Chapter 11

Elements of Potential Theory on Stratified Groups

In this chapter, we discuss elements of the potential theory and the theory of boundary layer operators in the setting of stratified groups. The main tools for this analysis are the fundamental solution for the sub-Laplacian and Green's identities established in Section 1.4.4.

From a different perspective than ours, comparable problems have been considered by Folland and Stein [FS74], Geller [Gel90], Jerison [Jer81], Romero [Rom91], Capogna, Garofalo and Nhieu [CGN08], Bonfiglioli, Lanconelli and Uguzzoni [BLU07] and a number of other people. A general setting of degenerate elliptic operators was considered by Bony [Bon69]. However, it seems that the potential theory on stratified group based on the layer potentials is an ingredient missing in the literature, and here our presentation follows the development of such a theory in [RS17c] on general stratified groups and in [RS16c] on the Heisenberg group.

Elements of the developed potential theory and the constructed layer potentials are then used in this chapter to discuss several applications. In particular, we describe new well-posed (i.e., solvable in the classical sense) boundary value problems in addition to using it in solving the known problems such as Dirichlet and Neumann problems for the sub-Laplacian. Furthermore, the developed potential theory is used to derive trace formulae for the Newton potential of the sub-Laplacian to piecewise smooth surfaces, and using these conditions to construct the analogue of Kac's boundary value problem in the setting of stratified groups which is related to Kac's "principle of not feeling the boundary" for the sub-Laplacian.

In this section $\mathbb{G}$ will be a stratified group of homogeneous dimension $Q \geq 3$. Since our analysis is based on the fundamental solution for the sub-Laplacian we will be restricting our discussion to the case of $Q \geq 3$. This is not restrictive since it effectively rules out only the spaces $\mathbb{R}$ and $\mathbb{R}^2$ where the fundamental solution assumes a different form and where most things presented in this chapter are already known.

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11.1 Boundary value problems on stratified groups

In this section, we show how Green's identities established in Theorem 1.4.6 can be used to provide simple proofs for the well-posedness for a number of boundary value problems. In particular, we provide examples of different boundary conditions, such as Dirichlet, Neumann, Robin, mixed Dirichlet and Robin, or different types of conditions on different parts of the boundary. For simplicity we restrict the considerations in this section to zero boundary conditions only, otherwise these problems may become very delicate due to the presence of characteristic points, and we can refer to [DGN06] for a thorough analysis in this direction, and to Remark 11.1.7 for some discussion.

We also note that in the subsequently considered boundary value problems, we can assume without loss of generality (in the proofs) that functions are real-valued since otherwise we can always take real and imaginary parts which would then satisfy the same equations.

In this section Ω is always an *admissible domain* as given in Definition 1.4.4.

The following result is known by other methods but given Green's first formula in Theorem 1.4.6 its proof becomes elementary.

Proposition 11.1.1 (Dirichlet boundary value problem for sub-Laplacian)**.** *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. Then the Dirichlet boundary value problem*

$$\mathcal{L}u(x) = 0, \ \ x \in \Omega, \tag{11.1}$$

$$u(x) = 0, \ \ x \in \partial\Omega, \tag{11.2}$$

has the unique trivial solution $u \equiv 0$ in the class of functions $C^2(\Omega) \bigcap C^1(\overline{\Omega})$.

Proof of Proposition 11.1.1. Setting $v = u$ in (1.86), by (11.1) and (11.2) we get

$$\int_\Omega \widetilde{\nabla} u u d\nu = \int_\Omega \left(\widetilde{\nabla} u u + u\mathcal{L}u\right) d\nu = \int_{\partial\Omega} u\langle \widetilde{\nabla} u, d\nu\rangle = 0.$$

Therefore

$$\int_\Omega \sum_{k=1}^{N_1} |X_k u|^2 d\nu = 0,$$

that is, $X_k u = 0$, $k = 1, \ldots, N_1$. Since any element of a Jacobian basis of $\mathbb{G}$ is represented by Lie brackets of $\{X_1, \ldots, X_{N_1}\}$, we obtain that u is a constant, so $u \equiv 0$ on Ω by (11.2). □

This result has the following simple extension to stationary Schrödinger operators:

Proposition 11.1.2 (Dirichlet boundary value problem for Schrödinger operator)**.** *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. Let $q : \Omega \to \mathbb{R}$ be a non-negative bounded function: $q \in L^\infty(\Omega)$ and $q(x) \geq 0$ for all $x \in \Omega$. Then the Dirichlet boundary value problem for the Schrödinger equation*

$$-\mathcal{L}u(x) + q(x)u(x) = 0, \;\; x \in \Omega, \tag{11.3}$$

$$u(x) = 0, \;\; x \in \partial\Omega, \tag{11.4}$$

has the unique trivial solution $u \equiv 0$ in the class of functions $C^2(\Omega) \bigcap C^1(\overline{\Omega})$.

