, then*

$$\left(\prod_{i=0}^{l-1} C_{p,2i+\alpha}\right) \int_{\mathbb{G}} |f|^2 dx \leq \left(\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^l f|^p}{|x|^{p\alpha}} dx\right)^{1/p} \left(\int_{\mathbb{G}} |f|^{p'} |x|^{p'(2l+\alpha)} dx\right)^{1/p'}. \tag{3.53}$$

(2) *if $k = 2l+1$, $l \geq 0$, and $\alpha \in (-(Q+p')/p', (Q-p(2l+1))/p)$ if $l \geq 1$ and $\alpha \in \mathbb{R}$ if $l = 0$, then*

$$\begin{aligned} &\left|\frac{\frac{Q-p(1+\alpha)}{p}}{C_{p,2l+1+\alpha}} \prod_{i=0}^{l} C_{p,2i+1+\alpha}\right| \int_{\mathbb{G}} |f|^2 dx \\ &\leq \left(\int_{\mathbb{G}} \frac{|\mathcal{R}\tilde{\mathcal{R}}^l f|^p}{|x|^{p\alpha}} dx\right)^{1/p} \left(\int_{\mathbb{G}} |f|^{p'} |x|^{p'(2l+1+\alpha)} dx\right)^{1/p'}. \end{aligned} \tag{3.54}$$

Proof of Corollary 3.1.17. By the Hölder inequality, we have

$$\int_{\mathbb{G}} |f|^2 dx = \int_{\mathbb{G}} \frac{|f|}{|x|^{2l+\alpha}} |f||x|^{2l+\alpha} dx \leq \left(\int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(2l+\alpha)}} dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} |f|^{p'} |x|^{p'(2l+\alpha)} dx\right)^{\frac{1}{p'}}.$$

Further, applying inequality (3.51) to this, we get (3.53) with $C_{2i+\alpha} > 0$ for $0 \leq i \leq l-1$ and $\alpha \in (-Q(p-1)/p, (Q-2pl)/p)$. Inequality (3.54) is proved in the same way. □

Combining Corollary 3.28 and Theorem 2.24, we obtain the weighted L^p-Rellich type inequality below which is an L^p-analogue of Theorem 3.1.14.

Theorem 3.1.18 (Higher-order L^p-weighted Rellich identities and inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any* *homogeneous quasi-norm on $\mathbb{G}$. Let $k, l \in \mathbb{N}$ be non-negative integers such that $k \geq l+1$ and let $p > 1$. Then for any $\alpha \in \mathbb{R}$ and for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$
\begin{aligned}
&\left\|\frac{\tilde{\mathcal{R}}^k f}{|x|^\alpha}\right\|^p_{L^p(\mathbb{G})} \\
&= \frac{p^p}{|Q-p\alpha|^p}\left|\prod_{i=0}^{k-l-1}\frac{(Q-p(2i+\alpha))(Q+p'(2i+\alpha))}{pp'}\right|^p \int_{\mathbb{G}} \frac{|\mathcal{R}\tilde{\mathcal{R}}^l f|^p}{|x|^{p(2(k-l)-1+\alpha)}}dx \\
&\quad + p\left|\prod_{i=0}^{k-l-2} C_{p,2i+\alpha}\right|^p \int_{\mathbb{G}} \frac{R_p\left(\frac{Q+p'(2(k-l-1)+\alpha)}{p'}\frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|}, \tilde{\mathcal{R}}^{l+1}f\right)}{|x|^{p(2(k-l-1)+\alpha)}}dx \\
&\quad + p\int_{\mathbb{G}} \frac{R_p\left(-C_{p,\alpha}\frac{\tilde{\mathcal{R}}^{k-1}f}{|x|^2}, \tilde{\mathcal{R}}^k f\right)}{|x|^{p\alpha}}dx \\
&\quad + p\sum_{j=1}^{k-l-2}\left|\prod_{i=0}^{j-1} C_{p,2i+\alpha}\right|^p \int_{\mathbb{G}} \frac{R_p\left(-C_{p,2j+\alpha}\frac{\tilde{\mathcal{R}}^{k-j-1}f}{|x|^2}, \tilde{\mathcal{R}}^{k-j}f\right)}{|x|^{p(2j+\alpha)}}dx \\
&\quad + B_{p,\alpha}(p-1)\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-2}}{|x|^{p(2+\alpha)-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{k-1}f| + \frac{Q-p(2+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{k-1}f|\right|^2 dx \\
&\quad + B_{p,\alpha}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{l-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-1}f}))^2}{|x|^{p(2+\alpha)-2}}dx \\
&\quad + \sum_{j=1}^{k-l-2}\left|\prod_{i=0}^{j-1} C_{p,2i+\alpha}\right|^p B_{p,\alpha+2j}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{k-j-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-j-1}f}))^2}{|x|^{p(2(j+1)+\alpha)-2}}dx \\
&\quad + (p-1)\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-2}}{|x|^{p(2(j+1)+\alpha)-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{k-j-1}f|\right. \\
&\quad \left. + \frac{Q-p(2(j+1)+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{k-j-1}f|\right|^2 dx,
\end{aligned}
\tag{3.55}
$$

where R_p is as in (2.22):

$$R_p(\xi,\eta) := \frac{1}{p}|\eta|^p + \frac{p-1}{p}|\xi|^p - \mathrm{Re}(|\xi|^{p-2}\xi\overline{\eta}),$$

$$B_{p,\alpha+2j} = p|C_{p,\alpha+2j}|^{p-2}C_{p,\alpha+2j},$$

and

$$C_{p,\alpha} = \frac{(Q-2p-p\alpha)(Q+p'\alpha)}{pp'}, \quad p' = \frac{p}{p-1}.$$

We also have

$$
\begin{aligned}
&\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^\alpha}\right\|^p_{L^p(\mathbb{G})}\\
&=\left|\prod_{i=0}^{k-l-1}\frac{(Q-p(2i+1+\alpha))(Q+p'(2i+1+\alpha))}{pp'}\right|^p\int_{\mathbb{G}}\frac{|\mathcal{R}\tilde{\mathcal{R}}^l f|^p}{|x|^{p(2(k-l)-1+\alpha)}}dx\\
&\quad+A_{p,\alpha}\left|\prod_{i=0}^{k-l-2}C_{p,2i+1+\alpha}\right|^p\int_{\mathbb{G}}\frac{R_p\left(\frac{Q+p'(2(k-l)-1+\alpha)}{p'}\frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|},\tilde{\mathcal{R}}^{l+1}f\right)}{|x|^{p(2(k-l)-1+\alpha)}}dx\\
&\quad+A_{p,\alpha}\int_{\mathbb{G}}\frac{R_p\left(-C_{p,1+\alpha}\frac{\tilde{\mathcal{R}}^{k-j-1}f}{|x|^2},\tilde{\mathcal{R}}^{k-j}f\right)}{|x|^{p(1+\alpha)}}dx\\
&\quad+A_{p,\alpha}\sum_{j=1}^{k-l-2}\left|\prod_{i=0}^{j-1}C_{p,2i+1+\alpha}\right|^p\int_{\mathbb{G}}\frac{R_p\left(-C_{p,2j+1+\alpha}\frac{\tilde{\mathcal{R}}^{k-j-1}f}{|x|^2},\tilde{\mathcal{R}}^{k-j}f\right)}{|x|^{p(2j+1+\alpha)}}dx\\
&\quad+A_{p,\alpha}|\frac{B_{p,\alpha+1}}{p}(p-1)\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-2}}{|x|^{p(3+\alpha)-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{k-1}f|+\frac{Q-p(3+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{k-1}f|\right|^2dx\\
&\quad+A_{p,\alpha}\frac{B_{p,\alpha+1}}{p}\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{k-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-1}f}))^2}{|x|^{p(3+\alpha)-2}}dx\\
&\quad+A_{p,\alpha}\sum_{j=1}^{k-l-2}\left|\prod_{i=0}^{j-1}C_{p,2i+1+\alpha}\right|^p\frac{B_{p,2j+\alpha+1}}{p}\\
&\qquad\times\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{k-j-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-j-1}f}))^2}{|x|^{p(2(j+1)+1+\alpha)-2}}dx\\
&\quad+(p-1)\int_{\mathbb{G}}\frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-2}}{|x|^{p(2(j+1)+1+\alpha)-2}}\\
&\qquad\times\left|\mathcal{R}|\tilde{\mathcal{R}}^{k-j-1}f|+\frac{Q-p(2(j+1)+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{k-j-1}f|\right|^2dx\\
&\quad+p\int_{\mathbb{G}}\frac{1}{|x|^{p\alpha}}R_p\left(-\frac{Q-p-p\alpha}{p}\frac{\tilde{\mathcal{R}}^k f}{|x|},\mathcal{R}\tilde{\mathcal{R}}^k f\right)dx,
\end{aligned}
\tag{3.56}
$$

where

$$A_{p,\alpha}=p\left|\frac{Q-p(1+\alpha)}{p}\right|^p.$$

Consequently, we obtain the following inequalities for all functions $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$: for any $\alpha\in(-Q(p-1)/p,(Q-2p(k-l-1))/2)$ if $k-l\geq 2$ and for any $\alpha\in\mathbb{R}$

if $k = l + 1$, we have

$$\frac{p}{|Q - p\alpha|}\left|\prod_{i=0}^{k-l-1}\frac{(Q - p(2i + \alpha))(Q + p'(2i + \alpha))}{pp'}\right|\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|^{(2(k-l)-1+\alpha)}}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\tilde{\mathcal{R}}^k f}{|x|^{\alpha}}\right\|_{L^p(\mathbb{G})}; \tag{3.57}$$

and for any $\alpha \in (-(Q + p')/p', (Q - p(2(k - l - 1) + 1))/p)$ if $k - l \geq 2$ and for any $\alpha \in \mathbb{R}$ if $k = l + 1$, we have

$$\left|\prod_{i=0}^{k-l-1}\frac{(Q - p(2i + 1 + \alpha))(Q + p'(2i + 1 + \alpha))}{pp'}\right|\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^l f}{|x|^{(2(k-l)-1+\alpha)}}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^{\alpha}}\right\|_{L^p(\mathbb{G})}. \tag{3.58}$$

Moreover, these inequalities (3.57) and (3.58) are sharp and equality in any of them holds if and only if $f = 0$.

Now let us present an extension of the critical Hardy inequality to the higher-order derivatives. To do this, still following [Ngu17] until the end of this section, we present a critical Rellich inequality for $\tilde{\mathcal{R}}$ as follows.

Theorem 3.1.19 (Critical Rellich identity and inequality for $\tilde{\mathcal{R}}$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ be any complex-valued function and denote*

$$f_R(x) := f(Rx/|x|)$$

for any $x \in \mathbb{G}$ and $R > 0$. Then we have

$$\begin{aligned}\left\|\frac{\tilde{\mathcal{R}}f}{|x|^{\frac{Q}{p}-2}}\right\|^p_{L^p(\mathbb{G})} &= \left(\frac{(p-1)(Q-2)}{p}\right)^p \int_{\mathbb{G}} \frac{|f - f_R|^p}{|x|^Q \left|\ln \frac{R}{|x|}\right|^p} dx \\ &\quad + p\int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} R_p\left((Q-2)\frac{\mathcal{R}f}{|x|}, \tilde{\mathcal{R}}f\right) dx \\ &\quad + p(Q-2)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-p}} R_p\left(-\frac{p-1}{p}\frac{f - f_R}{|x|\ln\frac{R}{|x|}}, \mathcal{R}f\right) dx,\end{aligned} \tag{3.59}$$

for any $1 < p < \infty$ and any $R > 0$. Here R_p is as in (2.22). As a consequence, for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have

$$\frac{(p-1)(Q-2)}{p} \sup_{R>0}\left\|\frac{f - f_R}{|x|^{\frac{Q}{p}}\left|\ln\frac{R}{|x|}\right|}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\tilde{\mathcal{R}}f}{|x|^{\frac{Q}{p}-2}}\right\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{3.60}$$

with the constant sharp $(p-1)(Q-2)/p$.

Proof of Theorem 3.1.19. Let $1 < p < \infty$. Let us first restate equality (2.48) in the form

$$\int_{\mathbb{G}} \frac{|\mathcal{R}f|^p}{|x|^{Q-p}}dx = \left(\frac{p-1}{p}\right)^p \int_{\mathbb{G}} \frac{|f-f_R|^p}{|x|^Q|\ln\frac{R}{|x|}|^p}dx + p\int_{\mathbb{G}} \frac{1}{|x|^{Q-p}} R_p\left(-\frac{p-1}{p}\frac{f-f_R}{|x|\ln\frac{R}{|x|}}, \mathcal{R}f\right)dx, \tag{3.61}$$

for any $R > 0$. It follows from Theorem 3.1.7 for $\alpha = (Q-2p)/p$ that

$$\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}f|^p}{|x|^{Q-2p}}dx = (Q-2)^p \int_{\mathbb{G}} \frac{|\mathcal{R}f|^p}{|x|^{Q-p}}dx + p\int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} R_p\left((Q-2)\frac{\mathcal{R}f}{|x|}, \tilde{\mathcal{R}}f\right)dx. \tag{3.62}$$

The combination of (3.61) and (3.62) gives (3.59), which in turn implies (3.60).

