Chapter 12

Hardy and Rellich Inequalities for Sums of Squares of Vector Fields

In this chapter, we demonstrate how some ideas originating in the analysis on groups can be applied in related settings without the group structure. In particular, in Chapter 7 we showed a number of Hardy and Rellich inequalities with weights expressed in terms of the so-called $\mathcal{L}$-gauge. There, the $\mathcal{L}$-gauge is a homogeneous quasi-norm on a stratified group which is obtained from the fundamental solution to the sub-Laplacian. At the same time, in Chapter 11 we used the fundamental solutions of the sub-Laplacian for the advancement of the potential theory on stratified groups, and in Section 7.3 fundamental solutions for the p-sub-Laplacian and their properties were used on polarizable Carnot groups for the derivation of further Hardy estimates in that setting.

The aim of this chapter is to show that given the existence of a fundamental solution one can use the ideas from the analysis on groups to establish a number of Hardy inequalities on spaces without group structure.

Thus, let M be a smooth manifold of dimension n with a volume form $d\nu$. Let $\{X_k\}_{k=1}^N$ be a family of real vector fields on M, and denote by $\mathcal{L}$ the sum of their squares:

$$\mathcal{L} := \sum_{k=1}^{N} X_k^2. \tag{12.1}$$

Identifying each vector field X with the derivative in its direction, second-order differential operators in the form (12.1) have been widely studied in the literature. For instance, by the well-known Hörmander sums of the squares theorem from [Hör67], the operator $\mathcal{L}$ is locally hypoelliptic if the iterated commutators of the vector fields $\{X_k\}_{k=1}^N$ generate the tangent space at each point. Such operators have been also investigated under weaker conditions or without the hypoellipticity property. There are many geometric considerations related to such operators, see, e.g., the seminal papers of Rothschild and Stein [RS76] and of Nagel, Stein and Wainger [NSW85].

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In this chapter our main assumption on the operator $\mathcal{L}$ in (12.1) will be that it has a local fundamental solution. In particular, this is the case when M is a stratified or a graded group, but the group assumption is, in principle, not necessary. In what follows we will give other examples of naturally appearing operators in other contexts having local or global fundamental solutions. The presentation of this chapters is based on the results obtained in [RS17d].

12.1 Assumptions

We start by formulating assumptions for the presentation in this chapter. Then we discuss several settings where these assumptions are satisfied. This will include stratified Lie groups and operators on $\mathbb{R}^n$ satisfying the Hörmander commutator condition.

Let M be a smooth manifold of dimension n with a volume form $d\nu$, and let $\mathcal{L}$ be an operator as in (12.1). At a point $y \in M$ we will be making the following assumption that we call (A_y), asking for the existence of a local fundamental solution at y:

(A_y) For $y \in M$, assume that there is an open set $T_y \subset M$ containing y such that the operator $-\mathcal{L}$ has a fundamental solution in T_y, that is, there exists a function $\Gamma_y \in C^2(T_y \setminus \{y\})$ such that

$$-\mathcal{L}\Gamma_y = \delta_y \quad \text{in} \quad T_y, \tag{12.2}$$

where δ_y is the Dirac δ-distribution at y.

When the point y is fixed, we will often use the notation $\Gamma(x,y) = \Gamma_y(x)$ or simply $\Gamma(x)$. Here C^2 stands for the space of functions with continuous second derivatives with respect to $\{X_k\}_{k=1}^N$. We note that among other things the existence of a fundamental solution implies that $\mathcal{L}$ is hypoelliptic.

Sometimes we will strengthen Assumption (A_y) to the following assumption that we call (A_y^+) asking for the local positivity of the fundamental solution:

(A_y^+) For $y \in M$, assume that (A_y) holds and, moreover, we have

$$\Gamma_y(x) > 0 \text{ in } T_y \setminus \{y\}, \text{ and } \frac{1}{\Gamma_y}(y) = 0.$$

The second part of the assumption is usually naturally satisfied since for a fundamental solution Γ_y, the quotient $\frac{1}{\Gamma_y}$ is usually well-defined and is equal to 0 at y since Γ_y normally blows up at y.

As before, we will be using the notation $\langle X_k, d\nu\rangle$ for the duality product of the vector field X_k with the volume form $d\nu$, that is, since $d\nu$ is an n-form, $\langle X_k, d\nu\rangle$ is an $(n-1)$-form on M.

It will be convenient to use the following notion of admissible domains in this chapter. We note that this notion here differs from the one in Definition 1.4.4.

However, there should be no confusion since the following definition will be used in this chapter only.

Definition 12.1.1 (Admissible domains)**.** We will say (in this chapter) that an open bounded set $\Omega \subset M$ is an *admissible domain* if its boundary $\partial\Omega$ has no self-intersections, and if the vector fields $\{X_k\}_{k=1}^N$ satisfy the equality

$$\sum_{k=1}^{N} \int_{\Omega} X_k f_k d\nu = \sum_{k=1}^{N} \int_{\partial\Omega} f_k \langle X_k, d\nu \rangle, \tag{12.3}$$

for all $f_k \in C^1(\Omega) \bigcap C(\overline{\Omega})$, $k = 1, \ldots, N$.

We will also say that an admissible domain Ω is *strongly admissible with* $y \in M$ if assumption (A_y) is satisfied, $\Omega \subset T_y$, and (12.3) holds for $f_k = vX_k\Gamma_y$ for all $v \in C^1(\Omega) \bigcap C(\overline{\Omega})$.

Although there are several conditions incorporated in the notion of a strongly admissible domain the examples below will actually show that in a number of natural settings, any open bounded set with a piecewise smooth boundary without self-intersections is strongly admissible, see Proposition 12.2.1. The condition that the boundary $\partial\Omega$ has no self-intersections implies that $\partial\Omega$ is orientable. For brevity, we will say that such boundaries are *simple*.

12.1.1 Examples

Let us now describe several rather general settings when bounded domains with simple boundaries are strongly admissible in the sense of Definition 12.1.1. Moreover, we discuss also the validity of assumptions (A_y) and (A_y^+).

For the examples (E2) and (E3) below we will need the following definition.

Definition 12.1.2 (Control distance and Hölder spaces)**.** The *control distance* $d_c(x, y)$ associated to the vector fields X_k is defined as the infimum of $T > 0$ such that there is a piecewise continuous integral curve γ of $X_1, \ldots, X_N$ such that $\gamma(0) = x$ and $\gamma(T) = y$.

The *Hölder space* $C^\alpha(\Omega)$ *with respect to the control distance* is then defined for $0 < \alpha \leq 1$ as the space of all functions u for which there is $C > 0$ such that

$$|u(x) - u(y)| \leq C d_c^\alpha(x, y)$$

holds for all $x, y \in \Omega$. Then, $u \in C^{1,\alpha}$ if $X_k u \in C^\alpha$ for all $k = 1, \ldots, N$, and the spaces $C^{r,\alpha}$ are defined inductively.

Example 12.1.3 (Examples of strongly admissible domains)**.** Let us give several examples.

(E1) Let M be a stratified Lie group, $n \geq 3$, and let $\{X_k\}_{k=1}^N$ be left invariant vector fields giving the first stratum of M. Then for any $y \in M$ the assumption (A_y^+) is satisfied with $T_y = M$. Moreover, any open bounded set $\Omega \subset M$ with a piecewise smooth simple boundary is strongly admissible.

(E2) Let $M = \mathbb{R}^n$, $n \geq 3$, and let the vector fields X_k, $k = 1, \ldots, N$, $N \leq n$, be of the form

$$X_k = \frac{\partial}{\partial x_k} + \sum_{m=N+1}^{n} a_{k,m}(x) \frac{\partial}{\partial x_m}, \tag{12.4}$$

where $a_{k,m}(x)$ are locally $C^{1,\alpha}$-regular for some $0 < \alpha \leq 1$, where $C^{1,\alpha}$ stands for the space of functions with X_k-derivatives in the Hölder space C^α with respect to the control distance defined by these vector fields. Assume also

$$\frac{\partial}{\partial x_k} = \sum_{1 \leq i < j \leq N} \lambda_k^{i,j}(x)[X_i, X_j] \tag{12.5}$$

for all $k = N + 1, \ldots, n$, with $\lambda_k^{i,j} \in L^\infty_{\text{loc}}(M)$. Then for any $y \in M$ the assumption (A_y^+) is satisfied. Moreover, any open bounded set $\Omega \subset M$ with a piecewise smooth simple boundary is strongly admissible.

(E3) More generally, let $M = \mathbb{R}^n$, $n \geq 3$, and let the vector fields X_k, $k = 1, \ldots, N$, $N \leq n$, satisfy the Hörmander commutator condition of step $r \geq 2$. Assume that all X_k, $k = 1, \ldots, N$, belong to $C^{r,\alpha}(U)$ for some $0 < \alpha \leq 1$ and $U \subset M$, and if $r = 2$ we assume $\alpha = 1$. Then for any $y \in M$ the assumption (A_y^+) is satisfied. Moreover, if X_k's are in the form (12.4), then any open bounded set $\Omega \subset M$ with a piecewise smooth simple boundary is strongly admissible.

Some remarks are in order.

Remark 12.1.4.

1. In Example (E1), the validity of Assumption (A_y^+) for any y follows from (1.74) and (1.75). The equality of (12.3) for (E1) and the strong admissibility for any domain with piecewise smooth simple boundary follows from Theorem 1.4.5.
2. In Example (E2), the existence of a local fundamental solution, that is (A_y) for any $y \in M$ was shown by Manfredini [Man12]. While the positivity of Γ_y does not seem to be explicitly stated there, see Sánchez-Calle [SC84], or Fefferman and Sánchez-Calle [FSC86] for the positivity, thus assuring that Assumption (A_y^+) holds. The validity of (12.3) and the strong admissibility for any domain with piecewise smooth simple boundary will follow from Theorem 12.2.1.
3. Condition (12.5) implies that the collection of vector fields $\{X_k\}_{k=1}^N$ satisfies Hörmander's commutator condition of step two.
4. The condition (12.4) on the vector fields in (E2) and (E3) is not restrictive. In fact, by a change of variables one can show that any collection of linearly independent vector fields which are locally $C^{r,\alpha}$-regular ($r \in \mathbb{N}$) can be transformed to a collection of the same regularity which satisfies condition (12.4), see Manfredini [Man12, page 975].

5. In Example (E3), the validity of condition (A_y^+) was studied by Bramanti, Brandolini, Manfredini and Pedroni [BBMP17, Theorem 4.8 and Theorem 5.9]. The validity of (12.3) and the strong admissibility for any domain with piecewise smooth simple boundary will follow from Theorem 12.2.1.
6. Assumptions (A_y) or (A_y^+) hold also in some other settings. The subject of the existence of local and global fundamental solutions for $\mathcal{L}$ is well studied when $\mathcal{L}$ is a hypoelliptic operator, see, e.g., [Man12, BLU04, FSC86, SC84, OR73] for more general and detailed discussions.
7. For both Examples (E2) and (E3) let us give the following explicit example: In $\mathbb{R}^3$ let $N = 2$ and let

$$X_1 = \frac{\partial}{\partial x_1} + a(x)\frac{\partial}{\partial x_3},$$
$$X_2 = \frac{\partial}{\partial x_2} + b(x)\frac{\partial}{\partial x_3},$$

be vector fields with coefficients

$$a(x) = x_2(1+|x_2|),\ b(x) = -x_1(1+|x_1|).$$

Clearly, these coefficients are not smooth. Then

$$[X_1, X_2] = -2(1+|x_1|+|x_2|)\frac{\partial}{\partial x_3}.$$

The vector fields X_1, X_2 are $C^{1,1}$ and satisfy Hörmander's commutator condition of step two, so that assumptions of Example (E2) hold. Replacing $|x_1|$, $|x_2|$ with $x_1|x_1|$, $x_2|x_2|$ we get $C^{2,1}$ vector fields, satisfying assumptions of Example (E3).

These examples and the corresponding sub-Laplacian $\mathcal{L} = X_1^2 + X_2^2$ were studied in [BBMP17, Section 6]. Other explicit examples can be built from the so-called Δ_λ-Laplacians, see, e.g., [KS16].

12.2 Divergence formula

For this, there is no need to make any assumptions on the step to which Hörmander's commutator condition is satisfied, whether it is satisfied or not, or on the existence of fundamental solutions as in (A_y). Thus, let us formulate this property as a general statement which shall be of interest on its own. The assumption for smoothness on X_k can be reduced here, e.g., to $a_{k,m} \in C^1$.

Theorem 12.2.1 (Divergence formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a piecewise smooth boundary that has no self-intersections. Let X_k, $k = 1, \ldots, N$, be C^1 vector fields in the form*

$$X_k = \frac{\partial}{\partial x_k} + \sum_{m=N+1}^{n} a_{k,m}(x)\frac{\partial}{\partial x_m}. \tag{12.6}$$

Let $f_k \in C^1(\Omega) \bigcap C(\overline{\Omega})$, $k = 1, \ldots, N$. *Then for each* $k = 1, \ldots, N$, *we have*

$$\int_\Omega X_k f_k d\nu = \int_{\partial\Omega} f_k \langle X_k, d\nu \rangle. \tag{12.7}$$

Consequently, we also have the divergence type formula

$$\int_\Omega \sum_{k=1}^N X_k f_k d\nu = \int_{\partial\Omega} \sum_{k=1}^N f_k \langle X_k, d\nu \rangle. \tag{12.8}$$

If $y \in \mathbb{R}^n$ *is such that* (A_y) *is satisfied, then we can also take* $f_k = vX_k\Gamma_y$ *in formulae above, for all* $v \in C^1(\Omega) \bigcap C(\overline{\Omega})$.

Formula (12.8) is exactly the one needed for the admissibility of a domain in Definition 12.1.1. For a discussion of other related versions of the divergence formula in the literature see Remark 1.4.7. The proof of Theorem 12.2.1 is similar to that of Theorem 1.4.5.

Proof of Theorem 12.2.1. For any function f we calculate the following differentiation formula

$$\begin{aligned} df &= \sum_{k=1}^N \frac{\partial f}{\partial x_k} dx_k + \sum_{m=N+1}^n \frac{\partial f}{\partial x_m} dx_m \\ &= \sum_{k=1}^N X_k f dx_k - \sum_{k=1}^N \sum_{m=N+1}^n a_{k,m}(x) \frac{\partial f}{\partial x_m} dx_k + \sum_{m=N+1}^n \frac{\partial f}{\partial x_m} dx_m \\ &= \sum_{k=1}^N X_k f dx_k + \sum_{m=N+1}^n \frac{\partial f}{\partial x_m} (-\sum_{k=1}^N a_{k,m}(x) dx_k + dx_m) \\ &= \sum_{k=1}^N X_k f dx_k + \sum_{m=N+1}^n \frac{\partial f}{\partial x_m} \theta_m, \end{aligned}$$

where we denote

$$\theta_m := -\sum_{k=1}^N a_{k,m}(x) dx_k + dx_m, \ m = N+1, \ldots, n. \tag{12.9}$$

That is, we have

$$df = \sum_{k=1}^N X_k f dx_k + \sum_{m=N+1}^n \frac{\partial f}{\partial x_m} \theta_m. \tag{12.10}$$

It is simple to see that

$$\langle X_s, dx_j \rangle = \frac{\partial}{\partial x_s} dx_j = \delta_{sj}, \ 1 \le s \le N, \ 1 \le j \le n,$$

where δ_{sj} is the Kronecker delta. Moreover, we have

$$\begin{aligned}
\langle X_s, \theta_m \rangle &= \left\langle \frac{\partial}{\partial x_s} + \sum_{g=N+1}^{n} a_{s,g}(x) \frac{\partial}{\partial x_g}, -\sum_{k=1}^{N} a_{k,m}(x) dx_k + dx_m \right\rangle \\
&= -\sum_{k=1}^{N} \left(\frac{\partial}{\partial x_s} a_{k,m}(x) \right) dx_k - \sum_{k=1}^{N} a_{k,m}(x) \frac{\partial}{\partial x_s} dx_k + \frac{\partial}{\partial x_s} dx_m \\
&\quad - \sum_{k=1}^{N} \sum_{g=N+1}^{n} a_{s,g}(x) \left(\frac{\partial}{\partial x_g} a_{k,m}(x) \right) dx_k - \sum_{k=1}^{N} \sum_{g=N+1}^{n} a_{s,g}(x) a_{k,m}(x) \frac{\partial}{\partial x_g} dx_k \\
&\quad + \sum_{g=N+1}^{n} a_{s,g}(x) \frac{\partial}{\partial x_g} dx_m \\
&= -\sum_{k=1}^{N} \left(\frac{\partial}{\partial x_s} a_{k,m}(x) \right) dx_k - \sum_{k=1}^{N} a_{k,m}(x) \delta_{sk} \\
&\quad - \sum_{k=1}^{N} \sum_{g=N+1}^{n} a_{s,g}(x) \left(\frac{\partial}{\partial x_g} a_{k,m}(x) \right) dx_k + \sum_{g=N+1}^{n} a_{s,g}(x) \delta_{gm} \\
&= -\sum_{k=1}^{N} \sum_{g=N+1}^{n} a_{s,g}(x) \left(\frac{\partial}{\partial x_g} a_{k,m}(x) \right) dx_k - \sum_{k=1}^{N} \left(\frac{\partial}{\partial x_s} a_{k,m}(x) \right) dx_k \\
&= -\sum_{k=1}^{N} \left[\sum_{g=N+1}^{n} a_{s,g}(x) \left(\frac{\partial}{\partial x_g} a_{k,m}(x) \right) + \frac{\partial}{\partial x_s} a_{k,m}(x) \right] dx_k.
\end{aligned}$$

That is, we have

$$\langle X_s, dx_j \rangle = \delta_{sj},$$

for $s = 1, \ldots, N,\ j = 1, \ldots, n$, and

$$\langle X_s, \theta_m \rangle = \sum_{k=1}^{N} \mathcal{C}_k(s,m) dx_k,$$

for $s = 1, \ldots, N,\ m = N+1, \ldots, n$, where we denote

$$\mathcal{C}_k(s,m) := -\sum_{g=N+1}^{n} a_{s,g}(x) \frac{\partial}{\partial x_g} a_{k,m}(x) - \frac{\partial}{\partial x_s} a_{k,m}(x).$$

We have

$$d\nu := d\nu(x) = \bigwedge_{j=1}^{N} dx_j = \bigwedge_{j=1}^{N} dx_j \bigwedge_{m=N+1}^{n} dx_m = \bigwedge_{j=1}^{N} dx_j \bigwedge_{m=N+1}^{n} \theta_m,$$

so that

$$\langle X_k, d\nu(x)\rangle = \bigwedge_{j=1, j\neq k}^{N} dx_j \bigwedge_{m=N+1}^{n} \theta_m. \tag{12.11}$$

Therefore, by using formula (12.10) we get

$$\begin{aligned} & d(f_s\langle X_s, d\nu(x)\rangle) \\ &\quad = df_s \wedge \langle X_s, d\nu(x)\rangle \\ &\quad = \sum_{k=1}^{N} X_k f_s dx_k \wedge \langle X_s, d\nu(x)\rangle + \sum_{m=N+1}^{n} \frac{\partial f_s}{\partial x_m}\theta_m \wedge \langle X_s, d\nu(x)\rangle \\ &\quad = \sum_{k=1}^{N} X_k f_s dx_k \wedge \bigwedge_{\substack{j=1,\\ j\neq k}}^{N} dx_j \bigwedge_{m=N+1}^{n} \theta_m + \sum_{m=N+1}^{n} \frac{\partial f_s}{\partial x_m}\theta_m \wedge \bigwedge_{\substack{j=1,\\ j\neq k}}^{N} dx_j \bigwedge_{m=N+1}^{n} \theta_m. \end{aligned}$$

The first term in the last line is equal to $X_s f_s d\nu(x)$ and the second term is zero by the wedge product rules. Therefore, we obtain

$$d(\langle f_s X_s, d\nu(x)\rangle) = X_s f_s d\nu(x), \quad s = 1, \ldots, N. \tag{12.12}$$

Now using the Stokes theorem (see, e.g., [DFN84, Theorem 26.3.1]) we obtain (12.7). Taking a sum over k we also obtain (12.8) for all $f_k \in C^1(\Omega)\bigcap C(\overline{\Omega})$.

As in the classical case, the formula (12.7) is still valid for the fundamental solution of $\mathcal{L}$ since Γ can be estimated by a distance function associated to $\{X_k\}$ (see, e.g., [Man12, Proposition 4.8]), or [FSC86, SC84] for such estimates in a more general setting. □

12.3 Green's identities for sums of squares

Similar to Theorem 1.4.6 the divergence formula in Theorem 12.2.1 implies the corresponding Green identities.

Theorem 12.3.1 (Green's identities)**.** *Let M be a smooth manifold of dimension n with a volume form $d\nu$ and let $\mathcal{L}$ be an operator as in* (12.1)*. Let $\Omega \subset M$ be an admissible domain.*

1. *Green's first identity: If $v \in C^1(\Omega)\bigcap C(\overline{\Omega})$ and $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ then we have*

$$\int_\Omega \left((\widetilde{\nabla} v)u + v\mathcal{L}u\right) d\nu = \int_{\partial\Omega} v\langle \widetilde{\nabla} u, d\nu\rangle, \tag{12.13}$$

where

$$\widetilde{\nabla} u = \sum_{k=1}^{N} (X_k u)\, X_k. \tag{12.14}$$

2. *Green's second identity: If* $u, v \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ *then we have*

$$\int_\Omega (u\mathcal{L}v - v\mathcal{L}u) d\nu = \int_{\partial\Omega} (u\langle\widetilde{\nabla}v, d\nu\rangle - v\langle\widetilde{\nabla}u, d\nu\rangle). \tag{12.15}$$

Moreover, if Ω *is strongly admissible, we can put* $u = \Gamma$ *in* (12.13), *and* $u = \Gamma$ *or* $v = \Gamma$ *in* (12.15).

As in Remark 1.4.7, Part 1, the notation (12.14) implies that for functions u and v we have

$$\left(\widetilde{\nabla}v\right) u = \widetilde{\nabla}vu = \sum_{k=1}^{N} (X_k v)(X_k u) = \sum_{k=1}^{N} X_k v X_k u = \left(\widetilde{\nabla}u\right) v \tag{12.16}$$

is a scalar.

Proof of Theorem 12.3.1. Taking $f_k = vX_k u$, we get

$$\sum_{k=1}^{N} X_k f_k = (\widetilde{\nabla}v)u + v\mathcal{L}u.$$

Since Ω is admissible we can use (12.3), so that we obtain

$$\begin{aligned}
\int_\Omega \left(\widetilde{\nabla}vu + v\mathcal{L}u\right) d\nu &= \int_\Omega \sum_{k=1}^{N} X_k f_k d\nu \\
&= \int_{\partial\Omega} \sum_{k=1}^{N} \langle f_k X_k, d\nu\rangle \\
&= \int_{\partial\Omega} \sum_{k=1}^{N} \langle vX_k u X_k, d\nu\rangle \\
&= \int_{\partial\Omega} v\langle\widetilde{\nabla}u, d\nu\rangle.
\end{aligned}$$

This proves (12.13). Then by rewriting (12.13) for interchanged functions u and v we have

$$\int_\Omega \left((\widetilde{\nabla}u)v + u\mathcal{L}v\right) d\nu = \int_{\partial\Omega} u\langle\widetilde{\nabla}v, d\nu\rangle,$$
$$\int_\Omega \left((\widetilde{\nabla}v)u + v\mathcal{L}u\right) d\nu = \int_{\partial\Omega} v\langle\widetilde{\nabla}u, d\nu\rangle.$$

By subtracting the second identity from the first one and using $(\widetilde{\nabla}u)v = (\widetilde{\nabla}v)u$ in view of (12.16), we obtain (12.15).

If Ω is strongly admissible, we can put Γ for u or v as stated since (12.3) holds in these cases as well. □

Remark 12.3.2. It is crucial that Green's identities are valid for the fundamental solution Γ. In the classical (Euclidean) case Green's identities are valid for the fundamental solution of the Laplacian and this fact is of fundamental importance in the classical theory as well.

12.3.1 Consequences of Green's identities

Let us now record several useful consequences of Theorem 12.3.1. Setting $v = 1$ we obtain the following analogue of Gauss' mean value type formulae:

Corollary 12.3.3 (Gauss' mean value formulae)**.** *Let $\Omega \subset M$ be an admissible domain. Then we have*

$$\mathcal{L}u \geq 0 \text{ in } \Omega \implies \int_{\partial\Omega} \langle \widetilde{\nabla} u, d\nu \rangle \geq 0$$

and

$$\mathcal{L}u \leq 0 \text{ in } \Omega \implies \int_{\partial\Omega} \langle \widetilde{\nabla} u, d\nu \rangle \leq 0.$$

Consequently, we also have

$$\mathcal{L}u = 0 \text{ in } \Omega \implies \int_{\partial\Omega} \langle \widetilde{\nabla} u, d\nu \rangle = 0.$$

Also, for a fixed $x \in \Omega$, taking $v = 1$ and $u(y) = \Gamma(x, y)$ in (12.13) we obtain:

Corollary 12.3.4. *Let $\Omega \subset M$ be a strongly admissible domain such that $\Omega \subset T_y$ for all $y \in \Omega$, and let $x \in \Omega$. Then we have*

$$\int_{\partial\Omega} \langle \widetilde{\nabla} \Gamma(x, y), d\nu(y) \rangle = -1,$$

where $\widetilde{\nabla}\Gamma(x, y) = \widetilde{\nabla}_y \Gamma(x, y)$ refers to the notation (12.14) with derivatives taken with respect to the variable y.

The assumption of $\Omega \subset T_y$ for all $y \in \Omega$ in Corollary 12.3.4 just assures that the family of Γ_y is defined over $y \in \Omega$.

Corollary 12.3.5 (Representation formulae)**.** *Let us assume the conditions of Corollary 12.3.4. Taking v in (12.15) to be the fundamental solution Γ we obtain the following representation formulae.*

1. *Let $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$. Then for all $x \in \Omega$ we have*

$$\begin{aligned} u(x) = & -\int_{\Omega} \Gamma(x, y) \mathcal{L}u(y) d\nu(y) \\ & - \int_{\partial\Omega} u(y) \langle \widetilde{\nabla} \Gamma(x, y), d\nu(y) \rangle + \int_{\partial\Omega} \Gamma(x, y) \langle \widetilde{\nabla} u(y), d\nu(y) \rangle. \end{aligned}$$

2. *Let* $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ *and* $\mathcal{L}u = 0$ *on* Ω*. Then for all* $x \in \Omega$ *we have*

$$u(x) = -\int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\Gamma(x,y), d\nu(y)\rangle + \int_{\partial\Omega} \Gamma(x,y)\langle \widetilde{\nabla}u(y), d\nu(y)\rangle.$$

3. *Let* $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ *and*

$$u(x) = 0, \ x \in \partial\Omega.$$

Then for all $x \in \Omega$ *we have*

$$u(x) = -\int_{\Omega} \Gamma(x,y)\mathcal{L}u(y)d\nu(y) + \int_{\partial\Omega} \Gamma(x,y)\langle \widetilde{\nabla}u(y), d\nu(y)\rangle.$$

4. *Let* $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ *and*

$$\sum_{j=1}^{N} X_j u\langle X_j, d\nu\rangle = 0 \ \ on \ \ \partial\Omega.$$

Then for all $x \in \Omega$ *we have*

$$u(x) = -\int_{\Omega} \Gamma(x,y)\mathcal{L}u(y)d\nu(y) - \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\Gamma(x,y), d\nu(y)\rangle.$$

12.3.2 Differential forms, perimeter and surface measures

In this section we briefly describe the relation between the forms $\langle X_j, d\nu\rangle$, perimeter measure, and the surface measure on the boundary $\partial\Omega$. In this we follow [RS17c] where this topic was discussed in the setting of stratified groups, and we would like to thank Nicola Garofalo and Valentino Magnani for discussions.

Definition 12.3.6 (Perimeter measure)**.** Let $\Omega \subset M$ be an open set with a piecewise smooth boundary. The *perimeter measure* on $\partial\Omega$ is defined by

$$\sigma_H(\partial\Omega) = \sup\left\{\sum_{i=1}^{N}\int_{\partial\Omega} \psi_i\langle X_i, d\nu\rangle : \ \psi = (\psi_1, \ldots, \psi_{N_1}), \ |\psi| \le 1, \ \psi \in C^1\right\}.$$

Then we have the following simple proof of the divergence formula in Theorem 12.2.1.