Let us now check the sharpness of (3.60). For small enough $\epsilon, \delta > 0$ and for $R > 2$, let us define the function

$$f_\delta(x) := \left(\ln\frac{R}{|x|}\right)^{1-\frac{1}{p}-\delta} g(|x|),$$

where g is the function as in the proof of Theorem 3.1.4. A straightforward calculation gives

$$\mathcal{R}f_\delta(r) = -\left(1-\frac{1}{p}-\delta\right)\frac{1}{r}\left(\ln\frac{R}{r}\right)^{-\frac{1}{p}-\delta} g(r) + \left(\ln\frac{R}{r}\right)^{1-\frac{1}{p}-\delta} g'(r)$$

and

$$\begin{aligned}\mathcal{R}^2 f_\delta(r) = & -2\left(1-\frac{1}{p}-\delta\right)\frac{1}{r}\left(\ln\frac{R}{r}\right)^{-\frac{1}{p}-\delta} g'(r) \\ & -\left(1-\frac{1}{p}-\delta\right)\left(\frac{1}{p}+\delta\right)\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{-1-\frac{1}{p}-\delta} g(r) \\ & +\left(1-\frac{1}{p}-\delta\right)\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{-\frac{1}{p}-\delta} g(r) + \left(\ln\frac{R}{r}\right)^{1-\frac{1}{p}-\delta} g''(r).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\tilde{\mathcal{R}}f_\delta(r) = & -2\left(1-\frac{1}{p}-\delta\right)\frac{1}{r}\left(\ln\frac{R}{r}\right)^{-\frac{1}{p}-\delta} g'(r) \\ & -(Q-2)\left(1-\frac{1}{p}-\delta\right)\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{-\frac{1}{p}-\delta} g(r) \\ & -\left(1-\frac{1}{p}-\delta\right)\left(\frac{1}{p}+\delta\right)\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{-1-\frac{1}{p}-\delta} g(r) + \left(\ln\frac{R}{r}\right)^{1-\frac{1}{p}-\delta} \tilde{\mathcal{R}}g(r).\end{aligned}$$

Since $(f_\delta)_R = 0$ we obtain

$$\begin{aligned}\int_{\mathbb{G}} \frac{|f_\delta - (f_\delta)_R|^p}{|x|^Q |\ln\frac{R}{|x|}|^p} dx &= |\wp| \int_0^2 \frac{1}{r}(\ln R - \ln r)^{-1-\delta p} g(r)^p dr \\ &\geq |\wp| \int_0^1 \frac{1}{r}(\ln R - \ln r)^{-1-\delta p} dr \\ &= \frac{1}{\delta p}(\ln R)^{p\sigma} |\wp|.\end{aligned}$$

Thus,

$$\lim_{\delta\to 0} \int_{\mathbb{G}} \frac{|f_\delta - (f_\delta)_R|^p}{|x|^Q |\ln\frac{R}{|x|}|^p} dx = \infty.$$

At the same time, direct calculations also give

$$\int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} \left| \frac{1}{|x|}\left(\ln\frac{R}{|x|}\right)^{-\frac{1}{p}-\delta} g'(|x|)\right|^p dx = O(1),$$

$$\int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} \left| \frac{1}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-1-\frac{1}{p}-\delta} g(|x|)\right|^p dx = O(1),$$

$$\int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} \left| \left(\ln\frac{R}{|x|}\right)^{1-\frac{1}{p}-\delta} \tilde{\mathcal{R}} g(|x|)\right|^p dx = O(1),$$

and

$$\begin{aligned}\int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} \left| \frac{1}{|x|}\left(\ln\frac{R}{|x|}\right)^{-\frac{1}{p}-\delta} g'(|x|)\right|^p dx &= |\wp| \int_0^2 \frac{1}{r}(\ln R - \ln r)^{-1-\delta p} g(r)^p dr \\ &= \int_{\mathbb{G}} \frac{|f_\delta - (f_\delta)_R|^p}{|x|^Q |\ln\frac{R}{|x|}|^p} dx\end{aligned}$$

Consequently, we obtain

$$\lim_{\delta\to 0} \frac{\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}} f_\delta|^p}{|x|^{Q-2p}} dx}{\int_{\mathbb{G}} \frac{|f_\delta - (f_\delta)_R|^p}{|x|^Q |\ln\frac{R}{|x|}|^p} dx} = \left(\frac{(p-1)(Q-2)}{p}\right)^p.$$

This proves the sharpness of (3.60). □

Consequently, the following identities hold true.

Theorem 3.1.20 (Higher-order critical Rellich identities for $\tilde{\mathcal{R}}$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ be any complex-valued function and denote*

$$f_R(x) := f(Rx/|x|)$$

for $x \in \mathbb{G}$ and $R > 0$.

Then for $2 \le k < Q/2$ we have

$$\left\| \frac{\tilde{\mathcal{R}}^k f}{|x|^{\frac{Q}{p}-2k}} \right\|^p_{L^p(\mathbb{G})} = \left(\frac{Q-2}{p'}\right)^p \left(\prod_{i=1}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{|f-f_R|^p}{|x|^Q| \left|\ln \frac{R}{|x|}\right|^p} dx \tag{3.63}$$
$$+ p \left(\prod_{i=1}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} R_p \left((Q-2)\frac{\mathcal{R}f}{|x|}, \tilde{\mathcal{R}}f\right) dx$$
$$+ p(Q-2)^p \left(\prod_{i=1}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-p}} R_p \left(-\frac{p-1}{p}\frac{f-f_R}{|x| \ln \frac{R}{|x|}}, \mathcal{R}f\right) dx$$
$$+ p \int_{\mathbb{G}} \frac{1}{|x|^{Q-2kp}} R_p \left(-a_{k-1,Q}\frac{\tilde{\mathcal{R}}^{k-1}f}{|x|^2}, \tilde{\mathcal{R}}^k f\right) dx$$
$$+ p \sum_{j=1}^{k-2} \left(\prod_{i=1}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-2(k-j)p}} R_p \left(-a_{k-j-1,Q}\frac{\tilde{\mathcal{R}}^{k-j-1}f}{|x|^2}, \tilde{\mathcal{R}}^{k-j} f\right) dx$$
$$+ p a^{p-1}_{k-1,Q}(p-1) \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-2}}{|x|^{Q-2(k-1)p-2}} \left| \mathcal{R}|\tilde{\mathcal{R}}^{k-1}f| + \frac{2(k-1)}{|x|}|\tilde{\mathcal{R}}^{k-1}f| \right|^2 dx$$
$$+ p a^{p-1}_{k-1,Q} \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{k-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-1}f}))^2}{|x|^{Q-2(k-1)p-2}} dx$$
$$+ p \sum_{j=1}^{l-1} \left(\prod_{i=k-j}^{k-1} a_{k-i-1,Q}\right)^p a^{p-1}_{k-i-1,Q}$$
$$\times \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-4}(\mathrm{Im}(\tilde{\mathcal{R}}^{k-j-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-j-1}f}))^2}{|x|^{Q-2(k-j-1)p-2}}$$
$$+ p \sum_{j=1}^{l-1} \left(\prod_{i=k-j}^{k-1} a_{k-i-1,Q}\right)^p a^{p-1}_{k-i-1,Q}(p-1)$$
$$\times \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-2}}{|x|^{Q-2(k-j-1)p-2}} \left| \mathcal{R}|\tilde{\mathcal{R}}^{k-j-1}f| + \frac{Q-2p(2(j+1)+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{k-j-1}f| \right|^2 dx,$$

where

$$a_{j,Q} = 2j(Q-2j-2)$$

and R_p is as in (2.22)*. For $1 \le k < (Q-1)/2$ we also have*

$$\left\| \frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^{\frac{Q}{p}-(2k+1)}} \right\|^p_{L^p(\mathbb{G})} = (2k)^p \left(\frac{(p-1)(Q-2)}{p}\right)^p \left(\prod_{i=1}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{|f-f_R|^p}{|x|^Q \left|\ln \frac{R}{|x|}\right|^p} dx$$
$$+ p(2k)^p \left(\prod_{i=1}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-2p}} R_p \left((Q-2)\frac{\mathcal{R}f}{|x|}, \tilde{\mathcal{R}}f\right) dx$$

$$
\begin{aligned}
&+ p(2k)^p(Q-2)^p \left(\prod_{i=1}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-p}} R_p\left(-\frac{p-1}{p}\frac{f-f_R}{|x|\ln\frac{R}{|x|}}, \mathcal{R}f\right) dx \\
&+ p(2k)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-2kp}} R_p\left(-a_{k-1,Q}\frac{\tilde{\mathcal{R}}^{k-1}f}{|x|^2}, \tilde{\mathcal{R}}^k f\right) dx \\
&+ p(2k)^p \sum_{j=1}^{k-2}\left(\prod_{i=k-j}^{k-1} a_{i,Q}\right)^p \int_{\mathbb{G}} \frac{1}{|x|^{Q-2(k-j)p}} R_p\left(-a_{k-j-1,Q}\frac{\tilde{\mathcal{R}}^{k-j-1}f}{|x|^2}, \tilde{\mathcal{R}}^{k-j} f\right) dx \\
&+ p(2k)^p a_{k-1,Q}^{p-1}(p-1)\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-2}}{|x|^{Q-2(k-1)p-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{k-1}f| + \frac{2(k-1)}{|x|}|\tilde{\mathcal{R}}^{k-1}f|\right|^2 dx \\
&+ p(2k)^p a_{k-1,Q}^{p-1}\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-1}f|^{p-4}(\operatorname{Im}(\tilde{\mathcal{R}}^{k-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-1}f}))^2}{|x|^{Q-2(k-1)p-2}} dx \\
&+ p(2k)^p \sum_{j=1}^{k-2}\left(\prod_{i=1}^{k-1} a_{k-i-1,Q}\right)^p a_{k-i-1,Q}^{p-1} \\
&\quad\times \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-4}(\operatorname{Im}(\tilde{\mathcal{R}}^{k-j-1}f\overline{\mathcal{R}\tilde{\mathcal{R}}^{k-j-1}f}))^2}{|x|^{Q-2(k-j-1)p-2}} dx \\
&+ p(2k)^p \sum_{j=1}^{k-2}\left(\prod_{i=1}^{k-1} a_{k-i-1,Q}\right)^p a_{k-i-1,Q}^{p-1}(p-1) \\
&\quad\times \int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^{k-j-1}f|^{p-2}}{|x|^{Q-2(k-j-1)p-2}}\left|\mathcal{R}|\tilde{\mathcal{R}}^{k-j-1}f| + \frac{Q-2p(2(j+1)+\alpha)}{p|x|}|\tilde{\mathcal{R}}^{k-j-1}f|\right|^2 dx \\
&+ p\int_{\mathbb{G}} \frac{1}{|x|^{Q-(2k+1)p}} R_p\left(-2k\frac{\tilde{\mathcal{R}}^k f}{|x|}, \mathcal{R}\tilde{\mathcal{R}}^k f\right) dx.
\end{aligned}
\tag{3.64}
$$

Proof of Theorem 3.1.20. Denoting $g = \tilde{\mathcal{R}}f$ and applying (3.49) to the function g with $l = k-1$, and then using (3.59), we arrive at (3.63). To prove (3.64) we use Theorem 2.1.8 with $\alpha = (Q-2(k+1)p)/p$ which gives

$$
\begin{aligned}
\left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^{\frac{Q}{p}-(2k+1)}}\right\|^p_{L^p(\mathbb{G})} &= (2k)^p\int_{\mathbb{G}} \frac{|\tilde{\mathcal{R}}^k f|^p}{|x|^{Q-2kp}} dx \\
&\quad + p\int_{\mathbb{G}} \frac{1}{|x|^{Q-(2k+1)p}} R_p\left(-2k\frac{\tilde{\mathcal{R}}^k f}{|x|}, \mathcal{R}\tilde{\mathcal{R}}^k f\right) dx.
\end{aligned}
\tag{3.65}
$$

Now, the combination of (3.65) and (3.63) implies (3.64). □

As a consequence of Theorem 3.1.20 we have the corresponding inequalities.

Corollary 3.1.21 (Higher-order critical Rellich inequalities for $\tilde{\mathcal{R}}$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and let $|\cdot|$ be a homogeneous* *quasi-norm on $\mathbb{G}$. Let $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ be any complex-valued function and denote*

$$f_R(x) := f(Rx/|x|)$$

with $x \in \mathbb{G}$ and $R > 0$. Let $1 < p < \infty$. Then for any $2 \leq k < Q/2$ we have

$$\frac{2^{k-1}(k-1)!(p-1)}{p}\prod_{i=0}^{k-1}(Q-2i-2)\sup_{R>0}\left\|\frac{f-f_R}{|x|^{\frac{Q}{p}}\left|\ln\frac{R}{|x|}\right|}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\tilde{\mathcal{R}}f}{|x|^{\frac{Q}{p}-2k}}\right\|_{L^p(\mathbb{G})}. \tag{3.66}$$

Furthermore, for any $1 \leq k \leq (Q-1)/2$ we have

$$\frac{2^k k!(p-1)}{p}\prod_{i=0}^{k-1}(Q-2i-2)\sup_{R>0}\left\|\frac{f-f_R}{|x|^{\frac{Q}{p}}\left|\ln\frac{R}{|x|}\right|}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\mathcal{R}\tilde{\mathcal{R}}^k f}{|x|^{\frac{Q}{p}-2(k+1)}}\right\|_{L^p(\mathbb{G})}. \tag{3.67}$$

Moreover, the constants in inequalities (3.66) and (3.67) are sharp.