Proposition 12.3.7 (Divergence formula)**.** *Let* X *be a vector field and let* $\langle X, d\nu\rangle$ *be the contraction of the volume form* $d\nu = dx_1 \wedge \cdots \wedge dx_n$ *by* X*. Then we have*

$$\int_{\Omega} X\varphi \, d\nu = \int_{\partial\Omega} \varphi\langle X, d\nu\rangle. \tag{12.17}$$

Proof of Proposition 12.3.7. Let L_X denote the Lie derivative with respect to the vector field X. The Cartan formula for L_X gives

$$L_X = d\,\imath_X + \imath_X\, d, \qquad \text{where} \qquad \imath_X d\nu = \langle X, d\nu\rangle.$$

Then we have

$$\int_\Omega X\varphi\, d\nu = \int_\Omega \operatorname{div}(\varphi X) d\nu = \int_\Omega L_{\varphi X} d\nu = \int_\Omega d(\imath_{\varphi X} d\nu) = \int_{\partial\Omega} \varphi\langle X, d\nu\rangle,$$

showing (12.17). □

Proposition 12.3.8 (Relation between forms, perimeter and surface measures)**.** *Let $\langle\cdot,\cdot\rangle_E$ denote the Euclidean scalar product. Then the perimeter measure $d\sigma_H$ and the surface measure dS on $\partial\Omega$ are related by*

$$\int_{\partial\Omega} \varphi\langle v, X_j\rangle_E dS = \int_{\partial\Omega} \varphi \frac{\langle v, X_j\rangle_E}{\left(\sum_{j=1}^N \langle v, X_j\rangle_E^2\right)^{\frac{1}{2}}}\, d\sigma_H = \int_{\partial\Omega} \varphi\langle X_j, d\nu\rangle, \tag{12.18}$$

for all outer unit vectors v and all $\varphi \in C^\infty(\partial\Omega)$.

Moreover, if $\mathfrak{g}$ denotes the vector space spanned by $\{X_j\}_{j=1}^N$ and X_j are orthonormal on $\mathfrak{g}$, then for any $f_j \in C^\infty(\partial\Omega)$ we have

$$\int_{\partial\Omega} \sum_{j=1}^N f_j\langle X_j, d\nu\rangle = \int_{\partial\Omega} \langle X, v_H\rangle_{\mathfrak{g}}\, d\sigma_H, \tag{12.19}$$

where $X = \sum_{j=1}^N f_j X_j$ and $v_H = \sum_{j=1}^N \langle v, X_j\rangle_E X_j$.

Proof of Proposition 12.3.8. For an outer unit vector v on $\partial\Omega$ let us write

$$|v_H| := \left(\sum_{j=1}^N \langle v, X_j\rangle_E^2\right)^{1/2} \qquad \text{and} \qquad |v_H|_j := \frac{\langle v, X_j\rangle_E}{|v_H|}.$$

If dS is the surface measure on $\partial\Omega$, we have

$$d\sigma_H = |v_H| dS,$$

and all these relations are well defined because the perimeter measure of the set of characteristic points of a smooth domain Ω is zero. We can now calculate

$$\begin{aligned}\int_\Omega X_j\varphi\, d\nu = \int_\Omega \operatorname{div}(\varphi X_j) d\nu &= \int_{\partial\Omega} \varphi\, \imath_{X_j}(d\nu) = \int_{\partial\Omega} \varphi\langle v, X_j\rangle_E dS \\ &= \int_{\partial\Omega} \varphi\frac{\langle v, X_j\rangle_E}{|v_H|}|v_H| dS = \int_{\partial\Omega} \varphi|v_H|_j\, d\sigma_H,\end{aligned}$$

giving one equality in (12.18). Combining this with (12.17) we obtain

$$\int_{\partial\Omega} \varphi\langle X_j, d\nu\rangle = \int_{\partial\Omega} \varphi|v_H|_j \, d\sigma_H, \tag{12.20}$$

the other equality in (12.18). Let us now assume that X_j are orthonormal on $\mathfrak{g}$, and let

$$X = \sum_{j=1}^{N} f_j X_j.$$

We write

$$v_H = \sum_{j=1}^{N} \langle v, X_j\rangle_E X_j$$

for a vector v with $|v_H| = 1$. Then we have

$$\langle X, v_H\rangle_{\mathfrak{g}} = \sum_{j=1}^{N} f_j |v_H|_j.$$

Now, applying (12.20) with $\varphi = f_j$ and summing over j, we get

$$\int_{\partial\Omega} \sum_{j=1}^{N} f_j \langle X_j, d\nu\rangle = \int_{\partial\Omega} \langle X, v_H\rangle_{\mathfrak{g}} \, d\sigma_H, \quad X = \sum_{j=1}^{N} f_j X_j,$$

which gives (12.19). □

12.4 Local Hardy inequalities

In this section we describe local versions of the Hardy inequality including boundary terms. The weights are formulated in terms of the fundamental solution and the proof relies on Green's first formula from Theorem 12.3.1. As usual, we denote

$$\nabla_X = (X_1, \ldots, X_N).$$

Theorem 12.4.1 (Local Hardy inequality with boundary terms)**.** *Let $y \in M$ be such that (A_y^+) holds with the fundamental solution $\Gamma = \Gamma_y$ in T_y. Let $\Omega \subset T_y$ be a strongly admissible domain such that $y \notin \partial\Omega$. Let $\alpha \in \mathbb{R}$, $\alpha > 2-\beta$, $\beta > 2$ and $R \geq e \sup_\Omega \Gamma^{\frac{1}{2-\beta}}$. Then for all $u \in C^1(\Omega) \bigcap C(\overline{\Omega})$ we have*

$$\begin{aligned} \int_\Omega \Gamma^{\frac{\alpha}{2-\beta}} |\nabla_X u|^2 \, d\nu \geq \left(\frac{\beta+\alpha-2}{2}\right)^2 \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\nu \\ + \frac{\beta+\alpha-2}{2(\beta-2)} \int_{\partial\Omega} \Gamma^{\frac{\alpha}{2-\beta}-1} |u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle, \end{aligned} \tag{12.21}$$

as well as its further refinement

$$\begin{aligned}\int_{\Omega}\Gamma^{\frac{\alpha}{2-\beta}}|\nabla_X u|^2\,d\nu \geq \left(\frac{\beta+\alpha-2}{2}\right)^2\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu \\ +\frac{1}{4}\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}|u|^2d\nu \\ +\frac{1}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha}{2-\beta}-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle \\ +\frac{\beta+\alpha-2}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle.\end{aligned} \tag{12.22}$$

Remark 12.4.2.

1. If $u=0$ on the boundary $\partial\Omega$, for example when $\operatorname{supp} u\subset\Omega$, then (12.21) can be regarded as a usual Hardy inequality (without boundary term). Inequality (12.22) can be regarded as a further refinement of (12.21) since it includes further positive interior terms as well as further boundary terms.
2. Even if $y\in\partial\Omega$, the estimates (12.21) and (12.22) of Theorem 12.4.1 remain true if $y\notin\partial\Omega\cap\operatorname{supp} u$.
3. In (12.21) the boundary term can be positive, see Remark 11.4.2, Part 2, i.e., we sometimes have

$$\frac{\beta+\alpha-2}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\geq 0, \tag{12.23}$$

for some u.
4. In the setting of Example (E1), i.e., when M is a stratified group, and X_j's are the vectors from the first stratum, then (12.21) is equivalent to (11.73) in Theorem 11.4.1, where this inequality was expressed in terms of the $\mathcal{L}$-gauge d, taking $\beta=Q\geq 3$, and $d(x)=\Gamma(x,0)^{\frac{1}{2-Q}}$, where Q is the homogeneous dimension of the group. For example, with $\alpha=0$ we get

$$\begin{aligned}\int_{\Omega}|\nabla_X u|^2\,d\nu \geq \left(\frac{Q-2}{2}\right)^2\int_{\Omega}\frac{|\nabla_X d|^2}{d^2}|u|^2\,d\nu \\ +\frac{1}{2}\int_{\partial\Omega}d^{Q-2}|u|^2\langle\widetilde{\nabla}d^{2-Q},d\nu\rangle,\end{aligned} \tag{12.24}$$

with the sharp constant $\left(\frac{Q-2}{2}\right)^2$.

Proof of Theorem 12.4.1. In the proof and in the subsequent analysis we follow [RS17d].

First, let us prove (12.21). By an argument of Remark 2.1.2, Part 3, we can assume that u is real-valued. In this case, recalling that

$$(\widetilde{\nabla}u)u = \sum_{k=1}^{N}(X_k u)X_k u = |\nabla_X u|^2,$$

inequality (12.21) reduces to

$$\begin{aligned}\int_\Omega \Gamma^{\frac{\alpha}{2-\beta}}(\widetilde{\nabla}u)u\,d\nu \geq \left(\frac{\beta+\alpha-2}{2}\right)^2 \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}(\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}}u^2\,d\nu \\ + \frac{\beta+\alpha-2}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha}{2-\beta}-1}u^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle,\end{aligned} \tag{12.25}$$

which we will now prove. Setting

$$u = d^\gamma q \tag{12.26}$$

for some real-valued functions $d > 0$, q, and a constant $\gamma \neq 0$ to be chosen later, we have

$$\begin{aligned}(\widetilde{\nabla}u)u &= (\widetilde{\nabla}d^\gamma q)d^\gamma q \\ &= \sum_{k=1}^{N} X_k(d^\gamma q)X_k(d^\gamma q) \\ &= \gamma^2 d^{2\gamma-2}\sum_{k=1}^{N}(X_k d)^2 q^2 + 2\gamma d^{2\gamma-1}q\sum_{k=1}^{N}X_k d\,X_k q + d^{2\gamma}\sum_{k=1}^{N}(X_k q)^2 \\ &= \gamma^2 d^{2\gamma-2}((\widetilde{\nabla}d)d)q^2 + 2\gamma d^{2\gamma-1}q(\widetilde{\nabla}d)q + d^{2\gamma}(\widetilde{\nabla}q)q.\end{aligned}$$

Multiplying both sides of this equality by d^α and applying Green's first formula from Theorem 12.3.1 to the second term in the last line we observe that

$$\begin{aligned}2\gamma\int_\Omega d^{\alpha+2\gamma-1}q(\widetilde{\nabla}d)q d\nu &= \frac{\gamma}{\alpha+2\gamma}\int_\Omega(\widetilde{\nabla}d^{\alpha+2\gamma})q^2 d\nu = \frac{\gamma}{\alpha+2\gamma}\int_\Omega(\widetilde{\nabla}q^2)d^{\alpha+2\gamma}d\nu \\ &= -\frac{\gamma}{\alpha+2\gamma}\int_\Omega q^2\mathcal{L}d^{\alpha+2\gamma}d\nu + \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega}q^2\langle\widetilde{\nabla}d^{\alpha+2\gamma}, d\nu\rangle,\end{aligned}$$

where we note that later on we will choose γ so that $d^{\alpha+2\gamma} = \Gamma$, and hence Theorem 12.3.1 is applicable. Consequently, we get

$$\begin{aligned}\int_\Omega d^\alpha(\widetilde{\nabla}u)u d\nu &= \gamma^2\int_\Omega d^{\alpha+2\gamma-2}((\widetilde{\nabla}d)d)\,q^2 d\nu + \frac{\gamma}{\alpha+2\gamma}\int_\Omega(\widetilde{\nabla}d^{\alpha+2\gamma})q^2 d\nu \\ &\quad + \int_\Omega d^{\alpha+2\gamma}(\widetilde{\nabla}q)q d\nu \\ &= \gamma^2\int_\Omega d^{\alpha+2\gamma-2}((\widetilde{\nabla}d)d)\,q^2 d\nu + \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega}q^2\langle\widetilde{\nabla}d^{\alpha+2\gamma}, d\nu\rangle \\ &\quad - \frac{\gamma}{\alpha+2\gamma}\int_\Omega q^2\mathcal{L}d^{\alpha+2\gamma}d\nu + \int_\Omega d^{\alpha+2\gamma}(\widetilde{\nabla}q)q d\nu\end{aligned}$$

$$\geq \gamma^2 \int_\Omega d^{\alpha+2\gamma-2}((\widetilde{\nabla} d)d)\, q^2 d\nu + \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega} q^2 \langle \widetilde{\nabla} d^{\alpha+2\gamma}, d\nu\rangle$$
$$- \frac{\gamma}{\alpha+2\gamma}\int_\Omega q^2 \mathcal{L} d^{\alpha+2\gamma} d\nu, \qquad (12.27)$$

since $d>0$ and $(\widetilde{\nabla} q)q = |\nabla_X q|^2 \geq 0$. On the other hand, it can be readily checked that for a vector field X we have

$$\frac{\gamma}{\alpha+2\gamma} X^2(d^{\alpha+2\gamma}) = \gamma X(d^{\alpha+2\gamma-1}Xd) = \frac{\gamma}{2-\beta} X(d^{\alpha+2\gamma+\beta-2}X(d^{2-\beta}))$$
$$= \frac{\gamma}{2-\beta}(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma+\beta-3}(Xd)X(d^{2-\beta}) + \frac{\gamma}{2-\beta} d^{\alpha+2\gamma+\beta-2}X^2(d^{2-\beta})$$
$$= \gamma(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma-2}(Xd)^2 + \frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}X^2(d^{2-\beta}).$$

Consequently, we get the equality

$$-\frac{\gamma}{\alpha+2\gamma}\mathcal{L}d^{\alpha+2\gamma} = -\gamma(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma-2}(\widetilde{\nabla} d)d - \frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}\mathcal{L}d^{2-\beta}. \qquad (12.28)$$

Since $q^2 = d^{-2\gamma}u^2$ in view of (12.26), substituting (12.28) into (12.27) we obtain

$$\int_\Omega d^\alpha (\widetilde{\nabla} u)u d\nu \geq (-\gamma^2 - \gamma(\alpha+\beta-2))\int_\Omega d^{\alpha-2}((\widetilde{\nabla} d)d)u^2 d\nu$$
$$-\frac{\gamma}{2-\beta}\int_\Omega (\mathcal{L}d^{2-\beta})d^{\alpha+\beta-2}u^2 dx + \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega} d^{-2\gamma}u^2\langle \widetilde{\nabla} d^{\alpha+2\gamma}, d\nu\rangle.$$

Taking $d = \Gamma^{\frac{1}{2-\beta}},\ \beta>2$, concerning the second term we observe that for $\alpha > 2-\beta$ and $\beta > 2$ we have

$$\int_\Omega (\mathcal{L}\Gamma)\Gamma^{\frac{\alpha+\beta-2}{2-\beta}}u^2 dx = 0, \qquad (12.29)$$

since $\Gamma = \Gamma_y$ is the fundamental solution to $\mathcal{L}$. Indeed, the above equality is clear when y is outside of Ω. If y belongs to Ω we have

$$\int_\Omega (\mathcal{L}\Gamma)\Gamma^{\frac{\alpha+\beta-2}{2-\beta}}u^2 dx = \Gamma^{\frac{\alpha+\beta-2}{2-\beta}}(y)u^2(y) = 0,$$

since conditions $\alpha > 2-\beta$ and $\beta > 2$ imply that $\frac{\alpha+\beta-2}{2-\beta} < 0$, and since $\frac{1}{\Gamma}(y) = 0$ by (A_y^+). Thus, with $d = \Gamma^{\frac{1}{2-\beta}},\ \beta > 2$, we get

$$\int_\Omega \Gamma^{\frac{\alpha}{2-\beta}}(\widetilde{\nabla} u)u\, d\nu \geq (-\gamma^2 - \gamma(\alpha+\beta-2))\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}(\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}}u^2\, d\nu$$
$$+ \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega}\Gamma^{-\frac{2\gamma}{2-\beta}}u^2\langle \widetilde{\nabla}\Gamma^{\frac{\alpha+2\gamma}{2-\beta}}, d\nu\rangle.$$

Taking $\gamma = \frac{2-\beta-\alpha}{2}$, we obtain (12.25). Finally, we note that with this γ, we have $d^{\alpha+2\gamma} = \Gamma$, so that the use of Theorem 12.3.1 is justified.