Proof of Corollary 3.1.21. Inequalities (3.66) and (3.67) are immediate consequences of identities (3.63) and (3.64), respectively, since we have $a_{i,Q} \geq 0$ for $1 \leq i \leq k-1$, as well as the equalities

$$\frac{(Q-2)(p-1)}{p}\prod_{i=1}^{k-1}a_{i,Q} = \frac{2^{k-1}(k-1)!}{p'}\prod_{i=0}^{k-1}(Q-2i-2),$$

and

$$\frac{2k(Q-2)(p-1)}{p}\prod_{i=1}^{k-1}a_{i,Q} = \frac{2^k k!(p-1)}{p}\prod_{i=0}^{k-1}(Q-2i-2).$$

To show the sharpness of the constants in (3.66) and (3.67) we can use the same argument as in the proof of Theorem 3.1.19 with the test function

$$f_\delta(x) := \left(\ln\frac{R}{|x|}\right)^{1-\frac{1}{p}-\delta} g(|x|).$$

This completes the proof. $\square$

3.2 Sobolev type inequalities

In this section we restate some of Hardy inequalities from the previous chapter in terms of the Euler operator $\mathbb{E}$ from Section 1.3.2, relating them to Sobolev type inequalities. We also briefly recast some of their proofs for the convenience of the reader since fixing the notation in terms of the Euler and radial operators will be useful in further arguments, especially for the analysis in Chapter 10.

The classical *Sobolev inequality* in $\mathbb{R}^n$ has the form

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C(p)\|\nabla g\|_{L^{p^*}(\mathbb{R}^n)}, \quad 1 < p, p^* < \infty, \tag{3.68}$$

where ∇ is the standard gradient in $\mathbb{R}^n$ and

$$\frac{1}{p} = \frac{1}{p^*} - \frac{1}{n}.$$

The aim of this section is to discuss another version of the Sobolev inequality (we call it *Sobolev type inequality*) with respect to the operator $x \cdot \nabla$ instead of ∇, that is, the inequality

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C'(p)\|x \cdot \nabla g\|_{L^q(\mathbb{R}^n)}. \tag{3.69}$$

For any $\lambda > 0$, by setting $g(x) = h(\lambda x)$ in (3.69), it is straightforward to see that $p = q$ is a necessary condition to have inequality (3.69). So, one may concentrate on the case $p = q$. In the case of $\mathbb{R}^n$ this inequality was analysed by Ozawa and Sasaki [OS09], now we concentrate on the setting of general homogeneous groups.

We also note that the classical Sobolev inequality (3.68) can be extended to nilpotent Lie groups: for stratified groups see Folland [Fol75], for graded groups see [FR17], with a general summary presented also in [FR16], and for another version on general homogeneous groups see Section 4.3.

3.2.1 Hardy and Sobolev type inequalities

The inequality (3.70) below is such an extension of (3.69) formulated in terms of the Euler operator $\mathbb{E}$. Moreover, it turns out to be possible to derive a formula for the remainder in this inequality. For indices $1 < p < Q$ this Sobolev type inequality implies the Hardy inequality and for $p = 2$ they are equivalent. All this is the subject of the following statement, an analogue of Theorem 2.1.1.

In this section, unless stated otherwise, $\mathbb{G}$ is a homogeneous group of homogeneous dimension $Q \geq 1$ and $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$.

Proposition 3.2.1 (Sobolev type and Hardy inequalities)**.** *We have the following properties.*

(i) *For all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$,*

$$\|f\|_{L^p(\mathbb{G})} \leq \frac{p}{Q}\|\mathbb{E}f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{3.70}$$

where the constant $\frac{p}{Q}$ is sharp and the equality is attained if and only if $f = 0$.

(ii) *Let*

$$u := u(x) = -\frac{p}{Q}\mathbb{E}f(x),\ v := v(x) = f(x).$$

Then we have the following expression for the remainder:

$$\|u\|^p_{L^p(\mathbb{G})} - \|v\|^p_{L^p(\mathbb{G})} = p\int_{\mathbb{G}} I_p(v,u)|v-u|^2 dx, \quad 1 < p < \infty, \tag{3.71}$$

for all real-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, *where*

$$I_p(h,g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

(iii) *For all complex-valued functions* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *the identity* (3.71) *with* $p = 2$ *holds and can be written in the form*

$$\|\mathbb{E}f\|^2_{L^2(\mathbb{G})} = \left(\frac{Q}{2}\right)^2 \|f\|^2_{L^2(\mathbb{G})} + \left\|\mathbb{E}f + \frac{Q}{2}f\right\|^2_{L^2(\mathbb{G})}, \quad Q \geq 1. \tag{3.72}$$

(iv) *In the case* $p = 2$ *and* $Q \geq 3$ *the inequality* (3.70) *is equivalent to Hardy's inequality, i.e., for any* $g \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *the inequality*

$$\left\|\frac{g}{|x|}\right\|_{L^2(\mathbb{G})} \leq \frac{2}{Q-2}\|\mathcal{R}g\|_{L^2(\mathbb{G})} \equiv \frac{2}{Q-2}\left\|\frac{1}{|x|}\mathbb{E}g\right\|_{L^2(\mathbb{G})}. \tag{3.73}$$

(v) *In the case* $1 < p < Q$ *the inequality* (3.70) *yields Hardy's inequality for any* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, *i.e., the inequality*

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p}\|\mathcal{R}f\|_{L^p(\mathbb{G})}. \tag{3.74}$$

Remark 3.2.2.

1. As mentioned above in the Euclidian case, for any $\lambda > 0$, substituting $g(x) = h(\lambda x)$ into the Sobolev type inequality

$$\|g\|_{L^p(\mathbb{G})} \leq C(p)\|\mathbb{E}g\|_{L^q(\mathbb{G})}, \quad 1 < p, q < \infty, \tag{3.75}$$

and using the fact that the Euler operator is a homogeneous operator of order zero, we obtain that $p = q$ is a necessary condition for having inequality (3.75).

2. In the Euclidean case $\mathbb{G} = \mathbb{R}^n$ the inequality (3.70) was observed in [BEHL08]: for any $n \geq 1$ and $1 \leq p < \infty$, for all $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\|f\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n}\|x \cdot \nabla f\|_{L^p(\mathbb{R}^n)}.$$

Indeed, this is a consequence of a simple integration by parts:

$$\begin{aligned} n\int_{\mathbb{R}^n}|f(x)|^p dx &= \int_{\mathbb{R}^n}\operatorname{div}(x)|f(x)|^p dx \\ &= -p\operatorname{Re}\int_{\mathbb{R}^n} x\cdot\nabla f(x)|f(x)|^{p-2}\overline{f(x)}dx \\ &\le p\left(\int_{\mathbb{R}^n}|x\cdot\nabla f(x)|^p dx\right)^{1/p}\left(\int_{\mathbb{R}^n}|f(x)|^p dx\right)^{\frac{p-1}{p}}, \end{aligned}$$

using Hölder's inequality in the last line.

There is a weighted version of the above inequality given in Theorem 6.6.1, see also Remark 6.6.2, Part 3.

3. The analysis in this section is based on [RSY18d].

Proof of Proposition 3.2.1. Let us first show that Part (ii) implies Part (i). By dropping non-negative term in the right-hand side of (3.71), we get

$$\|f\|_{L^p(\mathbb{G})}\le\frac{p}{Q}\|\mathbb{E}f\|_{L^p(\mathbb{G})},\quad 1<p<\infty,\ \ Q\ge 1, \tag{3.76}$$

for all real-valued functions $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$. Consequently, this inequality is valid for all complex-valued functions if we use the identity

$$\forall z\in\mathbb{C}:\ |z|^p=\left(\int_{-\pi}^{\pi}|\cos\theta|^p d\theta\right)^{-1}\int_{-\pi}^{\pi}|\operatorname{Re}(z)\cos\theta+\operatorname{Im}(z)\sin\theta|^p\, d\theta, \tag{3.77}$$

see (2.8).

So, the inequality (3.70) holds true and the expression for the remainder implies that the constant $\frac{p}{Q}$ is sharp.

Let us show that this constant is attained only for $f=0$. In view of the identity (3.77), it is enough to check this only for real-valued functions f. If the right-hand side of (3.71) is zero, then we must have the equality

$$-\frac{p}{Q}\mathbb{E}f(x)=f(x),$$

which yields that

$$\mathbb{E}f=-\frac{Q}{p}f.$$

By the property of the Euler operator in Lemma 1.3.1 this means that f is positively homogeneous function of order $-\frac{Q}{p}$, i.e., there exists a function $h:\wp\to\mathbb{C}$, where $\wp$ is defined by (1.12), such that

$$f(x)=|x|^{-\frac{Q}{p}}h\left(\frac{x}{|x|}\right). \tag{3.78}$$

In particular, (3.78) means that f cannot be compactly supported unless it is identically equal to zero. Therefore, we have shown that Part (ii), namely (3.71) implies Part (i) of Theorem 3.2.1.

Nevertheless, let us also give another direct proof of (3.70) for complex-valued functions without using formula (3.77) and without using the remainder formula in Part (ii).

Proof of Part (i). Introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \wp$ on $\mathbb{G}$, where the sphere $\wp$ is defined in (1.12), we now apply the polar decomposition formula (1.13). This and integrating by parts yield

$$\begin{aligned} \int_{\mathbb{G}} |f(x)|^p dx &= \int_0^\infty \int_{\wp} |f(ry)|^p r^{Q-1} d\sigma(y) dr \\ &= -\frac{p}{Q} \int_0^\infty r^Q \operatorname{Re} \int_{\wp} |f(ry)|^{p-2} f(ry) \overline{\frac{df(ry)}{dr}} d\sigma(y) dr \\ &= -\frac{p}{Q} \operatorname{Re} \int_{\mathbb{G}} |f(x)|^{p-2} f(x) \overline{\mathbb{E} f(x)} dx. \end{aligned} \tag{3.79}$$

By using the Hölder inequality with an index q such that $\frac{1}{p} + \frac{1}{q} = 1$ we obtain

$$\begin{aligned} \int_{\mathbb{G}} |f(x)|^p dx &= -\frac{p}{Q} \operatorname{Re} \int_{\mathbb{G}} |f(x)|^{p-2} f(x) \overline{\mathbb{E} f(x)} dx \\ &\leq \frac{p}{Q} \left(\int_{\mathbb{G}} \left| |f(x)|^{p-2} f(x) \right|^q dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{G}} |\mathbb{E} f(x)|^p \, dx \right)^{\frac{1}{p}} \\ &= \frac{p}{Q} \left(\int_{\mathbb{G}} |f(x)|^p dx \right)^{1-\frac{1}{p}} \|\mathbb{E} f\|_{L^p(\mathbb{G})}, \end{aligned}$$

which gives inequality (3.70) in Part (i).

Proof of Part (ii). With the notation

$$u := u(x) = -\frac{p}{Q} \mathbb{E} f \quad \text{and} \quad v := v(x) = f(x),$$

the formula (3.79) can be reformulated as

$$\|v\|^p_{L^p(\mathbb{G})} = \operatorname{Re} \int_{\mathbb{G}} |v|^{p-2} v \overline{u} dx. \tag{3.80}$$

For any real-valued function f formula (3.79) becomes

$$\int_{\mathbb{G}} |f(x)|^p dx = -\frac{p}{Q} \int_{\mathbb{G}} |f(x)|^{p-2} f(x) \mathbb{E} f(x) dx,$$

and (3.80) becomes

$$\|v\|^p_{L^p(\mathbb{G})} = \int_{\mathbb{G}} |v|^{p-2} v u dx. \tag{3.81}$$

Moreover, for all L^p-integrable real-valued functions u and v the following equalities hold

$$\begin{aligned}
&\|u\|^p_{L^p(\mathbb{G})} - \|v\|^p_{L^p(\mathbb{G})} + p\int_{\mathbb{G}}(|v|^p - |v|^{p-2}vu)dx \\
&= \int_{\mathbb{G}}(|u|^p + (p-1)|v|^p - p|v|^{p-2}vu)dx \\
&= p\int_{\mathbb{G}} I_p(v,u)|v-u|^2dx, \quad 1<p<\infty,
\end{aligned}$$

where

$$I_p(v,u) = (p-1)\int_0^1 |\xi v + (1-\xi)u|^{p-2}\xi d\xi.$$

Combining this with (3.81) we arrive at

$$\|u\|^p_{L^p(\mathbb{G})} - \|v\|^p_{L^p(\mathbb{G})} = p\int_{\mathbb{G}} I_p(v,u)|v-u|^2dx,$$

which proves the equality (3.71).