Let us now prove (12.22), with the proof similar to the above proof of (12.21). Recalling that

$$(\widetilde{\nabla} u)u = \sum_{k=1}^{N} (X_k u) X_k u = |\nabla_X u|^2,$$

inequality (12.22) reduces to

$$\begin{aligned}\int_\Omega \Gamma^{\frac{\alpha}{2-\beta}} (\widetilde{\nabla} u)u\, d\nu \geq \left(\frac{\beta+\alpha-2}{2}\right)^2 \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}} (\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}} u^2\, d\nu \\ + \frac{1}{4}\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}} (\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2} u^2 d\nu \\ + \frac{1}{2(\beta-2)}\int_{\partial\Omega} \Gamma^{\frac{\alpha}{2-\beta}-1} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1} u^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle \\ + \frac{\beta+\alpha-2}{2(\beta-2)} \int_{\partial\Omega} \Gamma^{\frac{\alpha}{2-\beta}-1} u^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle,\end{aligned} \tag{12.30}$$

which we will now prove. Let us recall the first part of (12.27) as

$$\begin{aligned}&\int_\Omega d^\alpha (\widetilde{\nabla} u) u d\nu \\ &= \gamma^2 \int_\Omega d^{\alpha+2\gamma-2}((\widetilde{\nabla} d)d)\, q^2 d\nu + \frac{\gamma}{\alpha+2\gamma}\int_\Omega (\widetilde{\nabla} d^{\alpha+2\gamma}) q^2 d\nu + \int_\Omega d^{\alpha+2\gamma}(\widetilde{\nabla} q) q d\nu \\ &= \gamma^2 \int_\Omega d^{\alpha+2\gamma-2}((\widetilde{\nabla} d)d)\, q^2 d\nu + \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega} q^2 \langle \widetilde{\nabla} d^{\alpha+2\gamma}, d\nu\rangle \\ &\quad - \frac{\gamma}{\alpha+2\gamma}\int_\Omega q^2 \mathcal{L} d^{\alpha+2\gamma} d\nu + \int_\Omega d^{\alpha+2\gamma}(\widetilde{\nabla} q) q d\nu.\end{aligned} \tag{12.31}$$

Since $q^2 = d^{-2\gamma} u^2$, substituting (12.28) into (12.31) we obtain

$$\begin{aligned}\int_\Omega d^\alpha (\widetilde{\nabla} u) u d\nu = (-\gamma^2 - \gamma(\alpha+\beta-2)) \int_\Omega d^{\alpha-2}((\widetilde{\nabla} d)d) u^2 d\nu \\ - \frac{\gamma}{2-\beta}\int_\Omega (\mathcal{L} d^{2-\beta}) d^{\alpha+\beta-2} u^2 dx \\ + \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega} d^{-2\gamma} u^2 \langle \widetilde{\nabla} d^{\alpha+2\gamma}, d\nu\rangle + \int_\Omega d^{\alpha+2\gamma}(\widetilde{\nabla} q) q d\nu.\end{aligned}$$

Using (12.29), with $d = \Gamma^{\frac{1}{2-\beta}}$, $\beta > 2$, we obtain

$$\begin{aligned}\int_\Omega \Gamma^{\frac{\alpha}{2-\beta}} (\widetilde{\nabla} u) u\, d\nu = (-\gamma^2 - \gamma(\alpha+\beta-2)) \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}} (\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}} u^2\, d\nu \\ + \frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega} \Gamma^{-\frac{2\gamma}{2-\beta}} u^2 \langle \widetilde{\nabla}\Gamma^{\frac{\alpha+2\gamma}{2-\beta}}, d\nu\rangle + \int_\Omega d^{\alpha+2\gamma} (\widetilde{\nabla} q) q d\nu.\end{aligned}$$

Taking $\gamma = \frac{2-\beta-\alpha}{2}$ we obtain

$$\int_{\Omega} \Gamma^{\frac{\alpha}{2-\beta}} (\widetilde{\nabla} u) u \, d\nu = \left(\frac{\beta+\alpha-2}{2} \right)^2 \int_{\Omega} \Gamma^{\frac{\alpha-2}{2-\beta}} (\widetilde{\nabla} \Gamma^{\frac{1}{2-\beta}}) \Gamma^{\frac{1}{2-\beta}} u^2 \, d\nu + \frac{\beta+\alpha-2}{2(\beta-2)} \int_{\partial\Omega} \Gamma^{\frac{\alpha}{2-\beta}-1} u^2 \langle \widetilde{\nabla} \Gamma, d\nu \rangle + \int_{\Omega} \Gamma (\widetilde{\nabla} q) q d\nu. \tag{12.32}$$

Let us now take

$$q = \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{1/2} \varphi,$$

that is,

$$\varphi = \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{-\frac{1}{2}} \Gamma^{-\frac{2-\beta-\alpha}{2(2-\beta)}} u.$$

A straightforward computation shows that

$$\begin{aligned}
\int_{\Omega} \Gamma (\widetilde{\nabla} q) q d\nu &= \sum_{j=1}^{N} \int_{\Omega} \Gamma \left(X_j \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{\frac{1}{2}} \varphi + \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{\frac{1}{2}} X_j \varphi \right)^2 d\nu \\
&= \frac{1}{4} \int_{\Omega} \Gamma^{\frac{-\beta}{2-\beta}} (\widetilde{\nabla} \Gamma^{\frac{1}{2-\beta}}) \Gamma^{\frac{1}{2-\beta}} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{-1} \varphi^2 d\nu \\
&\quad - \int_{\Omega} \Gamma^{1-\frac{1}{2-\beta}} \varphi (\widetilde{\nabla} \Gamma^{\frac{1}{2-\beta}}) \varphi d\nu + \int_{\Omega} \Gamma \ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} (\widetilde{\nabla} \varphi) \varphi d\nu \\
&= \frac{1}{4} \int_{\Omega} \Gamma^{\frac{-\beta}{2-\beta}} (\widetilde{\nabla} \Gamma^{\frac{1}{2-\beta}}) \Gamma^{\frac{1}{2-\beta}} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{-1} \varphi^2 d\nu \\
&\quad + \frac{1}{2(\beta-2)} \int_{\Omega} (\widetilde{\nabla} \Gamma) \varphi^2 d\nu + \int_{\Omega} \Gamma \ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} (\widetilde{\nabla} \varphi) \varphi d\nu \\
&= \frac{1}{4} \int_{\Omega} \Gamma^{\frac{-\beta}{2-\beta}} (\widetilde{\nabla} \Gamma^{\frac{1}{2-\beta}}) \Gamma^{\frac{1}{2-\beta}} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{-1} \varphi^2 d\nu \\
&\quad + \frac{1}{2(\beta-2)} \int_{\Omega} \mathcal{L} \Gamma \, \varphi^2 d\nu + \frac{1}{2(\beta-2)} \int_{\partial\Omega} \varphi^2 \langle \widetilde{\nabla} \Gamma, d\nu \rangle \\
&\quad + \int_{\Omega} \Gamma \ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} (\widetilde{\nabla} \varphi) \varphi d\nu.
\end{aligned} \tag{12.33}$$

Since the second integral term of the right-hand side vanishes and the last integral term is positive from (12.33) we obtain that

$$\int_{\Omega} \Gamma (\widetilde{\nabla} q) q d\nu \geq \frac{1}{4} \int_{\Omega} \Gamma^{\frac{-\beta}{2-\beta}} (\widetilde{\nabla} \Gamma^{\frac{1}{2-\beta}}) \Gamma^{\frac{1}{2-\beta}} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}} \right)^{-1} \varphi^2 d\nu + \frac{1}{2(\beta-2)} \int_{\partial\Omega} \varphi^2 \langle \widetilde{\nabla} \Gamma, d\nu \rangle$$

$$= \frac{1}{4}\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}(\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}u^2 d\nu$$
$$+\frac{1}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha}{2-\beta}-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}u^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle. \tag{12.34}$$

Finally, (12.32) and (12.34) imply (12.30). □

12.5 Anisotropic Hardy inequalities via Picone identities

In this section we discuss the anisotropic versions of local Hardy inequalities for general (real-valued) vector fields as in the previous sections. As in the most of this chapter, such weighted anisotropic Hardy type inequalities will also include the boundary terms, which of course disappear if one works with functions supported in the interior of the considered domain. The analysis is based on the anisotropic Picone type identities, analogous to those described in Section 6.10.1. As consequences, we also recover some of the Hardy type inequalities of the Euclidean space described earlier in the setting of the stratified groups. The presentation of this section is based on [RSS18c].

Throughout this and further sections, let M be a smooth manifold of dimension n equipped with a volume form $d\nu$, and let $\{X_k\}_{k=1}^N$, $N \le n$, be a family of real vector fields.

We start with the following weighted anisotropic Hardy type inequalities in admissible domains in the sense of Definition 12.1.1.

Theorem 12.5.1 (Weighted anisotropic Hardy type inequality)**.** *Let $\Omega \subset M$ be an admissible domain. Let $W_i(x)$, $H_i(x)$ be non-negative functions for $i = 1, \ldots, N$, such that for $v \in C^1(\Omega)\bigcap C(\overline{\Omega})$ satisfying $v > 0$ a.e. in Ω, we have*

$$-X_i(W_i(x)|X_i v|^{p_i-2}X_i v) \ge H_i(x)v^{p_i-1}, \quad i = 1, \ldots, N. \tag{12.35}$$

Then, for all non-negative functions $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ and the positive function $v \in C^1(\Omega)\bigcap C(\overline{\Omega})$ satisfying (12.35), we have

$$\sum_{i=1}^N\int_{\Omega}W_i(x)|X_i u|^{p_i}d\nu \ge \sum_{i=1}^N\int_{\Omega}H_i(x)|u|^{p_i}d\nu \tag{12.36}$$
$$+\sum_{i=1}^N\int_{\partial\Omega}\frac{u^{p_i}}{v^{p_i-1}}\langle\widetilde{\nabla}_i\left(W_i(x)|X_i v|^{p_i-2}X_i v\right), d\nu\rangle,$$

where $\widetilde{\nabla}_i f = X_i f X_i$ and $p_i > 1$, for $i = 1, \ldots, N$.

Before proving this inequality let us formulate several of its consequences, recovering and extending a number of known results, see Remark 12.5.3. In these examples of the weighted anisotropic Hardy type inequalities on M we express the weights in terms of the fundamental solution $\Gamma = \Gamma_y(x)$ in the assumption A_y. For brevity, we can just denote it by Γ, if we fix some $y \in M$ and the corresponding T_y and Γ_y.

Corollary 12.5.2 (Anisotropic Hardy inequalities and fundamental solutions)**.** *Let $\Omega \subset M$ be an admissible domain. Then we have the following estimates.*

(1) *Let $\alpha \in \mathbb{R}$, $1 < p_i < \beta + \alpha$, $i = 1, \ldots, N$, and $\gamma > -1, \beta > 2$. Then for all $u \in C_0^\infty(\Omega \backslash \{0\})$ we have*

$$\sum_{i=1}^{N} \int_{\Omega} \Gamma^{\frac{\alpha}{2-\beta}} |X_i \Gamma^{\frac{1}{2-\beta}}|^{\gamma} |X_i u|^{p_i} d\nu \geq \sum_{i=1}^{N} \left(\frac{\beta + \alpha - p_i}{p_i} \right)^{p_i} \int_{\Omega} \Gamma^{\frac{\alpha - p_i}{2-\beta}} |X_i \Gamma^{\frac{1}{2-\beta}}|^{p_i + \gamma} |u|^{p_i} d\nu. \tag{12.37}$$

(2) *Let $\alpha, \gamma \in \mathbb{R}$ and $\alpha \neq 0, \beta > 2$. Then for any $u \in C_0^1(\Omega)$ we have*

$$\sum_{i=1}^{N} \int_{\Omega} \Gamma^{\frac{\gamma + p_i}{2-\beta}} |X_i u|^{p_i} d\nu \geq \sum_{i=1}^{N} C_i(\alpha, \gamma, p_i)^{p_i} \int_{\Omega} \Gamma^{\frac{\gamma}{2-\beta}} |X_i \Gamma^{\frac{1}{2-\beta}}|^{p_i} |u|^{p_i} d\nu, \tag{12.38}$$

where $C_i(\alpha, \gamma, p_i) := \frac{(\alpha-1)(p_i-1)-\gamma-1}{p_i}$, $p_i > 1$, and $i = 1, \ldots, N$.

(3) *Let $\alpha \in \mathbb{R}, \beta > 2$, $1 < p_i < \beta + \alpha$ for $i = 1, \ldots, N$. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$\sum_{i=1}^{N} \int_{\Omega} \Gamma^{\frac{\alpha}{2-\beta}} |X_i u|^{p_i} d\nu \geq \sum_{i=1}^{N} C_i(\beta, \alpha, p_i) \int_{\Omega} \Gamma^{\frac{\alpha}{2-\beta}} \frac{|X_i \Gamma^{\frac{1}{2-\beta}}|^{p_i}}{\left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{p_i}} |u|^{p_i} d\nu, \tag{12.39}$$

where $C_i(\beta, \alpha, p_i) := \left(\frac{\beta+\alpha-p_i}{p_i-1}\right)^{p_i-1} (\beta + \alpha)$.

(4) *Let $\alpha \in \mathbb{R}, \beta > 2$, $1 < p_i < \beta + \alpha$ for $i = 1, \ldots, N$. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$\sum_{i=1}^{N} \int_{\Omega} \left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{\alpha(p_i-1)} |X_i u|^{p_i} d\nu \geq \sum_{i=1}^{N} C_i(\beta, p_i, \alpha) \int_{\Omega} \frac{|X_i \Gamma^{\frac{1}{2-\beta}}|^{p_i}}{\left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{(1-p_i)(1-\alpha)}} |u|^{p_i} d\nu, \tag{12.40}$$

where $C_i(\beta, p_i, \alpha) := \beta \left(\frac{p_i(\alpha-1)}{p_i-1}\right)^{p_i-1}$.