Now we prove Part (iii). If $p=2$, then the identity (3.80) can be rewritten as

$$\|v\|^2_{L^2(\mathbb{G})} = \mathrm{Re}\int_{\mathbb{G}} v\overline{u}dx.$$

Thus, we have

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{G})} - \|v\|^2_{L^2(\mathbb{G})} &= \|u\|^2_{L^2(\mathbb{G})} - \|v\|^2_{L^2(\mathbb{G})} + 2\int_{\mathbb{G}}(|v|^2 - \mathrm{Re}\, v\overline{u})dx \\
&= \int_{\mathbb{G}}(|u|^2 + |v|^2 - 2\mathrm{Re}\, v\overline{u})dx \\
&= \int_{\mathbb{G}} |u-v|^2dx,
\end{aligned}$$

which gives (3.72).

To show Part (iv) first we verify that the inequality (3.70) implies (3.73). A direct calculation shows

$$\begin{aligned}
\|\mathbb{E}f\|^2_{L^2(\mathbb{G})} &= \left\|\mathbb{E}\frac{g}{|x|}\right\|^2_{L^2(\mathbb{G})} \\
&= \int_0^\infty\int_{\wp}\left|\left(r\frac{d}{dr}\right)\frac{g(ry)}{r}\right|^2 r^{Q-1}d\sigma(y)dr \\
&= \left\|-\frac{g}{|x|} + \frac{d}{d|x|}g\right\|^2_{L^2(\mathbb{G})} \\
&= \left\|\frac{g}{|x|}\right\|^2_{L^2(\mathbb{G})} - 2\mathrm{Re}\int_{\mathbb{G}}\frac{g}{|x|}\overline{\frac{d}{d|x|}g}dx + \left\|\frac{d}{d|x|}g\right\|^2_{L^2(\mathbb{G})},
\end{aligned}$$

where $g=|x|f$. From

$$\begin{aligned}-2\mathrm{Re}\int_{\mathbb{G}}\frac{g}{|x|}\overline{\frac{d}{d|x|}g}dx &= -2\mathrm{Re}\int_0^\infty\int_{\wp}\frac{g(ry)}{r}\overline{\frac{d}{dr}g(ry)}r^{Q-1}d\sigma(y)dr\\ &= -\mathrm{Re}\int_0^\infty\int_{\wp}\frac{d}{dr}(|\overline{g}|^2)r^{Q-2}d\sigma(y)dr\\ &= (Q-2)\mathrm{Re}\int_0^\infty\int_{\wp}|\overline{g}|^2r^{Q-3}d\sigma(y)dr\\ &= (Q-2)\left\|\frac{g}{|x|}\right\|^2_{L^2(\mathbb{G})},\end{aligned}$$

it follows that

$$\|\mathbb{E}f\|^2_{L^2(\mathbb{G})}=(Q-1)\left\|\frac{g}{|x|}\right\|^2_{L^2(\mathbb{G})}+\left\|\frac{d}{d|x|}g\right\|^2_{L^2(\mathbb{G})}. \tag{3.82}$$

Combining (3.70) and (3.82) we obtain

$$\left\|\frac{g}{|x|}\right\|^2_{L^2(\mathbb{G})}\le\frac{4}{Q^2}\left((Q-1)\left\|\frac{g}{|x|}\right\|^2_{L^2(\mathbb{G})}+\left\|\frac{d}{d|x|}g\right\|^2_{L^2(\mathbb{G})}\right),$$

which gives (3.73).

Conversely, let us assume that (3.73) is valid. Then with the notation $f=g/|x|$ we get

$$\begin{aligned}\left\|\frac{d}{d|x|}(|x|f)\right\|^2_{L^2(\mathbb{G})} &= \|f+\mathbb{E}f\|^2_{L^2(\mathbb{G})}\\ &= \|f\|^2_{L^2(\mathbb{G})}+2\mathrm{Re}\int_{\mathbb{G}}f(x)\overline{\mathbb{E}f(x)}dx+\|\mathbb{E}f\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

Hence by (1.42) and (3.73) it follows that

$$\|f\|^2_{L^2(\mathbb{G})}\le\frac{4}{(Q-2)^2}\left(\|\mathbb{E}f\|^2_{L^2(\mathbb{G})}-(Q-1)\|f\|^2_{L^2(\mathbb{G})}\right),$$

which gives

$$\|f\|_{L^2(\mathbb{G})}\le\frac{2}{Q}\|\mathbb{E}f\|_{L^2(\mathbb{G})}.$$

Now it remains to prove Part (v). We will show that the inequality (3.70) gives (3.74). We have

$$\|\mathcal{R}(|x|f)\|_{L^p(\mathbb{G})}=\|\mathbb{E}f+f\|_{L^p(\mathbb{G})}\ge\|\mathbb{E}f\|_{L^p(\mathbb{G})}-\|f\|_{L^p(\mathbb{G})}.$$

Finally, by using the inequality (3.70) we establish

$$\|\mathcal{R}(|x|f)\|_{L^p(\mathbb{G})}\ge\frac{Q-p}{p}\|f\|_{L^p(\mathbb{G})},$$

which implies the Hardy inequality (3.74). $\square$

3.2.2 Weighted L^p-Sobolev type inequalities

Now we establish weighted L^p-Sobolev type inequalities on the homogeneous group $\mathbb{G}$ of homogeneous dimension $Q \geq 1$.

Theorem 3.2.3 (Weighted L^p-Sobolev type inequalities). *For all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, $1 < p < \infty$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ for $\alpha p \neq Q$ we have*

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \leq \left| \frac{p}{Q-\alpha p} \right| \left\| \frac{1}{|x|^\alpha} \mathbb{E} f \right\|_{L^p(\mathbb{G})} \quad \text{for all } \alpha \in \mathbb{R}. \tag{3.83}$$

If $\alpha p \neq Q$ then the constant $\left|\frac{p}{Q-\alpha p}\right|$ is sharp.

For $\alpha p = Q$ we have

$$\left\| \frac{f}{|x|^{\frac{Q}{p}}} \right\|_{L^p(\mathbb{G})} \leq p \left\| \frac{\log|x|}{|x|^{\frac{Q}{p}}} \mathbb{E} f \right\|_{L^p(\mathbb{G})}, \tag{3.84}$$

where the constant p is sharp.

Proof of Theorem 3.2.3. Using the integration by parts formula from Proposition 1.2.10, for $\alpha p \neq Q$ we obtain

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\alpha p}} dx = \int_0^\infty \int_{\wp} |f(ry)|^p r^{Q-1-\alpha p} d\sigma(y) dr$$
$$= -\frac{p}{Q-\alpha p} \int_0^\infty r^{Q-\alpha p} \operatorname{Re} \int_{\wp} |f(ry)|^{p-2} f(ry) \overline{\frac{df(ry)}{dr}} d\sigma(y) dr$$
$$\leq \left| \frac{p}{Q-\alpha p} \right| \int_{\mathbb{G}} \frac{|\mathbb{E} f(x)||f(x)|^{p-1}}{|x|^{\alpha p}} dx = \left| \frac{p}{Q-\alpha p} \right| \int_{\mathbb{G}} \frac{|\mathbb{E} f(x)||f(x)|^{p-1}}{|x|^{\alpha+\alpha(p-1)}} dx.$$

By Hölder's inequality, it follows that

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\alpha p}} dx \leq \left| \frac{p}{Q-\alpha p} \right| \left(\int_{\mathbb{G}} \frac{|\mathbb{E} f(x)|^p}{|x|^{\alpha p}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\alpha p}} dx \right)^{\frac{p-1}{p}},$$

which gives (3.83).

Now we show the sharpness of the constant. We need to check the equality condition in the above Hölder inequality. Let us consider the function

$$g(x) = \frac{1}{|x|^C},$$

where $C \in \mathbb{R}, C \neq 0$ and $\alpha p \neq Q$. Then by a direct calculation we obtain

$$\left| \frac{1}{C} \right|^p \left(\frac{|\mathbb{E} g(x)|}{|x|^\alpha} \right)^p = \left(\frac{|g(x)|^{p-1}}{|x|^{\alpha(p-1)}} \right)^{\frac{p}{p-1}},$$

which satisfies the equality condition in Hölder's inequality. This gives the sharpness of the constant $\left|\frac{p}{Q-\alpha p}\right|$ in (3.83).

Now let us prove (3.84). Using integration by parts, we have

$$\begin{aligned}\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q} dx &= \int_0^\infty \int_{\wp} |f(ry)|^p r^{Q-1-Q} d\sigma(y) dr \\ &= -p \int_0^\infty \log r \operatorname{Re} \int_{\wp} |f(ry)|^{p-2} f(ry) \overline{\frac{df(ry)}{dr}} d\sigma(y) dr \\ &\leq p \int_{\mathbb{G}} \frac{|\mathbb{E} f(x)||f(x)|^{p-1}}{|x|^Q} |\log|x|| dx = p \int_{\mathbb{G}} \frac{|\mathbb{E} f(x)| \log|x||}{|x|^{\frac{Q}{p}}} \frac{|f(x)|^{p-1}}{|x|^{\frac{Q(p-1)}{p}}} dx.\end{aligned}$$

By Hölder's inequality, it follows that

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q} dx \leq p \left(\int_{\mathbb{G}} \frac{|\mathbb{E} f(x)|^p |\log|x||^p}{|x|^Q} dx \right)^{1/p} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q} dx \right)^{(p-1)/p},$$

which gives (3.84).

Now we show the sharpness of the constant. We need to check the equality condition in the above Hölder inequality. Let us consider the function

$$h(x) = (\log|x|)^C,$$

where $C \in \mathbb{R}$ and $C \neq 0$. Then by a direct calculation we obtain

$$\left|\frac{1}{C}\right|^p \left(\frac{|\mathbb{E} h(x)|| \log|x||}{|x|^{\frac{Q}{p}}} \right)^p = \left(\frac{|h(x)|^{p-1}}{|x|^{\frac{Q(p-1)}{p}}} \right)^{p/(p-1)},$$

which satisfies the equality condition in Hölder's inequality. This gives the sharpness of the constant p in (3.84). $\square$

Let us consider separately the case $p = 2$, that is, let us restate Theorem 2.1.5 in terms of the operator $\mathbb{E}$:

Proposition 3.2.4 (An identity for Euler operator)**.** *For every complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left\| \frac{1}{|x|^\alpha} \mathbb{E} f \right\|^2_{L^2(\mathbb{G})} = \left(\frac{Q}{2} - \alpha \right)^2 \left\| \frac{f}{|x|^\alpha} \right\|^2_{L^2(\mathbb{G})} + \left\| \frac{1}{|x|^\alpha} \mathbb{E} f + \frac{Q-2\alpha}{2|x|^\alpha} f \right\|^2_{L^2(\mathbb{G})}, \quad (3.85)$$

for any $\alpha \in \mathbb{R}$.

Remark 3.2.5.

1. By dropping the non-negative last term in (3.85) we immediately get the following inequality for $\alpha \in \mathbb{R}$ with $Q - 2\alpha \neq 0$:

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{|Q-2\alpha|} \left\| \frac{1}{|x|^\alpha} \mathbb{E} f \right\|_{L^2(\mathbb{G})}, \quad (3.86)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, where the constant in (3.86) is sharp and the equality is attained if and only if $f = 0$. This statement on the constant and the equality follows by the same argument as that in Remark 2.1.7.

2. By iterating the established weighted Sobolev inequality (3.83) one obtains inequalities of higher order. Thus, for $1 < p < \infty$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $Q \neq \alpha p$ we have

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \leq \left| \frac{p}{Q - \alpha p} \right|^k \left\| \frac{\mathbb{E}^k f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \tag{3.87}$$

for any complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$.

3. For $k = 1$ (3.87) implies the weighted Sobolev inequality and for $k = 1$ and $\alpha = 0$ this gives the Sobolev inequality. In the case $k = 2$ this can be thought of as a (weighted) Sobolev–Rellich type inequality.

3.2.3 Stubbe type remainder estimates

The remainders in Hardy inequalities may be described in different ways: there may be equalities or estimates of different forms. These are discussed in some detail at various spaces of this book. Here, we give a remainder estimate in the most basic case of L^2. Such a type of inequalities have been analysed on $\mathbb{R}^n$ by Stubbe [Stu90], and here we give its general version on homogeneous groups.

In the proof of the following statement we will use the useful feature that some estimates involving radial derivatives of the Euler operator can be proved first for radial functions, and then extended to non-radial ones by a more abstract argument, see Section 1.3.3.

Theorem 3.2.6 (Stubbe type remainder estimate)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$. Let $|\cdot|$ be a homogeneous quasi-norm. Then we have for all $f \in C_0^\infty(\mathbb{G})$ and $0 \leq \delta < \frac{Q^2}{4}$, the inequality*

$$\int_{\mathbb{G}} |\mathbb{E}f(x)|^2 dx - \delta \int_{\mathbb{G}} |f(x)|^2 dx \geq \frac{\left(\frac{Q^2}{4} - \delta\right)^{\frac{Q-1}{Q}}}{\left(\frac{(Q-2)^2}{4}\right)^{\frac{Q-1}{Q}}} S_Q \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx \right)^{2/2^*} \tag{3.88}$$

with sharp constant, where

$$g(|x|) = \mathcal{M}(f)(|x|) := \frac{1}{|\wp|} \int_{\wp} f(|x|, y) d\sigma(y)$$

and

$$S_Q := |\wp|^{\frac{2}{Q}} Q^{\frac{Q-2}{Q}} (Q-2) \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{2/Q}. \tag{3.89}$$

Remark 3.2.7.

1. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $Q = n$, and inequality (3.88) becomes

$$\begin{aligned} &\int_{\mathbb{R}^n} |(x \cdot \nabla) f(x)|^2 dx - \delta \int_{\mathbb{R}^n} |f(x)|^2 dx \\ &\quad \geq \frac{\left(\frac{n^2}{4} - \delta\right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4}\right)^{\frac{n-1}{n}}} S_n \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx\right)^{\frac{2}{2^*}}. \end{aligned} \tag{3.90}$$

2. An interesting observation is that the constant in the above inequality on $\mathbb{R}^n$ is sharp for any quasi-norm $|\cdot|$, that is, it does not depend on the quasi-norm $|\cdot|$. Therefore, this inequality is new already in the Euclidean setting of $\mathbb{R}^n$. When $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the Euclidean distance, the inequality (3.90) was investigated in $\mathbb{R}^n$ in [BEHL08, Corollary 4.4] and in [Xia11, Theorem 1.1].

3. The following result, proved in [Bli30], will be useful in the proof: Let f be a non-negative function. Then for $s \geq 0$ and $q > p > 1$ we have

$$\left(\int_0^\infty \left|\int_0^s f(r) dr\right|^q r^{q/p-q-1} dr\right)^{p/q} \leq C_{p,q} \int_0^\infty |f(r)|^p dr, \tag{3.91}$$

where

$$C_{p,q} = (q - q/p)^{-p/q} \left(\frac{(q/p - 1)\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{(q-p)/q}$$

is sharp. Moreover, the equality in (3.91) is attained for functions of the form

$$f(r) = c_1(c_2 r^{q/p-1} + 1)^{q/(p-q)}, \; c_1 > 0, \; c_2 > 0. \tag{3.92}$$

Proof of Theorem 3.2.6. First we prove inequality (3.88) for $|\cdot|$-radial functions $f(x) = \widetilde{f}(|x|)$. Then we have $g(r) = \widetilde{f}(r)$ since

$$g(|x|) = \mathcal{M}(f)(|x|) := \frac{1}{|\wp|} \int_{\wp} f(|x|, y) d\sigma(y),$$

and we also have $\mathbb{E} f(x) = |x| \widetilde{f}'(|x|)$. We calculate

$$\begin{aligned} &\int_{\mathbb{G}} |\mathbb{E} \widetilde{f}(|x|)|^2 dx - \beta(Q - \beta) \int_{\mathbb{G}} |\widetilde{f}(|x|)|^2 dx \\ &\quad = |\wp| \left(\int_0^\infty |\widetilde{f}'(r)|^2 r^{Q+1} dr - \beta(Q - \beta) \int_0^\infty |\widetilde{f}(r)|^2 r^{Q-1} dr\right), \end{aligned} \tag{3.93}$$

where $0 \leq \beta < Q/2$. Using the notation $h(|x|) := |x|^{\beta}\widetilde{f}(|x|)$, and integrating by parts we obtain

$$\begin{aligned}
&\int_0^\infty |h'(r)|^2 r^{Q+1-2\beta}dr \\
&\quad = \int_0^\infty |\beta r^{\beta-1}\widetilde{f}(r) + r^{\beta}\widetilde{f}'(r)|^2 r^{Q+1-2\beta}dr \\
&\quad = \beta^2\int_0^\infty |\widetilde{f}(r)|^2 r^{Q-1}dr + \int_0^\infty |\widetilde{f}'(r)|^2 r^{Q+1}dr + \beta\int_0^\infty \frac{d}{dr}|\widetilde{f}(r)|^2 r^{Q}dr \\
&\quad = \int_0^\infty |\widetilde{f}'(r)|^2 r^{Q+1}dr - \beta(Q-\beta)\int_0^\infty |\widetilde{f}(r)|^2 r^{Q-1}dr.
\end{aligned} \tag{3.94}$$

Moreover, by changing the variables $s = r^{Q-2\beta}$ we get

$$\begin{aligned}
\int_0^\infty |h'(r)|^2 r^{Q+1-2\beta}dr &= \int_0^\infty |(Q-2\beta)s^{\frac{Q-2\beta-1}{Q-2\beta}}h'(s)|^2 s^{\frac{Q-2\beta+1}{Q-2\beta}}\frac{s^{\frac{1}{Q-2\beta}-1}ds}{Q-2\beta} \\
&= (Q-2\beta)\int_0^\infty s^2|h'(s)|^2ds.
\end{aligned} \tag{3.95}$$

Combining (3.94) and (3.95), we restate (3.93) as

$$\int_{\mathbb{G}} |\mathbb{E}\widetilde{f}(|x|)|^2dx - \beta(Q-\beta)\int_{\mathbb{G}} |\widetilde{f}(|x|)|^2dx = |\wp|(Q-2\beta)\left(\int_0^\infty s^2|h'(s)|^2ds\right). \tag{3.96}$$

Now setting $\phi(s) := h'(s)$ and $\psi(s) := s^{-2}\phi(s^{-1})$, and using (3.91) with $p = 2$ and $q = 2^*$, we obtain

$$\begin{aligned}
&\int_0^\infty s^2|h'(s)|^2ds = \int_0^\infty s^2|\phi(s)|^2ds = \int_0^\infty |\psi(s)|^2ds \\
&\quad \geq \left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}}\left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}}\left(\int_0^\infty \left|\int_0^s |\psi(t)|dt\right|^{2^*} s^{\frac{2-2Q}{Q-2}}ds\right)^{\frac{2}{2^*}} \\
&\quad = \left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}}\left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}}\left(\int_0^\infty \left|\int_{s^{-1}}^\infty |\phi(t)|dt\right|^{2^*} s^{\frac{2-2Q}{Q-2}}ds\right)^{\frac{2}{2^*}} \\
&\quad \geq \left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}}\left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}}\left(\int_0^\infty |h(s^{-1})|^{2^*} s^{\frac{2-2Q}{Q-2}}ds\right)^{\frac{2}{2^*}} \\
&\quad = \left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}}\left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}}\left(\int_0^\infty |h(s)|^{2^*} s^{\frac{2}{Q-2}}ds\right)^{\frac{2}{2^*}} \\
&\quad = (Q-2\beta)^{\frac{2}{2^*}}\left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}}\left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}}\left(\int_0^\infty |r\widetilde{f}(r)|^{2^*} r^{Q-1}dr\right)^{\frac{2}{2^*}},
\end{aligned}$$

where we have used $s = r^{Q-2\beta}$ and $h(r) = r^{\beta}\widetilde{f}(r)$ in the last line. Thus, now (3.96) implies that

$$\begin{aligned}
&\int_{\mathbb{G}} |\mathbb{E}\widetilde{f}(|x|)|^2 dx - \beta(Q-\beta)\int_{\mathbb{G}} |\widetilde{f}(|x|)|^2 dx \\
&\geq |\wp|(Q-2\beta)^{\frac{2Q-2}{Q}} \left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}} \left(\int_0^\infty |r\widetilde{f}(r)|^{2^*} r^{Q-1} dr\right)^{\frac{2}{2^*}} \\
&= |\wp|^{\frac{2}{Q}} (Q-2\beta)^{\frac{2Q-2}{Q}} \left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}} \left(\int_{\mathbb{G}} |x|^{2^*} |\widetilde{f}(|x|)|^{2^*} dx\right)^{\frac{2}{2^*}}.
\end{aligned} \tag{3.97}$$

Here, denoting $\beta = (Q - \sqrt{Q^2 - 4\delta})/2$ for $0 \leq \delta < Q^2/4$ and recalling that $g(|x|) = \widetilde{f}(|x|)$, we see that (3.97) (with (3.89)) yields

$$\begin{aligned}
&\int_{\mathbb{G}} |\mathbb{E}\widetilde{f}(|x|)|^2 dx - \delta\int_{\mathbb{G}} |\widetilde{f}(|x|)|^2 dx \\
&\geq |\wp|^{\frac{2}{Q}} (Q^2-4\delta)^{\frac{Q-1}{Q}} \left(\frac{Q}{Q-2}\right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)}\right)^{\frac{2}{Q}} \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx\right)^{\frac{2}{2^*}} \\
&= \left(\frac{Q^2-4\delta}{(Q-2)^2}\right)^{\frac{Q-1}{Q}} S_Q \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx\right)^{\frac{2}{2^*}}.
\end{aligned} \tag{3.98}$$

That is, we obtain (3.88) with sharp constant for all $|\cdot|$-radial functions $f \in C_0^\infty(\mathbb{G})$.

Finally, using Proposition 1.3.3, and in (3.98) with $g(|x|) = \widetilde{f}(|x|)$, we get (3.88) for non-radial functions. Clearly, the constant in (3.88) is sharp, since this constant is sharp for radial functions by Remark 3.2.7, Part (3). $\square$

3.3 Caffarelli–Kohn–Nirenberg inequalities

This section is devoted to deriving the Caffarelli–Kohn–Nirenberg inequalities in the setting of homogeneous groups. Here we will be working with the radial operators and general quasi-norms. The case of stratified groups with the horizontal gradient and weights will be discussed in Section 6.7.

First, we recall the classical Caffarelli–Kohn–Nirenberg inequalities on $\mathbb{R}^n$ due to Caffarelli, Kohn and Nirenberg [CKN84], with $|\cdot|$ denoting the usual Euclidean distance:

Theorem 3.3.1 (Classical Caffarelli–Kohn–Nirenberg inequality)**.** *Let $n \in \mathbb{N}$ and let p, q, r, a, b, d, $\delta \in \mathbb{R}$ be such that $p, q \geq 1$, $r > 0$, $0 \leq \delta \leq 1$, and*

$$\frac{1}{p} + \frac{a}{n},\ \frac{1}{q} + \frac{b}{n},\ \frac{1}{r} + \frac{c}{n} > 0, \tag{3.99}$$

where

$$c = \delta d + (1-\delta)b.$$

Then there exists a positive constant C such that

$$\||x|^c f\|_{L^r(\mathbb{R}^n)} \leq C\||x|^a|\nabla f|\|^{\delta}_{L^p(\mathbb{R}^n)}\||x|^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)} \tag{3.100}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

$$\frac{1}{r}+\frac{c}{n} = \delta\left(\frac{1}{p}+\frac{a-1}{n}\right)+(1-\delta)\left(\frac{1}{q}+\frac{b}{n}\right), \tag{3.101}$$

$$a-d \geq 0 \quad \textit{if} \quad \delta > 0, \tag{3.102}$$

$$a-d \leq 1 \quad \text{if} \quad \delta > 0 \quad \textit{and} \quad \frac{1}{r}+\frac{c}{n} = \frac{1}{p}+\frac{a-1}{n}. \tag{3.103}$$

Thus, in this section we are interested in inequalities of this type, and we show that some Caffarelli–Kohn–Nirenberg inequalities continue to hold in the setting of homogeneous groups, in particular, including the cases of anisotropic structures on $\mathbb{R}^n$, i.e., the quasi-norm $|\cdot|$ does not need to be the Euclidean norm $|\cdot|_E$ on $\mathbb{R}^n$.

Some inequalities will be obtained as a consequence of the weighted Hardy inequalities. As a particular case of such weighted inequalities we can think of the inequalities

$$\left(\int_{\mathbb{R}^n}|x|_E^{-p\beta}|f|^p dx\right)^{\frac{2}{p}} \leq C_{\alpha,\beta}\int_{\mathbb{R}^n}|x|_E^{-2\alpha}|\nabla f|^2 dx, \tag{3.104}$$

for $f \in C_0^\infty(\mathbb{R}^n)$, where for $n \geq 3$:

$$-\infty < \alpha < \frac{n-2}{2},\ \alpha \leq \beta \leq \alpha+1,\ \text{and}\ p = \frac{2n}{n-2+2(\beta-\alpha)},$$

and for $n = 2$:

$$-\infty < \alpha < 0,\ \alpha < \beta \leq \alpha+1,\ \text{and}\ p = \frac{2}{\beta-\alpha}.$$

Here

$$|x|_E = \sqrt{x_1^2+\cdots+x_n^2}$$

is the standard Euclidean norm. Moreover, we are interested in replacing the Euclidean norm by a general quasi-norm as well as extending such inequalities to general homogeneous groups.

As a special case we can highlight the case of $p = 2$ that was also studied by [WW03] in the Euclidean setting. Here, for all $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n}|x|_E^{-2(\alpha+1)}|f|^2 dx \leq \widetilde{C}_\alpha\int_{\mathbb{R}^n}|x|_E^{-2\alpha}|\nabla f|^2 dx,$$

with any $n \geq 2$ and $-\infty < \alpha < 0$, which in turn can be written for any $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ as

$$\left\| \frac{1}{|x|_E^{\alpha+1}} |f| \right\|_{L^2(\mathbb{R}^n)} \leq C_\alpha \left\| \frac{1}{|x|_E^{\alpha}} |\nabla f| \right\|_{L^2(\mathbb{R}^n)}, \tag{3.105}$$

for all $\alpha \in \mathbb{R}$.