(5) *Let* $\beta > 2$, $a, b > 0$ *and* $\alpha, \gamma, m \in \mathbb{R}$. *If* $\alpha\gamma > 0$ *and* $m \leq \frac{\beta-2}{2}$. *Then for all* $u \in C_0^\infty(\Omega)$ *we have*

$$\begin{aligned}\int_\Omega \frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^\gamma}{\Gamma^{\frac{2m}{2-\beta}}}|\nabla_X u|^2 d\nu \\ \geq C(\beta,m)^2 \int_\Omega \frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^\gamma}{\Gamma^{\frac{2m+2}{2-\beta}}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu \\ + C(\beta,m)\alpha\gamma b\int_\Omega \frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^{\gamma-1}}{\Gamma^{\frac{2m-\alpha+2}{2-\beta}}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu, \end{aligned} \tag{12.41}$$

where $C(\beta, m) := \frac{\beta-2m-2}{2}$ *and* $\nabla_X = (X_1, \ldots, X_N)$.

Remark 12.5.3.

1. In Theorem 12.5.1, if u vanishes on the boundary $\partial\Omega$ and if $p_i = p$, then we have the two-weighted Hardy type inequalities for general vector fields of the form

$$\int_\Omega W(x)|\nabla_X u|^p d\nu \geq \int_\Omega H(x)|u|^p d\nu, \tag{12.42}$$

   where $\nabla_X := (X_1, \ldots, X_N)$.

2. Inequality (12.37) is an analogue of the result of Wang and Niu [WN08], but now for general vector fields. Also, by taking $\gamma = 0$ and $p_i = 2$ we have the following inequality

$$\int_\Omega \Gamma^{\frac{\alpha}{2-\beta}}|\nabla_X u|^2 d\nu \geq \sum_{i=1}^N \left(\frac{\beta+\alpha-2}{2}\right)^2 \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu, \tag{12.43}$$

   for all $u \in C_0^\infty(\Omega)$ and where $\nabla_X = (X_1, \ldots, X_N)$, which gives (12.21) without the boundary term.

3. Inequality (12.38) recovers the result of D'Ambrosio in [D'A05, Theorem 2.7].
4. A Carnot group version of inequality (12.39) was established by Goldstein, Kombe and Yener in [GKY17].
5. The Carnot and Euclidean versions of inequality (12.40) were established in [GKY17] and [Skr13], respectively.
6. The Carnot and Euclidean versions of inequality (12.41) were established in [GKY17] and [GM11], respectively.

As in the setting of stratified groups let us first present the anisotropic Picone type identity, now for general vector fields.

Lemma 12.5.4 (Anisotropic Picone identity for general vector fields)**.** *Let* $\Omega \subset M$ *be an open set. Let* u, v *be differentiable a.e. in* Ω, $v > 0$ *a.e. in* Ω *and* $u \geq 0$.

Define

$$R(u,v) := \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right) |X_i v|^{p_i-2} X_i v, \tag{12.44}$$

$$\begin{aligned} L(u,v) := \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u \\ + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i}, \end{aligned} \tag{12.45}$$

where $p_i > 1$, $i = 1, \ldots, N$. Then

$$L(u,v) = R(u,v) \geq 0. \tag{12.46}$$

In addition, we have $L(u,v) = 0$ a.e. in Ω if and only if $u = cv$ a.e. in Ω with a positive constant $c > 0$.

Proof of Lemma 12.5.4. A direct calculation yields

$$\begin{aligned} R(u,v) &= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right) |X_i v|^{p_i-2} X_i v \\ &= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i} \\ &= L(u,v), \end{aligned}$$

which gives the equality in (12.46). Now we restate $L(u,v)$ in a different form, with the aim to show that $L(u,v) \geq 0$. Thus, we write

$$\begin{aligned} L(u,v) = & \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-1} |X_i u| + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i} \\ & + \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} \left(|X_i v||X_i u| - X_i v X_i u\right) \\ = & \; S_1 + S_2, \end{aligned}$$

where

$$\begin{aligned} S_1 := & \sum_{i=1}^{N} p_i \left[\frac{1}{p_i} |X_i u|^{p_i} + \frac{p_i - 1}{p_i} \left(\left(\frac{u}{v} |X_i v|\right)^{p_i-1}\right)^{\frac{p_i}{p_i-1}}\right] \\ & - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-1} |X_i u|, \end{aligned}$$

and

$$S_2 := \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} \left(|X_i v||X_i u| - X_i v X_i u\right).$$

Since

$$|X_i v||X_i u| \geq X_i v X_i u,$$

we have $S_2 \geq 0$. To check that $S_1 \geq 0$ we will use Young's inequality for $a \geq 0$ and $b \geq 0$ stating that

$$ab \leq \frac{a^{p_i}}{p_i} + \frac{b^{q_i}}{q_i}, \tag{12.47}$$

where $p_i > 1, q_i > 1$, and $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1, \ldots, N$. The equality in (12.47) holds if and only if $a^{p_i} = b^{q_i}$, i.e., if $a = b^{\frac{1}{p_i-1}}$. By setting

$$a := |X_i u| \ \text{ and } \ b := \left(\frac{u}{v}|X_i v|\right)^{p_i-1}$$

in (12.47), we get

$$p_i |X_i u| \left(\frac{u}{v}|X_i v|\right)^{p_i-1} \leq p_i \left[\frac{1}{p_i}|X_i u|^{p_i} + \frac{p_i-1}{p_i}\left(\left(\frac{u}{v}|X_i v|\right)^{p_i-1}\right)^{\frac{p_i}{p_i-1}}\right]. \tag{12.48}$$

This implies $S_1 \geq 0$ which proves that $L(u,v) = S_1 + S_2 \geq 0$.

It is straightforward to see that $u = cv$ implies $R(u,v) = 0$.

Now let us show that $L(u,v) = 0$ implies $u = cv$. Due to $u(x) \geq 0$ and $L(u,v)(x_0) = 0$, $x_0 \in \Omega$, we consider the two cases $u(x_0) > 0$ and $u(x_0) = 0$. For the case $u(x_0) > 0$ we conclude from $L(u,v)(x_0) = 0$ that $S_1 = 0$ and $S_2 = 0$. Then $S_1 = 0$ yields

$$|X_i u| = \frac{u}{v}|X_i v|, \quad i = 1, \ldots, N, \tag{12.49}$$

and $S_2 = 0$ implies

$$|X_i v||X_i u| - X_i v X_i u = 0, \quad i = 1, \ldots, N. \tag{12.50}$$

The combination of (12.49) and (12.50) gives

$$\frac{X_i u}{X_i v} = \frac{u}{v} = c, \quad \text{with} \quad c \neq 0, \quad i = 1, \ldots, N. \tag{12.51}$$

Let us denote

$$\Omega^* := \{x \in \Omega : \ u(x) = 0\}.$$

If $\Omega^* \neq \Omega$, then suppose that $x_0 \in \partial\Omega^*$. So there exists a sequence $x_k \notin \Omega^*$ such that $x_k \to x_0$. In particular, $u(x_k) \neq 0$, and hence by the first case we have $u(x_k) = cv(x_k)$. Passing to the limit we get $u(x_0) = cv(x_0)$. Since $u(x_0) = 0$ and $v(x_0) \neq 0$, we get that $c = 0$. But then by the first case again, since $u = cv$ and $u \neq 0$ in $\Omega\backslash\Omega^*$, it is impossible that $c = 0$. This contradiction implies that $\Omega^* = \Omega$. It completes the proof of Lemma 12.5.4. ☐

The established anisotropic Picone identity can be used to prove Theorem 12.5.1.

Proof of Theorem 12.5.1. In the following calculation, we will use the following properties: anisotropic Picone type identity (12.46), then we apply the divergence theorem and the hypothesis (12.35), respectively, finally yielding (12.36). Thus, we obtain

$$
\begin{aligned}
0 &\le \int_\Omega \sum_{i=1}^N W_i(x) L(u,v) d\nu = \int_\Omega \sum_{i=1}^N W_i(x) R(u,v) d\nu \\
&= \sum_{i=1}^N \int_\Omega W_i(x)|X_i u|^{p_i} d\nu - \sum_{i=1}^N \int_\Omega X_i\left(\frac{u^{p_i}}{v^{p_i-1}}\right) W_i(x)|X_i v|^{p_i-2} X_i v d\nu \\
&= \sum_{i=1}^N \int_\Omega W_i(x)|X_i u|^{p_i} d\nu + \sum_{i=1}^N \int_\Omega \frac{u^{p_i}}{v^{p_i-1}} X_i\left(W_i(x)|X_i v|^{p_i-2} X_i v\right) d\nu \\
&\quad - \sum_{i=1}^N \int_{\partial\Omega} \frac{u^{p_i}}{v^{p_i-1}} \langle \widetilde{\nabla}_i \left(W_i(x)|X_i v|^{p_i-2} X_i v\right), d\nu\rangle \\
&\le \sum_{i=1}^N \int_\Omega W_i(x)|X_i u|^{p_i} d\nu - \sum_{i=1}^N \int_\Omega H_i(x) u^{p_i} d\nu \\
&\quad - \sum_{i=1}^N \int_{\partial\Omega} \frac{u^{p_i}}{v^{p_i-1}} \langle \widetilde{\nabla}_i \left(W_i(x)|X_i v|^{p_i-2} X_i v\right), d\nu\rangle,
\end{aligned}
$$

where $\widetilde{\nabla}_i f = X_i f X_i$. The proof of Theorem 12.5.1 is complete. □

Finally, we prove Corollary 12.5.2.

Proof of Corollary 12.5.2. Part (1). Consider the functions W_i and v such that

$$
W_i = d^\alpha |X_i d|^\gamma \text{ and } v = \Gamma^{\frac{\psi}{2-\beta}} = d^\psi, \tag{12.52}
$$

where, to abbreviate the calculation, we denote

$$
d := \Gamma^{\frac{1}{2-\beta}} \quad \text{and} \quad \psi := -\frac{\beta+\alpha-p_i}{p_i}.
$$

Now we plug (12.52) in (12.35) to determine the candidate for the function H_i. For this, we first prepare several calculations. We can readily find

$$
\begin{aligned}
X_i v &= \psi d^{\psi-1} X_i d, \\
|X_i v|^{p_i-2} &= |\psi|^{p_i-2} d^{(\psi-1)(p_i-2)} |X_i d|^{p_i-2}, \\
W_i |X_i v|^{p_i-2} X_i v &= |\psi|^{p_i-2} \psi d^{\alpha+(\psi-1)(p_i-1)} |X_i d|^{\gamma+p_i-2} X_i d.
\end{aligned}
$$

Also, we get

$$\begin{aligned}\sum_{i=1}^{N} X_i^2 d^\alpha &= \sum_{i=1}^{N} X_i(X_i\Gamma^{\frac{\alpha}{2-\beta}}) = \sum_{i=1}^{N} X_i\left(\frac{\alpha}{2-\beta}\Gamma^{\frac{\alpha+\beta-2}{2-\beta}}X_i\Gamma\right)\\ &= \frac{\alpha(\alpha+\beta-2)}{(2-\beta)^2}\Gamma^{\frac{\alpha+2\beta-4}{2-\beta}}\sum_{i=1}^{N}|X_i\Gamma|^2 + \frac{\alpha}{2-\beta}\Gamma^{\frac{\alpha+\beta-2}{2-\beta}}\sum_{i=1}^{N}X_i^2\Gamma\\ &= \frac{\alpha(\alpha+\beta-2)}{(2-\beta)^2}d^{\alpha+2\beta-4}\sum_{i=1}^{N}|X_i d^{2-\beta}|^2\\ &= \alpha(\alpha+\beta-2)d^{\alpha-2}\sum_{i=1}^{N}|X_i d|^2. \end{aligned} \tag{12.53}$$

We observe that $\sum_{i=1}^{N} X_i^2\Gamma = 0$ outside y, since $\Gamma = \Gamma_y$ is the fundamental solution for $\mathcal{L}$. Also, we have

$$\begin{aligned} X_i|X_i d|^\gamma &= X_i((X_i d)^2)^{\gamma/2} = \gamma|X_i d|^{\gamma-2}X_i d X_i^2 d\\ &= \gamma(\beta-1)d^{-1}|X_i d|^\gamma X_i d. \end{aligned} \tag{12.54}$$

In the last line, we have used (12.53) with $\alpha = 1$. Using (12.53) and (12.54), we compute

$$\begin{aligned} &X_i(W_i|X_i v|^{p_i-2}X_i v)\\ &= |\psi|^{p_i-2}\psi X_i\left(d^{\alpha+(\psi-1)(p_i-1)}|X_i d|^{\gamma+p_i-2}X_i d\right)\\ &= |\psi|^{p_i-2}\psi\left((\alpha+(\psi-1)(p_i-1))d^{\alpha+(\psi-1)(p_i-1)-1}|X_i d|^{\gamma+p_i}\right)\\ &\quad + |\psi|^{p_i-2}\psi\left((\gamma+p_i-2)(\beta-1)d^{\alpha+(\psi-1)(p_i-1)-1}|X_i d|^{\gamma+p_i}\right)\\ &\quad + |\psi|^{p_i-2}\psi\left((\beta-1)d^{\alpha+(\psi-1)(p_i-1)-1}|X_i d|^{\gamma+p_i}\right)\\ &= |\psi|^{p_i-2}\psi\left(-\psi+(\gamma+p_i-2)(\beta-1)\right)d^{\alpha-p_i+\psi(p_i-1)}|X_i d|^{\gamma+p_i}\\ &= -|\psi|^{p_i}d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1}\\ &\quad + |\psi|^{p_i-2}\psi(\gamma+p_i-2)(\beta-1)d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1}. \end{aligned}$$

If we put back the value of ψ, we get

$$\begin{aligned} &-X_i(W_i|X_i v|^{p_i-2}X_i v)\\ &= \left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i} d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1}\\ &\quad + \left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i-2}\left(\frac{\beta+\alpha-p_i}{p_i}\right)(\gamma+p_i-2)(\beta-1)d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1} \end{aligned}$$

$$\geq \left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i} d^{\alpha-p_i}|X_i d|^{\gamma+p_i} v^{p_i-1} \geq H_i(x) v^{p_i-1},$$

the last inequality being the desired one. So having satisfied the hypothesis, we plug the values of these functions W_i and

$$H_i = \left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i} \Gamma^{\frac{\alpha-p_i}{2-\beta}} |X_i \Gamma^{\frac{1}{2-\beta}}|^{\gamma+p_i},$$

in (12.36), which completes the proof of Part (1).