A homogeneous group version of the inequality (3.105) was obtained in Corollary 2.1.6, that is, it was proved that if $\mathbb{G}$ is a homogeneous group of homogeneous dimension Q, then for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ and for every $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have

$$\frac{|Q-2-2\alpha|}{2} \left\| \frac{f}{|x|^{\alpha+1}} \right\|_{L^2(\mathbb{G})} \leq \left\| \frac{1}{|x|^\alpha} \mathcal{R} f \right\|_{L^2(\mathbb{G})} \quad \text{for all } \alpha \in \mathbb{R}, \tag{3.106}$$

where $\mathcal{R}$ is the radial derivative operator with respect to the norm $|\cdot|$. Note that if $\alpha \neq \frac{Q-2}{2}$, then the constant in (3.106) was shown to be sharp for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$.

Remark 3.3.2.

1. An alternative formulation of Theorem 3.3.1 emphasizing the appearing indices was given by D'Ancona and Luca [DL12].
2. The improved versions of the Caffarelli–Kohn–Nirenberg inequality for radially symmetric functions with respect to the range of parameters were investigated in [NDD12]. In [ZHD15] and [HZ11], weighted Hardy type inequalities were obtained for the generalized Baouendi–Grushin vector fields: for $\gamma = 0$ it gives the standard gradient in $\mathbb{R}^n$. We also refer to [HNZ11], [Han15] for weighted Hardy inequalities on the Heisenberg group, to [HZD11] and [ZHD14] on the H-type groups, and a recent paper [Yac18] on Lie groups of polynomial growth.
3. The analysis in this section is based on [ORS18] as well as on [RSY17b] and [RSY18b].

3.3.1 L^p-Caffarelli–Kohn–Nirenberg inequalities

In this section we generalize inequality (3.106) to L^p-cases for all $1 < p < \infty$. Since all the inequalities are of similar type we will keep calling them the Caffarelli–Kohn–Nirenberg inequalities.

Theorem 3.3.3 (Caffarelli–Kohn–Nirenberg inequality for L^p-norms)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then we have*

$$\frac{|Q-\gamma|}{p} \left\| \frac{f}{|x|^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{G})}^p \leq \left\| \frac{1}{|x|^\alpha} \mathcal{R} f \right\|_{L^p(\mathbb{G})} \left\| \frac{f}{|x|^{\frac{\beta}{p-1}}} \right\|_{L^p(\mathbb{G})}^{p-1}, \tag{3.107}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$, $1 < p < \infty$, *and all* $\alpha, \beta \in \mathbb{R}$ *with*

$$\gamma = \alpha + \beta + 1.$$

If $\gamma \neq Q$ *then the constant* $\frac{|Q-\gamma|}{p}$ *is sharp.*

Before proving this theorem let us point out some of its implications.

Remark 3.3.4.

1. In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$, Theorem 3.3.3 gives the inequality

$$\frac{|n-\gamma|}{p}\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{R}^n)} \leq \left\|\frac{1}{|x|^\alpha}\frac{df}{d|x|}\right\|_{L^p(\mathbb{R}^n)}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{R}^n)},$$

with the optimal constant. In particular, for the standard Euclidean distance $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$, by using the Cauchy–Schwarz inequality, it follows that

$$\frac{|n-\gamma|}{p}\left\|\frac{f}{|x|_E^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{R}^n)} \leq \left\|\frac{1}{|x|_E^\alpha}\nabla f\right\|_{L^p(\mathbb{R}^n)}\left\|\frac{f}{|x|_E^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{R}^n)},$$

for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, with the sharp constant.

2. In the case $\alpha = 0$, $\beta = p-1$, and $1 < p < Q$, the inequality (3.107) implies the homogeneous group version of the L^p-Hardy inequality

$$\left\|\frac{1}{|x|}f\right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p}\left\|\mathcal{R}f\right\|_{L^p(\mathbb{G})}, \tag{3.108}$$

again with $\frac{p}{Q-p}$ being the best constant, see Section 2.1.1.

3. For $\mathbb{G} = (\mathbb{R}^n, +)$, $n \geq 3$, inequality (3.108) gives

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\left\|\frac{df}{d|x|}\right\|_{L^p(\mathbb{R}^n)}. \tag{3.109}$$

For the Euclidean distance, by the Cauchy–Schwarz inequality, it implies the classical Hardy inequality:

$$\left\|\frac{f}{|x|_E}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\left\|\nabla f\right\|_{L^p(\mathbb{R}^n)},$$

for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.

4. For the Euclidean distance, the exact formulae of the difference between the right-hand side and the left-hand side of inequality (3.109) were investigated by Ioku, Ishiwata and Ozawa [IIO16b], see also Machihara, Ozawa and Wadade [MOW17a] as well as [IIO16a].

5. One can obtain a number of Heisenberg–Pauli–Weyl type uncertainty inequities directly from the inequality (3.107) which have various consequences and applications. For instance, if $\alpha p = \alpha + \beta + 1$, inequality (3.107) gives

$$\frac{|Q - \alpha p|}{p} \left\| \frac{f}{|x|^\alpha} \right\|^p_{L^p(\mathbb{G})} \leq \left\| \frac{\mathcal{R} f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \left\| |x|^{\frac{1}{p-1}} \frac{f}{|x|^\alpha} \right\|^{p-1}_{L^p(\mathbb{G})}. \tag{3.110}$$

For $\alpha + \beta + 1 = 0$ and $\alpha = -p$, inequality (3.107) gives

$$\frac{Q}{p} \|f\|^p_{L^p(\mathbb{G})} \leq \| |x|^p \mathcal{R} f \|_{L^p(\mathbb{G})} \left\| \frac{f}{|x|} \right\|^{p-1}_{L^p(\mathbb{G})}, \tag{3.111}$$

all with sharp constants.

The Hardy inequality immediately implies a version of the Heisenberg–Pauli–Weyl uncertainty principle.

Corollary 3.3.5 (Heisenberg–Pauli–Weyl type uncertainty principle)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then we have*

$$\|f\|^2_{L^2(\mathbb{G})} \leq \frac{p}{Q-p} \|\mathcal{R} f\|_{L^p(\mathbb{G})} \| |x| f \|_{L^{\frac{p}{p-1}}(\mathbb{G})}, \quad 1 < p < Q, \tag{3.112}$$

for all $f \in C_0^\infty(\mathbb{G} \backslash \{0\})$.

Proof of Corollary 3.3.5*.* By applying the Hölder inequality to (3.108) with $1 < p < Q$, we get

$$\|f\|^2_{L^2(\mathbb{G})} \leq \left\| \frac{1}{|x|} f \right\|_{L^p(\mathbb{G})} \| |x| f \|_{L^{\frac{p}{p-1}}(\mathbb{G})} \leq \frac{p}{Q-p} \|\mathcal{R} f\|_{L^p(\mathbb{G})} \| |x| f \|_{L^{\frac{p}{p-1}}(\mathbb{G})}, \tag{3.113}$$

giving (3.112). ∎

Remark 3.3.6.

1. For the Euclidean space $\mathbb{G} = (\mathbb{R}^n, +)$ and $p = 2$, inequality (3.112) implies the uncertainty principle for any homogeneous quasi-norm $|x|$ on $\mathbb{R}^n$:

$$\left(\int_{\mathbb{R}^n} |f(x)|^2 dx \right)^2 \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} \left| \frac{df(x)}{d|x|} \right|^2 dx \int_{\mathbb{R}^n} |x|^2 |f(x)|^2 dx. \tag{3.114}$$

In turn this gives the classical uncertainty principle on $\mathbb{R}^n$ with the standard Euclidean distance:

$$\left(\int_{\mathbb{R}^n} |f(x)|^2 dx \right)^2 \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} |\nabla f(x)|^2 dx \int_{\mathbb{R}^n} |x|^2_E |f(x)|^2 dx,$$

which is the Heisenberg–Pauli–Weyl uncertainty principle on the Euclidean spaces $\mathbb{R}^n$, $n > 2$.

2. The stratified groups version of Corollary 3.3.5 will be discussed in Section 4.7.

Proof of Theorem 3.3.3. In the case $\gamma = Q$ the inequality is trivial, so we may assume that $\gamma \neq Q$. To use the polar decomposition in Proposition 1.2.10 we denote $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \wp$ on $\mathbb{G}$, where $\wp$ is the unit quasi-sphere. Then a direct calculation gives that

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^\gamma} dx &= \int_0^\infty \int_\wp \frac{|f(ry)|^p}{r^\gamma} r^{Q-1} d\sigma(y) dr \\
&= \frac{1}{Q-\gamma} \int_0^\infty \int_\wp |f(ry)|^p \frac{d\, r^{Q-\gamma}}{dr} d\sigma(y) dr \\
&= -\frac{1}{Q-\gamma} \mathrm{Re} \int_0^\infty \int_\wp p f(ry) |f(ry)|^{p-2} \left(\overline{\frac{df(ry)}{dr}} \right) \frac{1}{r^{\gamma-1}} r^{Q-1} d\sigma(y) dr \\
&= -\frac{p}{Q-\gamma} \mathrm{Re} \int_{\mathbb{G}} f(x) \frac{|f(x)|^{p-2}}{|x|^{\gamma-1}} \left(\overline{\frac{d}{d|x|} f(x)} \right) dx \\
&\leq \left| \frac{p}{Q-\gamma} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x|^{\gamma-1}} \left| \frac{d}{d|x|} f(x) \right| dx \\
&= \left| \frac{p}{Q-\gamma} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x|^{\alpha+\beta}} |\mathcal{R} f(x)|\, dx \\
&\leq \left| \frac{p}{Q-\gamma} \right| \left(\int_{\mathbb{G}} \frac{|\mathcal{R} f(x)|^p}{|x|^{\alpha p}} dx \right)^{1/p} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\frac{\beta p}{p-1}}} dx \right)^{(p-1)/p},
\end{aligned}$$

using the Hölder inequality in the last line. Thus, we obtain

$$\left| \frac{Q-\gamma}{p} \right| \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^\gamma} dx \leq \left(\int_{\mathbb{G}} \frac{|\mathcal{R} f(x)|^p}{|x|^{\alpha p}} dx \right)^{1/p} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\frac{\beta p}{p-1}}} dx \right)^{(p-1)/p},$$

yielding (3.107).

Now it remains to show the sharpness of the constant. To do it we have to examine the equality condition in the Hölder inequality. Let us consider the following function

$$g(x) = \begin{cases} e^{-\frac{C}{\lambda}|x|^\lambda}, & \lambda := \alpha - \frac{\beta}{p-1} + 1 \neq 0, \\ \frac{1}{|x|^C}, & \alpha - \frac{\beta}{p-1} + 1 = 0, \end{cases}$$

where $C = \left| \frac{Q-\gamma}{p} \right|$ and $\gamma \neq Q$. Then one can readily check that

$$\left| \frac{p}{Q-\gamma} \right|^p \frac{|\mathcal{R} g(x)|^p}{|x|^{\alpha p}} = \frac{|g(x)|^p}{|x|^{\frac{\beta p}{p-1}}},$$

satisfying the equality condition in the Hölder inequality. This means that the constant $|(Q-\gamma)/p|$ is sharp. $\square$

3.3.2 Higher-order L^p-Caffarelli–Kohn–Nirenberg inequalities

By iterating the L^p-Caffarelli–Kohn–Nirenberg type inequalities from Theorem 3.3.3 we can establish higher-order inequalities.

Theorem 3.3.7 (Higher-order L^p-Caffarelli–Kohn–Nirenberg inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then for all $k, m \in \mathbb{N}$ and all $1 < p < \infty$ we have*

$$\frac{|Q-\gamma|}{p}\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{G})} \leq \widetilde{A}_{\alpha,m}\widetilde{A}_{\beta,k}\left\|\frac{1}{|x|^{\alpha-m}}\mathcal{R}^{m+1}f\right\|_{L^p(\mathbb{G})}\left\|\frac{1}{|x|^{\frac{\beta}{p-1}-k}}\mathcal{R}^k f\right\|^{p-1}_{L^p(\mathbb{G})}, \tag{3.115}$$

for all complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$,

$$\gamma = \alpha + \beta + 1,$$

and $\alpha \in \mathbb{R}$ such that $\prod_{j=0}^{m-1}|Q - p(\alpha - j)| \neq 0$, and

$$\widetilde{A}_{\alpha,m} := p^m\left[\prod_{j=0}^{m-1}|Q-p(\alpha-j)|\right]^{-1},$$

as well as $\beta \in \mathbb{R}$ such that $\prod_{j=0}^{k-1}\left|Q - p(\frac{\beta}{p-1} - j)\right| \neq 0$, and

$$\widetilde{A}_{\beta,k} := p^{k(p-1)}\left[\prod_{j=0}^{k-1}\left|Q-p\left(\frac{\beta}{p-1}-j\right)\right|\right]^{-(p-1)}.$$

For $p = 2$ the above constants are sharp.