Part (2) can be proved by the same approach as the previous case by considering the functions

$$W_i = \Gamma^{\frac{\gamma+p_i}{2-\beta}} \qquad \text{and} \qquad v = \Gamma^{-\frac{(\alpha-1)(p_i-1)-\gamma-1}{(2-\beta)p_i}}.$$

Part (3) can be proved by the same approach as the previous cases by considering the functions

$$W_i = \Gamma^{\frac{\alpha}{2-\beta}} \qquad \text{and} \qquad v = \left(1+\Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{-\frac{\beta+\alpha-p_i}{p_i}}.$$

Part (4) can be proved by the same approach as the previous case by considering the functions

$$W_i = \left(1+\Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{\alpha(p_i-1)} \qquad \text{and} \qquad v = \left(1+\Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{1-\alpha}.$$

Part (5) can be proved by the same approach for $p_i = 2$, $i = 1, \ldots, N$, as the previous cases by considering the functions

$$W = \frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^{\gamma}}{\Gamma^{\frac{2m}{2-\beta}}} \qquad \text{and} \qquad v = \Gamma^{-\frac{\beta-2m-2}{2(2-\beta)}}.$$

This completes the proof of Corollary 12.5.2. □

12.6 Local uncertainty principles

As usual, Hardy inequalities imply uncertainty principles, and we now formulate such consequences of Theorem 12.4.1 and Theorem 12.5.1.

Corollary 12.6.1 (Local uncertainty principles for sums of squares). *Let $y \in M$ be such that (A_y^+) holds with the fundamental solution $\Gamma = \Gamma_y$ in T_y. Let $\Omega \subset T_y$ be an admissible domain and let $\beta > 2$. Then for all $u \in C^1(\Omega) \bigcap C(\overline{\Omega})$ we have*

$$\begin{aligned}
&\int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \int_\Omega |\nabla_X u|^2 d\nu \\
&\quad \geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu\right)^2 \\
&\qquad + \frac{1}{2}\int_{\partial\Omega} \Gamma^{-1}|u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu, \qquad (12.55)
\end{aligned}$$

and also

$$\int_\Omega \frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2}|u|^2 d\nu \int_\Omega |\nabla_X u|^2 d\nu \tag{12.56}$$
$$\geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_\Omega |u|^2 d\nu\right)^2 + \frac{1}{2}\int_{\partial\Omega} \Gamma^{-1}|u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle \int_\Omega \frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2}|u|^2 d\nu.$$

Remark 12.6.2.

1. As in Remark 12.4.2, Part 3, the last (boundary) terms in (12.55) and (12.56) can also be positive, thus providing refined uncertainty principles with respect to the boundary conditions.
2. One can readily check that (12.55) extends the classical Hardy inequality. Indeed, in the case of $M = \mathbb{R}^n$ and $X_k = \frac{\partial}{\partial x_k}$, $k = 1, \ldots, n$, taking $\alpha = 0$ and $\beta = n \geq 3$, the fundamental solution for the Laplacian is given by $\Gamma(x) = C_n |x|^{2-n}$ for some constant C_n and $|x|_E$ being the Euclidean norm, so that (12.55) reduces to the classical Hardy inequality
$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|_E^2} dx, \quad n \geq 3, \tag{12.57}$$
where ∇ is the standard gradient in $\mathbb{R}^n$, $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, and the constant $\left(\frac{n-2}{2}\right)^2$ is known to be sharp. The constant C_n does not enter (12.57) due to the scaling invariance of the inequality (12.55) with respect to the multiplication of Γ by positive constants.
3. Further to the Euclidean example (12.57), with $\Gamma^{\frac{1}{2-\beta}}(x) = C|x|_E$ we have $|\nabla \Gamma^{\frac{1}{2-\beta}}| = C$, and hence both (12.55) and (12.56) reduce to the classical uncertainty principle for $\Omega \subset \mathbb{R}^n$ if $u = 0$ on $\partial\Omega$ (for example, for $u \in C_0^\infty(\Omega)$):
$$\int_\Omega |x|_E^2 |u(x)|^2 dx \int_\Omega |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \left(\int_\Omega |u(x)|^2 dx\right)^2, \quad n \geq 3.$$
4. In the example of stratified Lie groups with $\beta = Q \geq 3$ being the homogeneous dimension of the group, and $\Gamma^{\frac{1}{2-\beta}}(x) = d(x)$ being the $\mathcal{L}$-gauge, inequality (12.55) reduces to
$$\int_\Omega d^2 |\nabla_X d|^2 |u|^2 d\nu \int_\Omega |\nabla_X u|^2 d\nu$$
$$\geq \left(\frac{Q-2}{2}\right)^2 \left(\int_\Omega |\nabla_X d|^2 |u|^2 d\nu\right)^2$$
$$+ \frac{1}{2}\int_{\partial\Omega} d^{Q-2}|u|^2 \langle \widetilde{\nabla} d^{2-Q}, d\nu\rangle \int_\Omega d^2 |\nabla_X d|^2 |u|^2 d\nu,$$
which gives inequality (12.24).

Proof of Corollary 12.6.1. Taking $\alpha = 0$ in inequality (12.21) we get

$$\begin{aligned}
&\int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \int_\Omega |\nabla_X u|^2 d\nu \\
&\quad \geq \left(\frac{\beta-2}{2}\right)^2 \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \int_\Omega \frac{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}} |u|^2 \, d\nu \\
&\qquad + \frac{1}{2} \int_{\partial\Omega} \Gamma^{-1} |u|^2 \langle \widetilde{\nabla} \Gamma, d\nu \rangle \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \\
&\quad \geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu\right)^2 \\
&\qquad + \frac{1}{2} \int_{\partial\Omega} \Gamma^{-1} |u|^2 \langle \widetilde{\nabla} \Gamma, d\nu \rangle \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu,
\end{aligned}$$

where we have used the Hölder inequality in the last line. This shows (12.55). The proof of (12.56) is similar. □

Inequality (12.22) also implies the following refinement of Corollary 12.6.1.

Corollary 12.6.3 (Refined local uncertainty principles for sums of squares)**.** *Let $y \in M$ be such that (A_y^+) holds with the fundamental solution $\Gamma = \Gamma_y$ in T_y. Let $\Omega \subset T_y$, $y \notin \partial\Omega$, be an admissible domain and let $\beta > 2$. Then for all $u \in C^1(\Omega) \bigcap C(\overline{\Omega})$ we have*

$$\begin{aligned}
&\int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \int_\Omega |\nabla_X u|^2 d\nu \\
&\quad \geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu\right)^2 \\
&\qquad + \frac{1}{4} \int_\Omega \frac{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2} |u|^2 d\nu \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \\
&\qquad + \frac{1}{2(\beta-2)} \int_{\partial\Omega} \Gamma^{-1} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1} |u|^2 \langle \widetilde{\nabla} \Gamma, d\nu \rangle \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \\
&\qquad + \frac{1}{2} \int_{\partial\Omega} \Gamma^{-1} |u|^2 \langle \widetilde{\nabla} \Gamma, d\nu \rangle \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu, \qquad (12.58)
\end{aligned}$$

and also

$$\begin{aligned}
&\int_\Omega \frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2} |u|^2 d\nu \int_\Omega |\nabla_X u|^2 d\nu \\
&\quad \geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_\Omega |u|^2 d\nu\right)^2 \\
&\qquad + \frac{1}{4} \int_\Omega \frac{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2} |u|^2 d\nu \int_\Omega \frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2} |u|^2 d\nu
\end{aligned}$$

$$+\frac{1}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\int_\Omega\frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|u|^2d\nu$$
$$+\frac{1}{2}\int_{\partial\Omega}\Gamma^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\int_\Omega\frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|u|^2d\nu. \qquad (12.59)$$

Proof of Corollary 12.6.3. Taking $\alpha=0$ in inequality (12.22) we get

$$\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu\int_\Omega|\nabla_X u|^2d\nu$$
$$\geq\left(\frac{\beta-2}{2}\right)^2\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu\int_\Omega\frac{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}}|u|^2\,d\nu$$
$$+\frac{1}{4}\int_\Omega\frac{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}|u|^2d\nu\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu$$
$$+\frac{1}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu$$
$$+\frac{1}{2}\int_{\partial\Omega}\Gamma^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu$$
$$\geq\left(\frac{\beta-2}{2}\right)^2\left(\int_\Omega|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu\right)^2$$
$$+\frac{1}{4}\int_\Omega\frac{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}|u|^2d\nu\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu$$
$$+\frac{1}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu$$
$$+\frac{1}{2}\int_{\partial\Omega}\Gamma^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\int_\Omega\Gamma^{\frac{2}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu,$$

where we have used the Hölder inequality. This shows (12.58). The proof of (12.59) is similar. □

Remark 12.6.4.

1. In the Euclidean case $M=\mathbb{R}^n$ with $\beta=n\geq 3$, we have $\Gamma^{\frac{1}{2-\beta}}(x)=C|x|_E$ is a constant multiple of the Euclidean distance, so that $|\nabla\Gamma^{\frac{1}{2-\beta}}|=C$. Consequently both (12.58) and (12.59) reduce to the improved uncertainty principle for $\Omega\subset\mathbb{R}^n$ if $u=0$ on $\partial\Omega$ (for example, usually one takes $u\in C_0^\infty(\Omega)$):
$$\int_\Omega|x|^2|u(x)|^2dx\int_\Omega|\nabla u(x)|^2dx$$
$$\geq\left(\frac{n-2}{2}\right)^2\left(\int_\Omega|u(x)|^2dx\right)^2$$
$$+\frac{1}{4}\int_\Omega\frac{1}{|x|^2}\left(\ln\frac{R}{|x|}\right)^{-2}|u(x)|^2d\nu\int_\Omega|x|^2|u(x)|^2d\nu,\quad n\geq 3.$$

2. In the Example of stratified Lie groups with $\beta = Q \geq 3$ being the homogeneous dimension of the group $\mathbb{G}$, and $\Gamma^{\frac{1}{2-\beta}}(x) = d(x)$ being the $\mathcal{L}$-gauge, inequality (12.58) reduces to

$$\begin{aligned}
&\int_\Omega d^2|\nabla_X d|^2|u|^2 d\nu \int_\Omega |\nabla_X u|^2 d\nu \\
&\geq \left(\frac{Q-2}{2}\right)^2 \left(\int_\Omega |\nabla_X d|^2|u|^2 d\nu\right)^2 \\
&\quad + \frac{1}{4}\int_\Omega \frac{|\nabla_X d|^2}{d^2}\left(\ln\frac{R}{d}\right)^{-2}|u|^2 d\nu \int_\Omega d^2|\nabla_X d|^2|u|^2 d\nu \\
&\quad + \frac{1}{2(Q-2)}\int_{\partial\Omega} d^{Q-2}\left(\ln\frac{R}{d}\right)^{-1}|u|^2\langle\widetilde{\nabla} d^{2-Q}, d\nu\rangle \int_\Omega d^2|\nabla_X d|^2|u|^2 d\nu \\
&\quad + \frac{1}{2}\int_{\partial\Omega} d^{Q-2}|u|^2\langle\widetilde{\nabla} d^{2-Q}, d\nu\rangle \int_\Omega d^2|\nabla_X d|^2|u|^2 d\nu.
\end{aligned}$$

Again, if $u \in C_0^\infty(\mathbb{G})$, the last terms disappear, and one obtains the improved uncertainty principle on stratified Lie groups compared to the statement of Corollary 11.4.3.

Theorem 12.5.1 also implies the following uncertainty principles:

Corollary 12.6.5 (Further uncertainty inequalities)**.** *Let $\Omega \subset M$ be an admissible domain. Let $\beta > 2$. Then we have the following uncertainty inequalities:*

(1) *For all $u \in C_0^\infty(\Omega)$ we have*

$$\frac{\beta^2}{4}\left(\int_\Omega |u|^2 d\nu\right)^2 \leq \left(\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{-2}|\nabla_X u|^2 d\nu\right)\left(\int_\Omega \Gamma^{\frac{2}{2-\beta}}|u|^2 d\nu\right). \tag{12.60}$$

(2) *For all $u \in C_0^\infty(\Omega)$ we have*

$$\left(\int_\Omega |\nabla_X u|^2 d\nu\right)\left(\int_\Omega \Gamma^{\frac{2}{2-\beta}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu\right) \geq \frac{\beta^2}{4}\left(\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu\right)^2. \tag{12.61}$$

(3) *For all $u \in C_0^\infty(\Omega)$ we have*

$$\begin{aligned}
&\left(\int_\Omega |\nabla_X u|^2 d\nu\right)\left(\int_\Omega \Gamma^{\frac{2}{2-\beta}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu\right) \\
&\geq \frac{(\beta-1)^2}{4}\left(\int_\Omega \Gamma^{-\frac{1}{2-\beta}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu\right)^2.
\end{aligned} \tag{12.62}$$

The Carnot group versions of these uncertainty principles in were established in [Kom10] and [GKY17], and in our proof we follow [RSS18c].

Proof of Corollary 12.6.5. Part (1). In Theorem 12.5.1, by letting

$$W(x) = |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{-2} \qquad \text{and} \qquad v = e^{-\alpha\Gamma^{\frac{2}{2-\beta}}}$$

with $\alpha \in \mathbb{R}$, we obtain the inequality

$$-4\alpha^2 \int_\Omega \Gamma^{\frac{2}{2-\beta}} |u|^2 d\nu + 2\alpha\beta \int_\Omega |u|^2 d\nu - \int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{-2} |\nabla_X u|^2 d\nu \leq 0.$$

This inequality is of the form $a\alpha^2 + b\alpha + c \leq 0$, if we denote by

$$a := -4 \int_\Omega \Gamma^{\frac{2}{2-\beta}} |u|^2 d\nu, \qquad b := 2\beta \int_\Omega |u|^2 d\nu,$$

and

$$c := -\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{-2} |\nabla_X u|^2 d\nu.$$

Thus, we must have $b^2 - 4ac \leq 0$ which proves (12.60).

Part (2). Setting

$$W = 1 \text{ and } v = e^{-\alpha\Gamma^{\frac{2}{2-\beta}}}$$

with $\alpha \in \mathbb{R}$, we obtain

$$\int_\Omega |\nabla_X u|^2 d\nu \geq 2\alpha\beta \int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu - 4\alpha^2 \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu.$$

Using the same technique as in Part (1) we prove (12.61).