Proof of Theorem 3.3.7. First, let us consider in (3.107) the case

$$\beta = \gamma\left(1-\frac{1}{p}\right).$$

In this case we have $\beta = (\alpha+1)(p-1)$ and $\gamma = p(\alpha+1)$, so that inequality (3.107) becomes

$$\left\|\frac{f}{|x|^{\alpha+1}}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{|Q-p(\alpha+1)|}\left\|\frac{1}{|x|^\alpha}\mathcal{R}f\right\|_{L^p(\mathbb{G})}, \quad 1<p<\infty, \tag{3.116}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ and every $\alpha \in \mathbb{R}$ with $\alpha \neq \frac{Q}{p} - 1$.

Now taking $\mathcal{R}f$ instead of f and $\alpha-1$ instead of α in (3.116) we consequently obtain

$$\left\|\frac{\mathcal{R}f}{|x|^{\alpha}}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{|Q-p\alpha|}\left\|\frac{1}{|x|^{\alpha-1}}\mathcal{R}^2 f\right\|_{L^p(\mathbb{G})},$$

for $\alpha \neq \frac{Q}{p}$. Combining it with (3.116) we get

$$\left\|\frac{f}{|x|^{\alpha+1}}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{|Q-p(\alpha+1)|}\frac{p}{|Q-p\alpha|}\left\|\frac{1}{|x|^{\alpha-1}}\mathcal{R}^2 f\right\|_{L^p(\mathbb{G})},$$

for every $\alpha \in \mathbb{R}$ such that $\alpha \neq \frac{Q}{p}-1$ and $\alpha \neq \frac{Q}{p}$. This iteration process yields

$$\left\|\frac{f}{|x|^{\theta+1}}\right\|_{L^p(\mathbb{G})} \leq A_{\theta,k}\left\|\frac{1}{|x|^{\theta+1-k}}\mathcal{R}^k f\right\|_{L^p(\mathbb{G})}, \quad 1<p<\infty, \tag{3.117}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ and all $\theta \in \mathbb{R}$ such that $\prod_{j=0}^{k-1}|Q-p(\theta+1-j)| \neq 0$, and

$$A_{\theta,k} := p^k\left[\prod_{j=0}^{k-1}|Q-p(\theta+1-j)|\right]^{-1}.$$

Similarly, we get

$$\left\|\frac{\mathcal{R}f}{|x|^{\vartheta+1}}\right\|_{L^p(\mathbb{G})} \leq A_{\vartheta,m}\left\|\frac{1}{|x|^{\vartheta+1-m}}\mathcal{R}^{m+1} f\right\|_{L^p(\mathbb{G})}, \quad 1<p<\infty, \tag{3.118}$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ and all $\vartheta \in \mathbb{R}$ such that $\prod_{j=0}^{m-1}|Q-p(\vartheta+1-j)| \neq 0$, and

$$A_{\vartheta,m} := p^m\left[\prod_{j=0}^{m-1}|Q-p(\vartheta+1-j)|\right]^{-1}.$$

Now putting $\vartheta+1=\alpha$ and $\theta+1=\frac{\beta}{p-1}$ into (3.118) and (3.117), respectively, we arrive at (3.115).

Let us now show the sharpness of the constants in the case $p=2$. This will follow from having an exact form of the remainder in these inequalities. Recall Theorem 3.1.10 saying that if $Q \geq 3$, $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $\prod_{j=0}^{k-1}\left|\frac{Q-2}{2}-(\alpha+j)\right| \neq 0$, then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have

$$\left\|\frac{f}{|x|^{k+\alpha}}\right\|_{L^2(\mathbb{G})} \leq \left[\prod_{j=0}^{k-1}\left|\frac{Q-2}{2}-(\alpha+j)\right|\right]^{-1}\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^k f\right\|_{L^2(\mathbb{G})}, \tag{3.119}$$

where the constant is sharp. In addition, from (3.34) we have the identity

$$\begin{aligned}\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^{k}f\right\|^{2}_{L^{2}(\mathbb{G})} &= \left[\prod_{j=0}^{k-1}\left(\frac{Q-2}{2}-(\alpha+j)\right)^{2}\right]\left\|\frac{f}{|x|^{k+\alpha}}\right\|^{2}_{L^{2}(\mathbb{G})}\\ &\quad+\sum_{l=1}^{k-1}\left[\prod_{j=0}^{l-1}\left(\frac{Q-2}{2}-(\alpha+j)\right)^{2}\right]\\ &\quad\times\left\|\frac{1}{|x|^{l+\alpha}}\mathcal{R}^{k-l}f+\frac{Q-2(l+1+\alpha)}{2|x|^{l+1+\alpha}}\mathcal{R}^{k-l-1}f\right\|^{2}_{L^{2}(\mathbb{G})}\\ &\quad+\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}^{k}f+\frac{Q-2-2\alpha}{2|x|^{1+\alpha}}\mathcal{R}^{k-1}f\right\|^{2}_{L^{2}(\mathbb{G})},\end{aligned}\tag{3.120}$$

for all $k\in\mathbb{N}$ and $\alpha\in\mathbb{R}$. When $p=2$, Theorem 3.3.3 can be restated that for each $f\in C_0^{\infty}(\mathbb{G}\backslash\{0\})$ we have

$$\frac{|Q-\gamma|}{2}\left\|\frac{f}{|x|^{\frac{\gamma}{2}}}\right\|^{2}_{L^{2}(\mathbb{G})}\leq\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f\right\|_{L^{2}(\mathbb{G})}\left\|\frac{f}{|x|^{\beta}}\right\|_{L^{2}(\mathbb{G})},\quad\forall\alpha,\,\beta\in\mathbb{R},\tag{3.121}$$

where $\gamma=\alpha+\beta+1$. The sharpness then follows from the following remark. □

Remark 3.3.8. Combining (3.121) with (3.119) (or with (3.120)), one can obtain a number of inequalities with sharp constants, for example:

$$\frac{|Q-\gamma|}{2}\left\|\frac{f}{|x|^{\frac{\gamma}{2}}}\right\|^{2}_{L^{2}(\mathbb{G})}\leq C_j(\beta,k)\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f\right\|_{L^{2}(\mathbb{G})}\left\|\frac{1}{|x|^{\beta-k}}\mathcal{R}^{k}f\right\|_{L^{2}(\mathbb{G})},\tag{3.122}$$

for $\gamma=\alpha+\beta+1$ and all $\alpha,\,\beta\in\mathbb{R}$ and $k\in\mathbb{N}$, such that,

$$C_j(\beta,k):=\left[\prod_{j=0}^{k-1}\left|\frac{Q-2}{2}-(\beta-k+j)\right|\right]^{-1}\neq0,$$

as well as

$$\frac{|Q-\gamma|}{2}\left\|\frac{f}{|x|^{\frac{\gamma}{2}}}\right\|^{2}_{L^{2}(\mathbb{G})}\leq C_j(\alpha,k)\left\|\frac{1}{|x|^{\alpha-k}}\mathcal{R}^{k+1}f\right\|_{L^{2}(\mathbb{G})}\left\|\frac{f}{|x|^{\beta}}\right\|_{L^{2}(\mathbb{G})},\tag{3.123}$$

for $\gamma=\alpha+\beta+1$ and all $\alpha,\,\beta\in\mathbb{R}$ and $k\in\mathbb{N}$, such that,

$$C_j(\alpha,k):=\left[\prod_{j=0}^{k-1}\left|\frac{Q-2}{2}-(\alpha+k+j)\right|\right]^{-1}\neq0.$$

It follows from (3.120) that these constants $C_j(\beta,k)$ and $C_j(\alpha,k)$ in (3.122) and (3.123) are sharp.

3.3.3 New type of L^p-Caffarelli–Kohn–Nirenberg inequalities

In this section, we introduce new Caffarelli–Kohn–Nirenberg type inequalities on homogeneous groups.

Theorem 3.3.9 (New types of L^p-Caffarelli–Kohn–Nirenberg inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$. Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $0 < \delta < 1$. Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left\| \frac{f}{|x|^{\frac{Q-p}{p}+\delta}} \right\|_{L^p(\mathbb{G})} \leq p^\delta \left\| \frac{\log|x|}{|x|^{\frac{Q-p}{p}}} \mathcal{R}f \right\|_{L^p(\mathbb{G})}^{\delta} \left\| \frac{f}{|x|^{\frac{Q-p}{p}}} \right\|_{L^p(\mathbb{G})}^{1-\delta}, \quad 1 < p < \infty. \tag{3.124}$$

Moreover, we have

$$\left\| \frac{f}{|x|^{\frac{Q-p}{p}+\delta}} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{\mathcal{R}f}{|x|^{\frac{Q-2p}{p}}} \right\|_{L^p(\mathbb{G})}^{1-\delta} \left\| \frac{f}{|x|^{\frac{Q}{p}}} \right\|_{L^p(\mathbb{G})}^{\delta}, \quad 1 < p < \infty. \tag{3.125}$$

Remark 3.3.10. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, (3.124) implies a new type of the Caffarelli–Kohn–Nirenberg inequality for any quasi-norm on $\mathbb{R}^n$: For any function $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ and any $1 < p < \infty$ we have

$$\left\| \frac{f}{|x|^{\frac{n-p}{p}+\delta}} \right\|_{L^p(\mathbb{R}^n)} \leq p^\delta \left\| \frac{\log|x|}{|x|^{\frac{n-p}{p}}} \left(\frac{x}{|x|} \cdot \nabla f \right) \right\|_{L^p(\mathbb{R}^n)}^{\delta} \left\| \frac{f}{|x|^{\frac{n-p}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{1-\delta}. \tag{3.126}$$

By the Schwarz inequality with the standard Euclidean distance given by $|x|_E = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, we obtain the Euclidean form of the Caffarelli–Kohn–Nirenberg type inequality for any quasi-norm on $\mathbb{R}^n$, for $1 < p < \infty$, and for all functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$:

$$\left\| \frac{f}{|x|_E^{\frac{n-p}{p}+\delta}} \right\|_{L^p(\mathbb{R}^n)} \leq p^\delta \left\| \frac{\log|x|_E}{|x|_E^{\frac{n-p}{p}}} \nabla f \right\|_{L^p(\mathbb{R}^n)}^{\delta} \left\| \frac{f}{|x|_E^{\frac{n-p}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{1-\delta}, \tag{3.127}$$

where ∇ is the standard gradient in $\mathbb{R}^n$. Similarly, we can write the inequality (3.125) in the Euclidean case as

$$\left\| \frac{f}{|x|_E^{\frac{n-p}{p}+\delta}} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla f}{|x|_E^{\frac{n-2p}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{1-\delta} \left\| \frac{f}{|x|_E^{\frac{n}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{\delta}, \quad 1 < p < \infty. \tag{3.128}$$

Note that since $\frac{1}{p} + (-\frac{n}{p})\frac{1}{n} = 0$, the inequality (3.128) does not follow from the Caffarelli–Kohn–Nirenberg inequality in Theorem 3.3.3, thus providing an extension of (3.107) in terms of indices but also in terms of a possibility of choosing any homogeneous quasi-norm on $\mathbb{R}^n$.

Proof of Theorem 3.3.9. A direct calculation shows that we have

$$\left\|\frac{f}{|x|^{\frac{Q-p}{p}+\delta}}\right\|^p_{L^p(\mathbb{G})} = \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{Q-p+\delta p}}dx = \int_{\mathbb{G}} \frac{|f(x)|^{\delta p}}{|x|^{\delta Q}} \cdot \frac{|f(x)|^{p(1-\delta)}}{|x|^{(Q-p)(1-\delta)}}dx.$$

Using Hölder's inequality it follows that

$$\left\|\frac{f}{|x|^{\frac{Q-p}{p}+\delta}}\right\|^p_{L^p(\mathbb{G})} \leq \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}dx\right)^{\delta} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{Q-p}}dx\right)^{1-\delta}. \tag{3.129}$$

By Theorem 3.2.3, we have

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}dx \leq p^p \int_{\mathbb{G}} \frac{(\log|x|)^p}{|x|^Q}|\mathbb{E}f(x)|^p dx, \quad 1 < p < \infty,$$

where $\mathbb{E} = |x|\mathcal{R}$ is the Euler operator. It implies that

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}dx \leq p^p \int_{\mathbb{G}} \frac{(\log|x|)^p}{|x|^{Q-p}}|\mathcal{R}f(x)|^p dx, \quad 1 < p < \infty.$$

Using this in (3.129), one obtains

$$\left\|\frac{f}{|x|^{\frac{Q-p}{p}+\delta}}\right\|^p_{L^p(\mathbb{G})} \leq p^{p\delta} \left(\int_{\mathbb{G}} \frac{(\log|x|)^p}{|x|^{Q-p}}|\mathcal{R}f(x)|^p dx\right)^{\delta} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{Q-p}}dx\right)^{1-\delta},$$

which implies (3.124).