We can prove Part (3) by the same approach, considering the pair

$$W = 1 \text{ and } v = e^{-\alpha\Gamma^{\frac{1}{2-\beta}}}.$$

The proof is complete. □

12.7 Local Rellich inequalities

In this section we present local refined versions of Rellich inequalities with additional boundary terms on the right-hand side, in the way analogous to Hardy inequalities and uncertainty principles in the previous sections. As before, we use the notation

$$\nabla_X = (X_1, \ldots, X_N).$$

Theorem 12.7.1 (Local Rellich inequalities for sums of squares)**.** *Let* $y \in M$ *be such that* (A_y^+) *holds with the fundamental solution* $\Gamma = \Gamma_y$ *in* T_y*. Let* $\Omega \subset T_y$ *be* *a strongly admissible domain such that* $y \notin \partial\Omega$*. Let* $\alpha \in \mathbb{R}$*,* $\beta > \alpha > 4-\beta$*,* $\beta > 2$ *and* $R \geq e\sup_{\Omega}\Gamma^{\frac{1}{2-\beta}}$*. Then for all* $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *we have*

$$\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu \geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16}\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu$$
$$+\frac{(\beta+\alpha-4)^2(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u), \tag{12.63}$$

as well as its further refinement

$$\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu$$
$$\geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16}\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{8}\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}|u|^2 d\nu$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle$$
$$+\frac{(\beta+\alpha-4)^2(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u), \tag{12.64}$$

where

$$\mathcal{C}(u) := \frac{\alpha-2}{2-\beta}\int_{\partial\Omega}u^2\Gamma^{\frac{\alpha-2}{2-\beta}-1}\langle\widetilde{\nabla}\Gamma, d\nu\rangle - 2\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}u\langle\widetilde{\nabla}u, d\nu\rangle.$$

Proof of Theorem 12.7.1. Let us prove (12.63) first. A direct calculation shows that

$$\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}} = \sum_{k=1}^{N}X_k^2\Gamma^{\frac{\alpha-2}{2-\beta}} = (\alpha-2)\sum_{k=1}^{N}X_k\left(\Gamma^{\frac{\alpha-3}{2-\beta}}X_k\Gamma^{\frac{1}{2-\beta}}\right)$$
$$= (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=1}^{N}\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2 + (\alpha-2)\Gamma^{\frac{\alpha-3}{2-\beta}}\sum_{k=1}^{N}X_k\left(X_k\Gamma^{\frac{1}{2-\beta}}\right)$$
$$= (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=1}^{N}\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\alpha-3}{2-\beta}}\sum_{k=1}^{N}X_k\left(\Gamma^{\frac{\beta-1}{2-\beta}}X_k\Gamma\right)$$

$$
\begin{aligned}
&= (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=1}^{N}\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2 \\
&\quad + \frac{(\alpha-2)(\beta-1)}{2-\beta}\Gamma^{\frac{\alpha-3}{2-\beta}}\Gamma^{-1}\sum_{k=1}^{N}(X_k\Gamma^{\frac{1}{2-\beta}})(X_k\Gamma) \\
&\quad + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma = (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=1}^{N}\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2 \\
&\quad + (\alpha-2)(\beta-1)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=1}^{N}(X_k\Gamma^{\frac{1}{2-\beta}})(X_k\Gamma^{\frac{1}{2-\beta}}) + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma \\
&= (\beta+\alpha-4)(\alpha-2)\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma,
\end{aligned}
$$

that is, we have

$$
\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}} = (\beta+\alpha-4)(\alpha-2)\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma. \tag{12.65}
$$

As in the proof of Theorem 12.4.1 we can assume that u is real-valued. Multiplying both sides of (12.65) by u^2 and integrating over Ω, since Γ is the fundamental solution of $\mathcal{L}$ and $\beta+\alpha-4>0$, we obtain

$$
\int_\Omega u^2\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}}\,d\nu = (\beta+\alpha-4)(\alpha-2)\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2u^2\,d\nu. \tag{12.66}
$$

On the other hand, by using Green's second formula (12.15) we have

$$
\begin{aligned}
\int_\Omega u^2\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}}\,d\nu &= \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}\mathcal{L}u^2\,d\nu + \int_{\partial\Omega}u^2\langle\widetilde{\nabla}\Gamma^{\frac{\alpha-2}{2-\beta}},d\nu\rangle - \int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}\langle\widetilde{\nabla}u^2,d\nu\rangle \\
&= \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}(2u\mathcal{L}u + 2|\nabla_X u|^2)\,d\nu + \mathcal{C}(u),
\end{aligned} \tag{12.67}
$$

where

$$
\mathcal{C}(u) := \frac{\alpha-2}{2-\beta}\int_{\partial\Omega}u^2\Gamma^{\frac{\alpha-2}{2-\beta}-1}\langle\widetilde{\nabla}\Gamma,d\nu\rangle - \int_{\partial\Omega}2\Gamma^{\frac{\alpha-2}{2-\beta}}u\langle\widetilde{\nabla}u,d\nu\rangle.
$$

Combining (12.66) and (12.67) we obtain

$$
\begin{aligned}
-2\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}u\mathcal{L}u d\nu + (\beta+\alpha-4)(\alpha-2)\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\,u^2 d\nu \\
= 2\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\nu + \mathcal{C}(u).
\end{aligned} \tag{12.68}
$$

By using (12.21) we have

$$\begin{aligned}&-2\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}u\mathcal{L}u d\nu+(\beta+\alpha-4)(\alpha-2)\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\,|u|^2d\nu\\&\quad\geq 2\left(\frac{\beta+\alpha-4}{2}\right)^2\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu\\&\qquad+\frac{\beta+\alpha-4}{\beta-2}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle+\mathcal{C}(u).\end{aligned}$$

It follows that

$$\begin{aligned}-\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}u\mathcal{L}u d\nu\geq&\left(\frac{\beta+\alpha-4}{2}\right)\left(\frac{\beta-\alpha}{2}\right)\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu\\&+\frac{\beta+\alpha-4}{2(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle+\frac{1}{2}\mathcal{C}(u).\end{aligned}\tag{12.69}$$

On the other hand, for any $\epsilon>0$, Hölder's and Young's inequalities give

$$\begin{aligned}-\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}u\mathcal{L}u d\nu&\leq\left(\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu\right)^{1/2}\left(\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2d\nu\right)^{1/2}\\&\leq\epsilon\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2d\nu+\frac{1}{4\epsilon}\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2d\nu.\end{aligned}\tag{12.70}$$

Inequalities (12.70) and (12.69) imply that

$$\begin{aligned}\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2d\nu\geq&\left(-4\epsilon^2+(\beta+\alpha-4)(\beta-\alpha)\epsilon\right)\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu\\&+\frac{2(\beta+\alpha-4)\epsilon}{\beta-2}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle+2\epsilon\mathcal{C}(u).\end{aligned}$$

Taking $\epsilon=\frac{(\beta+\alpha-4)(\beta-\alpha)}{8}$, we obtain

$$\begin{aligned}&\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2d\nu\\&\quad\geq\frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16}\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu\\&\qquad+\frac{(\beta+\alpha-4)^2(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u),\end{aligned}$$

which proves (12.63).

Let us now prove (12.64). From (12.68), if we use (12.22), we get

$$\begin{aligned}
&-2\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u d\nu + (\beta+\alpha-4)(\alpha-2)\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 \, |u|^2 d\nu \\
&\geq 2\left(\frac{\beta+\alpha-4}{2}\right)^2 \int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\nu \\
&\quad + \frac{1}{2}\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2} |u|^2 d\nu \\
&\quad + \frac{1}{(\beta-2)}\int_{\partial\Omega} \Gamma^{\frac{\alpha-2}{2-\beta}-1} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1} |u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle \\
&\quad + \frac{\beta+\alpha-4}{\beta-2}\int_{\partial\Omega} \Gamma^{\frac{\alpha-2}{2-\beta}-1} |u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle + \mathcal{C}(u).
\end{aligned}$$

It follows that

$$\begin{aligned}
-\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u d\nu \geq &\left(\frac{\beta+\alpha-4}{2}\right)\left(\frac{\beta-\alpha}{2}\right) \int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\nu \\
&+ \frac{1}{4}\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2} |u|^2 d\nu \\
&+ \frac{1}{2(\beta-2)}\int_{\partial\Omega} \Gamma^{\frac{\alpha-2}{2-\beta}-1} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1} |u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle \\
&+ \frac{\beta+\alpha-4}{2(\beta-2)}\int_{\partial\Omega} \Gamma^{\frac{\alpha-2}{2-\beta}-1} |u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle + \frac{1}{2}\mathcal{C}(u). \qquad (12.71)
\end{aligned}$$

Inequalities (12.70) and (12.71) imply that

$$\begin{aligned}
\int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 d\nu \geq &\left(-4\epsilon^2 + (\beta+\alpha-4)(\beta-\alpha)\epsilon\right) \int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\nu \\
&+ \epsilon \int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2} |u|^2 d\nu \\
&+ \frac{2\epsilon}{(\beta-2)}\int_{\partial\Omega} \Gamma^{\frac{\alpha-2}{2-\beta}-1} \left(\ln \frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1} |u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle \\
&+ \frac{2(\beta+\alpha-4)\epsilon}{\beta-2}\int_{\partial\Omega} \Gamma^{\frac{\alpha-2}{2-\beta}-1} |u|^2 \langle \widetilde{\nabla}\Gamma, d\nu\rangle + 2\epsilon\mathcal{C}(u).
\end{aligned}$$

Taking $\epsilon = \frac{(\beta+\alpha-4)(\beta-\alpha)}{8}$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 d\nu \\
&\geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16}\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\nu
\end{aligned}$$

$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{8}\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}|u|^2d\nu$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle$$
$$+\frac{(\beta+\alpha-4)^2(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u).$$

This completes the proof of Theorem 12.7.1. □

By a modification and refinement of the proof of Theorem 12.7.1 we can obtain another alternative of an improved Rellich inequality with boundary terms.

Theorem 12.7.2 (Refined local Rellich inequalities for sums of squares)**.** *Let $y\in M$ be such that (A_y^+) holds with the fundamental solution $\Gamma=\Gamma_y$ in T_y. Let $\Omega\subset T_y$ be a strongly admissible domain such that $y\notin\partial\Omega$. Let $\alpha\in\mathbb{R}$, $\beta>\alpha>\frac{8-\beta}{3}$, $\beta>2$ and $R\geq e\sup_{\Omega}\Gamma^{\frac{1}{2-\beta}}$. Then for all $u\in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ we have*

$$\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2d\nu\geq\frac{(\beta-\alpha)^2}{4}\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_Xu|^2d\nu$$
$$+\frac{(\beta+3\alpha-8)(\beta+\alpha-4)(\beta-\alpha)}{8(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u),\tag{12.72}$$

and its further refinement

$$\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2d\nu$$
$$\geq\frac{(\beta-\alpha)^2}{4}\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_Xu|^2d\nu$$
$$+\frac{(\beta+3\alpha-8)(\beta-\alpha)}{16}\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}|u|^2d\nu$$
$$+\frac{(\beta+3\alpha-8)(\beta-\alpha)}{8(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle$$
$$+\frac{(\beta+3\alpha-8)(\beta+\alpha-4)(\beta-\alpha)}{8(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u),\tag{12.73}$$

where

$$\mathcal{C}(u):=\frac{\alpha-2}{2-\beta}\int_{\partial\Omega}u^2\Gamma^{\frac{\alpha-2}{2-\beta}-1}\langle\widetilde{\nabla}\Gamma,d\nu\rangle-2\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}u\langle\widetilde{\nabla}u,d\nu\rangle.$$

Proof of Theorem 12.7.2. Let us first prove (12.72). Let us rewrite (12.68) in the form

$$\frac{1}{2}\mathcal{C}(u)+\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\nu \tag{12.74}$$
$$=-\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}u\mathcal{L}u d\nu+\frac{(\beta+\alpha-4)(\alpha-2)}{2}\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\,|u|^2 d\nu.$$

Also recalling (12.70) we have

$$-\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}u\mathcal{L}u d\nu\le\epsilon\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu+\frac{1}{4\epsilon}\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu. \tag{12.75}$$

Inequalities (12.75) and (12.74) imply that

$$\frac{1}{2}\mathcal{C}(u)+\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\nu$$
$$\le\left(\frac{(\beta+\alpha-4)(\alpha-2)}{2}+\epsilon\right)\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu$$
$$+\frac{1}{4\epsilon}\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu. \tag{12.76}$$

The already obtained inequality (12.63) can be rewritten as

$$\frac{16}{(\beta+\alpha-4)^2(\beta-\alpha)^2}\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu-\frac{4}{(\beta+\alpha-4)(\beta-\alpha)}\mathcal{C}(u)$$
$$-\frac{4}{(\beta-\alpha)(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle\ge\int_{\Omega}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\,d\nu.$$

Combining this with (12.76) we obtain

$$\frac{1}{2}\mathcal{C}(u)+\left(\frac{(\beta+\alpha-4)(\alpha-2)}{2}+\epsilon\right)\frac{4}{(\beta+\alpha-4)(\beta-\alpha)}\mathcal{C}(u)$$
$$+\left(\frac{(\beta+\alpha-4)(\alpha-2)}{2}+\epsilon\right)\frac{4}{(\beta-\alpha)(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma,d\nu\rangle$$
$$+\int_{\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\nu$$
$$\le\left(\frac{16\epsilon}{(\beta+\alpha-4)^2(\beta-\alpha)^2}+\frac{8(\alpha-2)}{(\beta+\alpha-4)(\beta-\alpha)^2}+\frac{1}{4\epsilon}\right)$$
$$\times\int_{\Omega}\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu.$$

Taking $\epsilon = \frac{(\beta+\alpha-4)(\beta-\alpha)}{8}$ this implies

$$\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u) + \frac{(\beta+3\alpha-8)(\beta+\alpha-4)(\beta-\alpha)}{8(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle$$
$$+\frac{(\beta-\alpha)^2}{4}\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\nu \le \int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu,$$

which shows (12.72).

Let us now prove (12.73). Inequality (12.64) can be rewritten as

$$\frac{16}{(\beta+\alpha-4)^2(\beta-\alpha)^2}\int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu - \frac{16}{(\beta+\alpha-4)^2(\beta-\alpha)^2}\mathcal{D}$$
$$\ge \int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\, d\nu,$$

where

$$\mathcal{D} := \frac{(\beta+\alpha-4)(\beta-\alpha)}{8}\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-2}|u|^2 d\nu$$
$$+\frac{(\beta+\alpha-4)(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}\left(\ln\frac{R}{\Gamma^{\frac{1}{2-\beta}}}\right)^{-1}|u|^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle$$
$$+\frac{(\beta+\alpha-4)^2(\beta-\alpha)}{4(\beta-2)}\int_{\partial\Omega}\Gamma^{\frac{\alpha-2}{2-\beta}-1}|u|^2\langle\widetilde{\nabla}\Gamma, d\nu\rangle + \frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\mathcal{C}(u).$$

Combining it with (12.76) we obtain

$$\frac{1}{2}\mathcal{C}(u) + \left(\frac{(\beta+\alpha-4)(\alpha-2)}{2} + \epsilon\right)\frac{16}{(\beta+\alpha-4)^2(\beta-\alpha)^2}\mathcal{D} + \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\nu$$
$$\le \left(\frac{16\epsilon}{(\beta+\alpha-4)^2(\beta-\alpha)^2} + \frac{8(\alpha-2)}{(\beta+\alpha-4)(\beta-\alpha)^2} + \frac{1}{4\epsilon}\right)\int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu.$$

Taking $\epsilon = \frac{(\beta+\alpha-4)(\beta-\alpha)}{8}$ we obtain

$$\frac{(\beta-\alpha)^2}{8}\mathcal{C}(u) + \frac{\beta+3\alpha-8}{2(\beta+\alpha-4)}\mathcal{D} + \frac{(\beta-\alpha)^2}{4}\int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\nu$$
$$\le \int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu,$$

completing the proof. □

Remark 12.7.3. Let us formulate several consequences of the described estimates for the setting of functions $u \in C_0^\infty(\Omega)$, so that we have $\mathcal{C}(u) = 0$. Thus if $M = \mathbb{G}$ is a stratified group of homogeneous dimension $Q \ge 3$, we take $\beta = Q$, so that $\Gamma^{\frac{1}{2-\beta}}(x) = d(x)$ is the $\mathcal{L}$-gauge on the group $\mathbb{G}$. Then the above estimates give refinements compared to known estimates, as, e.g., in Kombe [Kom10], with respect to the inclusion of boundary terms. Thus, for all $u \in C_0^\infty(\Omega)$, we have that

1. Estimate (12.63) is reduced to

$$\int_\Omega \frac{d^\alpha}{|\nabla_X d|^2}|\mathcal{L}u|^2 d\nu \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_\Omega d^{\alpha-4}|\nabla_X d|^2|u|^2\, d\nu,$$

for $Q > \alpha > 4-Q$.

2. Estimate (12.64) is reduced to

$$\int_\Omega \frac{d^\alpha}{|\nabla_X d|^2}|\mathcal{L}u|^2 d\nu \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_\Omega d^{\alpha-4}|\nabla_X d|^2|u|^2\, d\nu + \frac{(Q+\alpha-4)(Q-\alpha)}{8}\int_\Omega d^{\alpha-4}|\nabla_X d|^2\left(\ln\frac{R}{d}\right)^{-2}|u|^2 d\nu,$$

for $Q > \alpha > 4-Q$.

3. Estimate (12.72) is reduced to

$$\int_\Omega \frac{d^\alpha}{|\nabla_X d|^2}|\mathcal{L}u|^2 d\nu \geq \frac{(Q-\alpha)^2}{4}\int_\Omega d^{\alpha-2}|\nabla_X u|^2 d\nu,$$

for $Q > \alpha > \frac{8-Q}{3}$.

4. Estimate (12.73) is reduced to

$$\int_\Omega \frac{d^\alpha}{|\nabla_X d|^2}|\mathcal{L}u|^2 d\nu \geq \frac{(Q-\alpha)^2}{4}\int_\Omega d^{\alpha-2}|\nabla_X u|^2 d\nu + \frac{(Q+3\alpha-8)(Q-\alpha)}{16}\int_\Omega d^{\alpha-4}|\nabla_X d|^2\left(\ln\frac{R}{d}\right)^{-2}|u|^2 d\nu,$$

for $Q > \alpha > \frac{8-Q}{3}$.

For unweighted versions (with $\alpha = 0$) inequalities (12.63)–(12.64) work under the condition $Q \geq 5$ which is usually appearing in Rellich inequalities, while (12.72)–(12.73) work for homogeneous dimensions $Q \geq 9$.

12.8 Rellich inequalities via Picone identities

In this section we discuss weighted Rellich inequalities in the spirit of Hardy inequalities from Section 12.5, relying on an appropriate version of Picone identities.

Theorem 12.8.1 (Weighted anisotropic Rellich type inequality)**.** *Let $\Omega \subset M$ be an admissible domain. Let $W_i(x) \in C^2(\Omega)$ and $H_i(x) \in L^1_{\mathrm{loc}}(\Omega)$ be the non-negative weight functions. Let $v > 0$, $v \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ with*

$$X_i^2 \left(W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v\right) \geq H_i(x) v^{p-1} \quad \text{and} \quad -X_i^2 v > 0, \tag{12.77}$$

a.e. in Ω, for all $i = 1, \ldots, N$. Then for every $0 \leq u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ we have the following inequality

$$\begin{aligned} \sum_{i=1}^N \int_\Omega H_i(x)|u|^{p_i} d\nu \leq & \sum_{i=1}^N \int_\Omega W_i(x)|X_i^2 u|^{p_i} d\nu \\ & + \sum_{i=1}^N \int_{\partial\Omega} W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v \langle \widetilde{\nabla}_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right), d\nu\rangle \\ & - \sum_{i=1}^N \int_{\partial\Omega} \left(\frac{u^{p_i}}{v^{p_i-1}}\right) \langle \widetilde{\nabla}_i (W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v), d\nu\rangle, \end{aligned} \tag{12.78}$$

where $1 < p_i < N$ for $i = 1, \ldots, N$, and $\widetilde{\nabla}_i u = X_i u X_i$.

The proof of Theorem 12.8.1 will rely on first establishing an appropriate version of the second-order Picone identity. Before giving its formulation and the proofs, let us make some remarks and also formulate several of its consequences.

Remark 12.8.2.

1. A Carnot group version of Theorem 12.8.1 was obtained by Goldstein, Kombe and Yener in [GKY18]. In our exposition for general vector fields we follow [RSS18c], also allowing one to include boundary terms into the inequality.
2. Note that the function v from the assumption (12.77) appears in the boundary terms (12.78), which seems a new effect unlike known particular cases of Theorem 12.8.1.

As a consequence of Theorem 12.8.1 we can obtain several Rellich type inequalities involving the sum of squares operator.

Corollary 12.8.3 (Rellich inequalities for sums of squares)**.** *Let $\Omega \subset M$ be an admissible domain, and let the operator $\mathcal{L}$ is the sum of squares of vector fields:*

$$\mathcal{L} := \sum_{i=1}^N X_i^2.$$

Then we have the following estimates:

(1) *Let $\beta > 2$, $\alpha \in \mathbb{R}, \beta+\alpha > 4$ and $\beta > \alpha$. Then we have*

$$\int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2 d\nu \geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16}\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 d\nu,$$

for all $u \in C_0^\infty(\Omega\backslash\{0\})$.

(2) *Let $1 < p < \infty$ and $2-\beta < \alpha < \min\{(\beta-2)(p-1),(\beta-2)\}$. Then for all $u \in C_0^\infty(\Omega\backslash\{0\})$ we have*

$$\int_\Omega \frac{\Gamma^{\frac{\alpha+2p-2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{2p-2}}|\mathcal{L}u|^p d\nu \geq C(\beta,\alpha,p)^p \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^p d\nu, \quad (12.79)$$

where $C(\beta,\alpha,p) := \frac{(\beta+\alpha-2)}{p}\frac{(\beta-2)(p-1)-\alpha}{p}$.

Proof of Corollary 12.8.3. To prove Part (1), we take $\gamma = -\frac{\beta+\alpha-4}{2}$, and choose the functions $W(x)$ and v such that

$$W(x) = \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|X_i\Gamma^{\frac{1}{2-\beta}}|^2} \quad \text{and} \quad v = \Gamma^{\frac{\gamma}{2-\beta}},$$

and apply Theorem 12.8.1.

To prove Part (2), we set

$$W(x) = \frac{\Gamma^{\frac{\alpha+2p-2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{2p-2}} \text{ and } v = \gamma^{-\frac{\beta+\alpha-2}{p(2-\beta)}},$$

and apply Theorem 12.8.1. □

Now let us prove the following anisotropic (second-order) Picone type identity, extending its horizontal version in Lemma 6.10.4. Then, as a consequence, we obtain Theorem 12.8.1.

Lemma 12.8.4 (Second-order Picone identity)**.** *Let $\Omega \subset \mathbb{G}$ be an open set. Let u, v be twice differentiable a.e. in Ω and satisfying the following conditions:*

$$u \geq 0,\ v > 0,\ X_i^2 v < 0 \ \ a.e.\ in\ \ \Omega.$$

Let $p_i > 1$, $i = 1, \ldots, N$. Then we have

$$L_1(u,v) = R_1(u,v) \geq 0, \quad (12.80)$$

where

$$R_1(u,v) := \sum_{i=1}^N |X_i^2 u|^{p_i} - \sum_{i=1}^N X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i^2 v|^{p_i-2}X_i^2 v,$$

and

$$\begin{aligned}L_1(u,v) :=& \sum_{i=1}^{N} |X_i^2 u|^{p_i} - \sum_{i=1}^{N} p_i \left(\frac{u}{v}\right)^{p_i-1} X_i^2 u X_i^2 v |X_i^2 v|^{p_i-2} \\ &+ \sum_{i=1}^{N} (p_i-1)\left(\frac{u}{v}\right)^{p_i} |X_i^2 v|^{p_i} \\ &- \sum_{i=1}^{N} p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}} |X_i^2 v|^{p_i-2} X_i^2 v \left(X_i u - \frac{u}{v} X_i v\right)^2.\end{aligned}$$

Proof of Lemma 12.8.4. A direct computation yields

$$\begin{aligned}X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right) &= X_i\left(p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i u - (p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i v\right) \\ &= p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-2}}\left(\frac{(X_i u)v - u(X_i v)}{v^2}\right) X_i u + p_i \frac{u^{p_i-1}}{v^{p_i-1}} X_i^2 u \\ &\quad - p_i(p_i-1)\frac{u^{p_i-1}}{v^{p_i-1}}\left(\frac{(X_i u)v - u(X_i v)}{v^2}\right) X_i v - (p_i-1)\frac{u^{p_i}}{v^{p_i}} X_i^2 v \\ &= p_i(p_i-1)\left(\frac{u^{p_i-2}}{v^{p_i-1}}|X_i u|^2 - 2\frac{u^{p_i-1}}{v^{p_i}} X_i v X_i u + \frac{u^{p_i}}{v^{p_i+1}}|X_i v|^2\right) \\ &\quad + p_i\frac{u^{p_i-1}}{v^{p_i-1}} X_i^2 u - (p_i-1)\frac{u^{p_i}}{v^{p_i}} X_i^2 v \\ &= p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}\left(X_i u - \frac{u}{v}X_i v\right)^2 + p_i\frac{u^{p_i-1}}{v^{p_i-1}} X_i^2 u - (p_i-1)\frac{u^{p_i}}{v^{p_i}} X_i^2 v,\end{aligned}$$

which gives the equality in (12.80). By Young's inequality we have

$$\frac{u^{p_i-1}}{v^{p_i-1}} X_i^2 u X_i^2 v |X_i^2 v|^{p_i-2} \le \frac{|X_i^2 u|^{p_i}}{p_i} + \frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2 v|^{p_i}, \quad i = 1,\ldots,N,$$

where $p_i > 1$ and $q_i > 1$ with $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Since $X_i^2 v < 0$, $i = 1,\ldots,N$, we arrive at

$$\begin{aligned}L_1(u,v) \ge& \sum_{i=1}^{N} |X_i^2 u|^{p_i} + \sum_{i=1}^{N}(p_i-1)\frac{u^{p_i}}{v^{p_i}}|X_i^2 v|^{p_i} - \sum_{i=1}^{N} p_i\left(\frac{|X_i^2 u|^{p_i}}{p_i} + \frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2 v|^{p_i}\right) \\ &- \sum_{i=1}^{N} p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2 v|^{p_i-2} X_i^2 v \left|X_i u - \frac{u}{v}X_i v\right|^2 \\ =& \sum_{i=1}^{N}\left(p_i - 1 - \frac{p_i}{q_i}\right)\frac{u^{p_i}}{v^{p_i}}|X_i^2 v|^{p_i} \\ &- \sum_{i=1}^{N} p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2 v|^{p_i-2} X_i^2 v \left|X_i u - \frac{u}{v}X_i v\right|^2 \ge 0.\end{aligned}$$

This completes the proof of Lemma 12.8.4. □

Proof of Theorem 12.8.1. Let us give a brief outline of the following proof as it is similar to the proof of Theorem 12.5.1: we start by using the Picone type identity (12.80), then we apply an analogue of Green's second formula and the hypothesis (12.77), respectively. Finally, we arrive at (12.78) by using $H_i(x) \geq 0$. Summarizing, we have

$$\begin{aligned}
0 \leq & \int_\Omega W_i(x) L_1(u,v) d\nu = \int_\Omega W_i(x) R_1(u,v) d\nu \\
&= \int_\Omega W_i(x) |X_i^2 u|^{p_i} d\nu - \int_\Omega X_i^2 \left(\frac{u^{p_i}}{v^{p_i-1}} \right) W_i(x) |X_i^2 v|^{p_i-2} X_i^2 v d\nu \\
&= \int_\Omega W_i(x) |X_i^2 u|^{p_i} d\nu - \int_\Omega \frac{u^{p_i}}{v^{p_i-1}} X_i^2 \left(W_i(x) |X_i^2 v|^{p_i-2} X_i^2 v \right) d\nu \\
&\quad + \int_{\partial\Omega} \Bigg(W_i(x) |X_i^2 v|^{p_i-2} X_i^2 v \langle \widetilde{\nabla}_i \left(\frac{u^{p_i}}{v^{p_i-1}} \right), d\nu \rangle \\
&\quad - \left(\frac{u^{p_i}}{v^{p_i-1}} \right) \langle \widetilde{\nabla}_i (W_i(x) |X_i^2 v|^{p_i-2} X_i^2 v), d\nu \rangle \Bigg) \\
&\leq \int_\Omega W_i(x) |X_i^2 u|^{p_i} d\nu - \int_\Omega H_i(x) |u|^{p_i} d\nu \\
&\quad + \int_{\partial\Omega} \Bigg(W_i(x) |X_i^2 v|^{p_i-2} X_i^2 v \langle \widetilde{\nabla}_i \left(\frac{u^{p_i}}{v^{p_i-1}} \right), d\nu \rangle \\
&\quad - \left(\frac{u^{p_i}}{v^{p_i-1}} \right) \langle \widetilde{\nabla}_i (W_i(x) |X_i^2 v|^{p_i-2} X_i^2 v), d\nu \rangle \Bigg).
\end{aligned}$$

In the last line, we have used (12.77) which leads to (12.78). □