Now let us prove (3.125). Using Theorem 3.2.3, one has

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{Q-p}}dx \leq \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)|^p}{|x|^{Q-p}}dx, \quad 1 < p < \infty.$$

Then, using this in (3.130), we obtain

$$\begin{aligned}\left\|\frac{f}{|x|^{\frac{Q-p}{p}+\delta}}\right\|^p_{L^p(\mathbb{G})} &\leq \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}dx\right)^{\delta} \left(\int_{\mathbb{G}} \frac{|\mathbb{E}f(x)|^p}{|x|^{Q-p}}dx\right)^{1-\delta} \\ &= \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}dx\right)^{\delta} \left(\int_{\mathbb{G}} \frac{|\mathcal{R}f(x)|^p}{|x|^{Q-2p}}dx\right)^{1-\delta},\end{aligned}$$

which gives (3.125), completing the proof. □

3.3.4 Extended Caffarelli–Kohn–Nirenberg inequalities

In this section, we extend the range of indices for Theorem 3.3.1. Again, we work in the setting of general homogeneous groups: $\mathbb{G}$ is a homogeneous group of homogeneous dimension $Q \geq 1$ and $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$.

Theorem 3.3.11 (Extended Caffarelli–Kohn–Nirenberg inequalities)**.** *Let* $1 < p$, $q < \infty$, $0 < r < \infty$, *with* $p + q \geq r$. *Let* $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$ *and* a, b, $c \in \mathbb{R}$. *Assume that*

$$\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1 \quad and \quad c = \delta(a-1) + b(1-\delta).$$

Then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have the following inequalities:*

(i) *If* $Q \neq p(1-a)$, *then we have*

$$\||x|^c f\|_{L^r(\mathbb{G})} \leq \left|\frac{p}{Q-p(1-a)}\right|^\delta \||x|^a \mathcal{R} f\|^\delta_{L^p(\mathbb{G})} \left\||x|^b f\right\|^{1-\delta}_{L^q(\mathbb{G})}. \tag{3.130}$$

(ii) *If* $Q = p(1-a)$, *then we have*

$$\||x|^c f\|_{L^r(\mathbb{G})} \leq p^\delta \||x|^a \log|x| \mathcal{R} f\|^\delta_{L^p(\mathbb{G})} \left\||x|^b f\right\|^{1-\delta}_{L^q(\mathbb{G})}. \tag{3.131}$$

The constant in the inequality (3.130) *is sharp for* $p = q$ *with* $a - b = 1$ *or* $p \neq q$ *with* $p(1-a) + bq \neq 0$. *Moreover, the constants in* (3.130) *and* (3.131) *are sharp for* $\delta = 0$ *or* $\delta = 1$.

To compare these inequalities with those in Theorem 3.3.1 let us first formulate the isotropic version of Theorem 3.3.11 in the usual setting of $\mathbb{R}^n$, and its further implication in the case of the Euclidean norm.

Corollary 3.3.12. *Let* $|\cdot|$ *be a homogeneous quasi-norm on* $\mathbb{R}^n$, $n \in \mathbb{N}$. *Let* $1 < p, q < \infty$, $0 < r < \infty$, *with* $p+q \geq r$, $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$ *and* a, b, $c \in \mathbb{R}$. *Assume that*

$$\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1 \quad and \quad c = \delta(a-1) + b(1-\delta).$$

Then we have the following estimates:

(i) *If* $n \neq p(1-a)$, *then for any function* $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ *we have*

$$\||x|^c f\|_{L^r(\mathbb{R}^n)} \leq \left|\frac{p}{n-p(1-a)}\right|^\delta \left\||x|^a \left(\frac{x}{|x|} \cdot \nabla f\right)\right\|^\delta_{L^p(\mathbb{R}^n)} \left\||x|^b f\right\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \tag{3.132}$$

(ii) *In the critical case* $n = p(1-a)$ *for any function* $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ *we have*

$$\||x|^c f\|_{L^r(\mathbb{R}^n)} \leq p^\delta \left\||x|^a \log|x| \left(\frac{x}{|x|} \cdot \nabla f\right)\right\|^\delta_{L^p(\mathbb{R}^n)} \left\||x|^b f\right\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \tag{3.133}$$

(iii) *If* $|\cdot|_E$ *is the Euclidean norm on* $\mathbb{R}^n$, *inequalities* (3.132) *and* (3.133) *imply, respectively,*

$$\||x|^c_E f\|_{L^r(\mathbb{R}^n)} \leq \left|\frac{p}{n-p(1-a)}\right|^\delta \||x|^a_E \nabla f\|^\delta_{L^p(\mathbb{R}^n)} \left\||x|^b_E f\right\|^{1-\delta}_{L^q(\mathbb{R}^n)} \tag{3.134}$$

for $n \neq p(1-a)$, *and*

$$\left\| |x|_E^c f \right\|_{L^r(\mathbb{R}^n)} \leq p^{\delta} \left\| |x|_E^a \log |x| \nabla f \right\|^{\delta}_{L^p(\mathbb{R}^n)} \left\| |x|_E^b f \right\|^{1-\delta}_{L^q(\mathbb{R}^n)}, \tag{3.135}$$

for $n = p(1-a)$.

The inequality (3.132) *holds for any homogeneous quasi-norm* $|\cdot|$, *and the constant* $\left|\frac{p}{n-p(1-a)}\right|^{\delta}$ *is sharp for* $p = q$ *with* $a - b = 1$, *or for* $p \neq q$ *with* $p(1-a) + bq \neq 0$. *Furthermore, the constants* $\left|\frac{p}{n-p(1-a)}\right|^{\delta}$ *and* p^{δ} *are sharp for* $\delta = 0, 1$.

Remark 3.3.13.

1. If the conditions (3.99) on the parameters hold, then the inequality (3.134) is contained in the inequalities in Theorem 3.3.1. However, already in this case, if we require $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1-a) + bq \neq 0$, then (3.134) yields the inequality (3.100) with sharp constant. Moreover, the constants $\left|\frac{p}{n-p(1-a)}\right|^{\delta}$ and p^{δ} are sharp for $\delta = 0$ or $\delta = 1$. The conditions

$$\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1 \quad \text{and} \quad c = \delta(a-1) + b(1-\delta)$$

imply the condition (3.101) of Theorem 3.3.1, as well as conditions (3.102)–(3.103) which are all necessary for having estimates of this type, at least under the conditions (3.99).

2. If the conditions (3.99) are not satisfied, then the inequality (3.134) is not covered by Theorem 3.3.1. So, this gives an extension of Theorem 3.3.1 with respect to the range of parameters. Indeed, let us take, for example,

$$1 < p = q = r < \infty, \ a = -\frac{n-2p}{p}, \ b = -\frac{n}{p}, \ c = -\frac{n-\delta p}{p}.$$

Then by (3.134), for all $f \in C_0^{\infty}(\mathbb{R}^n \backslash \{0\})$ we have the inequalities

$$\left\| \frac{f}{|x|_E^{\frac{n-\delta p}{p}}} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla f}{|x|_E^{\frac{n-2p}{p}}} \right\|^{\delta}_{L^p(\mathbb{R}^n)} \left\| \frac{f}{|x|_E^{\frac{n}{p}}} \right\|^{1-\delta}_{L^p(\mathbb{R}^n)}, \tag{3.136}$$

for all $1 < p < \infty$ and $0 \leq \delta \leq 1$, where ∇ is the standard gradient in $\mathbb{R}^n$. Since we have

$$\frac{1}{q} + \frac{b}{n} = \frac{1}{p} + \frac{1}{n}\left(-\frac{n}{p}\right) = 0,$$

we see that conditions (3.99) fail, so that the inequality (3.136) is not covered by Theorem 3.3.1.

Proof of Theorem 3.3.11. Case $\delta = 0$. In this case we have $q = r$ and $b = c$ by $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a-1) + b(1-\delta)$, respectively. Then, the inequalities (3.130) and (3.131) are equivalent to the trivial estimate

$$\||x|^b f\|_{L^q(\mathbb{G})} \le \left\||x|^b f\right\|_{L^q(\mathbb{G})},$$

with clearly a sharp constant.

Case $\delta = 1$. In this case we have $p = r$ and $a - 1 = c$. By Theorem 3.2.3, we have for $Q + pc = Q + p(a-1) \neq 0$ the inequality

$$\||x|^c f\|_{L^r(\mathbb{G})} \le \left|\frac{p}{Q+pc}\right| \||x|^c \mathbb{E} f\|_{L^r(\mathbb{G})},$$

where $\mathbb{E} = |x|\mathcal{R}$ is the Euler operator. Using this estimate we get

$$\||x|^c f\|_{L^r(\mathbb{G})} \le \left|\frac{p}{Q+pc}\right| \||x|^{c+1} \mathcal{R} f\|_{L^r(\mathbb{G})} = \left|\frac{p}{Q-p(1-a)}\right| \||x|^a \mathcal{R} f\|_{L^p(\mathbb{G})},$$

which implies (3.130). For $Q + pc = Q + p(a-1) = 0$ by Theorem 3.2.3 we obtain

$$\begin{aligned}\||x|^c f\|_{L^r(\mathbb{G})} &\le p\||x|^c \log|x| \mathbb{E} f\|_{L^r(\mathbb{G})}\\ &= p\||x|^{c+1} \log|x| \mathcal{R} f\|_{L^r(\mathbb{G})}\\ &= p\||x|^a \log|x| \mathcal{R} f\|_{L^p(\mathbb{G})},\end{aligned}$$

which gives (3.131). In this case, the constants in (3.130) and (3.131) are sharp, since the constants in Theorem 3.2.3 are sharp.

Case $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$. Using $c = \delta(a-1) + b(1-\delta)$, a direct calculation gives

$$\||x|^c f\|_{L^r(\mathbb{G})} = \left(\int_{\mathbb{G}} |x|^{cr} |f(x)|^r dx\right)^{1/r} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|x|^{\delta r(1-a)}} \frac{|f(x)|^{(1-\delta)r}}{|x|^{-br(1-\delta)}} dx\right)^{1/r}.$$

Since we have $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$ and $p + q \ge r$, then by using Hölder's inequality for $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, we obtain

$$\begin{aligned}\||x|^c f\|_{L^r(\mathbb{G})} &\le \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{p(1-a)}} dx\right)^{\delta/p} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|x|^{-bq}} dx\right)^{(1-\delta)/q}\\ &= \left\|\frac{f}{|x|^{1-a}}\right\|^{\delta}_{L^p(\mathbb{G})} \left\|\frac{f}{|x|^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{G})}.\end{aligned} \tag{3.137}$$

Here we note that when $p = q$ and $a - b = 1$, the equality in Hölder's inequality holds for any function. We also note that in the case $p \neq q$ the function

$$h(x) = |x|^{\frac{1}{(p-q)}(p(1-a)+bq)} \tag{3.138}$$

satisfies Hölder's equality condition

$$\frac{|h|^p}{|x|^{p(1-a)}} = \frac{|h|^q}{|x|^{-bq}}.$$

If $Q \neq p(1-a)$, then by Theorem 3.2.3 we have

$$\begin{aligned}\left\|\frac{f}{|x|^{1-a}}\right\|^{\delta}_{L^p(\mathbb{G})} &\leq \left|\frac{p}{Q-p(1-a)}\right|^{\delta}\left\|\frac{\mathbb{E}f}{|x|^{1-a}}\right\|^{\delta}_{L^p(\mathbb{G})} \\ &= \left|\frac{p}{Q-p(1-a)}\right|^{\delta}\left\|\frac{\mathcal{R}f}{|x|^{-a}}\right\|^{\delta}_{L^p(\mathbb{G})}, \quad 1<p<\infty.\end{aligned} \tag{3.139}$$

Putting this in (3.125), one has

$$\||x|^c f\|_{L^r(\mathbb{G})} \leq \left|\frac{p}{Q-p(1-a)}\right|^{\delta}\left\|\frac{\mathcal{R}f}{|x|^{-a}}\right\|^{\delta}_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{G})}.$$

We note that in the case of $p=q$ and $a-b=1$, Hölder's equality condition of the inequalities (3.137) and (3.139) holds true for functions $\frac{1}{|x|^c}$, $c \in \mathbb{R}\backslash\{0\}$. Moreover, in the case of $p \neq q$ and $p(1-a)+bq \neq 0$, Hölder's equality condition of the inequalities (3.137) and (3.139) holds true for the function $h(x)$ in (3.138). Therefore, the constant in (3.130) is sharp when $p=q$ and $a-b=1$, or when $p \neq q$ and $p(1-a)+bq \neq 0$.

Now let us consider the case $Q=p(1-a)$. Using Theorem 3.2.3, one has

$$\left\|\frac{f}{|x|^{1-a}}\right\|^{\delta}_{L^p(\mathbb{G})} \leq p^{\delta}\left\|\frac{\log|x|}{|x|^{1-a}}\mathbb{E}f\right\|^{\delta}_{L^p(\mathbb{G})}, \quad 1<p<\infty.$$

Then, putting this in (3.137), we obtain

$$\begin{aligned}\||x|^c f\|_{L^r(\mathbb{G})} &\leq p^{\delta}\left\|\frac{\log|x|}{|x|^{1-a}}\mathbb{E}f\right\|^{\delta}_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{G})} \\ &= p^{\delta}\left\|\frac{\log|x|}{|x|^{-a}}\mathcal{R}f\right\|^{\delta}_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{G})},\end{aligned}$$

completing the proof. □