Proof of Proposition 11.1.2*.* Using Green's formula (1.86), from (11.3) and (11.4) we obtain

$$\begin{aligned}\int_\Omega \widetilde{\nabla} u u d\nu &= \int_\Omega \left(\widetilde{\nabla} u u + u\mathcal{L}u\right) d\nu - \int_\Omega q(y)|u(y)|^2 d\nu \\ &= \int_{\partial\Omega} u\langle \widetilde{\nabla} u, d\nu\rangle - \int_\Omega q(y)|u(y)|^2 d\nu \\ &= -\int_\Omega q(y)|u(y)|^2 d\nu.\end{aligned}$$

Therefore, we have

$$0 \leq \int_\Omega \sum_{k=1}^{N_1} |X_k u|^2 d\nu = -\int_\Omega q(y)|u(y)|^2 d\nu \leq 0,$$

that is, we must have $u \equiv 0$ because of (11.4). □

Similarly, we can treat von Neumann type boundary conditions.

Proposition 11.1.3 (Neumann boundary value problem for sub-Laplacian)**.** *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. Then the boundary value problem*

$$\mathcal{L}u(x) = 0, \;\; x \in \Omega \subset \mathbb{G}, \tag{11.5}$$

$$\sum_{j=1}^{N_1} X_j u \langle X_j, d\nu\rangle = 0 \quad on \;\; \partial\Omega, \tag{11.6}$$

has only constant solutions $u \equiv \mathrm{const}$ in the class of functions $C^2(\Omega) \bigcap C^1(\overline{\Omega})$.

Remark 11.1.4. Note that von Neumann type boundary value problems for the sub-Laplacian have been known and studied, see, e.g., [DGN06]. However, Proposition 11.1.3 provides a new measure type condition for the von Neumann type boundary value problem.

Proof of Proposition 11.1.3. Set $v = u$ in (1.86), then by (11.5) and (11.6) we get

$$\begin{aligned}\int_\Omega \widetilde{\nabla} u u d\nu &= \int_\Omega \left(\widetilde{\nabla} u u + u\mathcal{L}u\right) d\nu \\ &= \int_{\partial\Omega} u\langle \widetilde{\nabla} u, d\nu\rangle \\ &= \int_{\partial\Omega} u \sum_{j=1}^{N_1} X_j u \langle X_j, d\nu\rangle \\ &= 0.\end{aligned}$$

Therefore

$$\int_\Omega \sum_{k=1}^{N_1} |X_k u|^2 d\nu = 0,$$

that is, $X_k u = 0,\ k = 1, \ldots, N_1$. Since all vector fields in $\mathfrak{g}$ are represented by Lie brackets of $\{X_1, \ldots, X_{N_1}\}$, we obtain that u is a constant. □

Similarly, the Robin type boundary conditions can be also considered.

Proposition 11.1.5 (Robin boundary value problem for sub-Laplacian)**.** *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. Let $a_k : \partial\Omega \to \mathbb{R}$, $k = 1, \ldots, N_1$, be bounded functions such that the measure*

$$\sum_{j=1}^{N_1} a_j \langle X_j, d\nu\rangle \geq 0 \tag{11.7}$$

is non-negative on $\partial\Omega$. Then the boundary value problem

$$\mathcal{L}u(x) = 0,\ x \in \Omega \subset \mathbb{G}, \tag{11.8}$$

$$\sum_{j=1}^{N_1} (a_j u + X_j u)\langle X_j, d\nu\rangle = 0 \quad \text{on} \quad \partial\Omega, \tag{11.9}$$

has only constant solutions $u \equiv \mathrm{const}$ in the class of functions $C^2(\Omega) \bigcap C^1(\overline{\Omega})$.

Moreover, if the integral of the measure (11.7) is positive, i.e., if

$$\int_{\partial\Omega} \sum_{j=1}^{N_1} a_j \langle X_j, d\nu\rangle > 0, \tag{11.10}$$

then the boundary value problem (11.8)–(11.9) has the unique trivial solution $u \equiv 0$ in the class of functions $C^2(\Omega) \bigcap C^1(\overline{\Omega})$.

Proof of Proposition 11.1.5. Set $v = u$ in (1.86), then by (11.8) and (11.9) we get

$$\begin{aligned}\int_\Omega \widetilde{\nabla} u u d\nu &= \int_\Omega \left(\widetilde{\nabla} u u + u\mathcal{L}u\right) d\nu \\ &= \int_{\partial\Omega} u\langle \widetilde{\nabla} u, d\nu\rangle \\ &= \int_{\partial\Omega} u \sum_{j=1}^{N_1} X_j u \langle X_j, d\nu\rangle \\ &= -\int_{\partial\Omega} u^2 \sum_{j=1}^{N_1} a_j \langle X_j, d\nu\rangle.\end{aligned}$$

This means that

$$\int_\Omega \sum_{k=1}^{N_1} |X_k u|^2 d\nu = -\int_{\partial\Omega} u^2 \sum_{j=1}^{N_1} a_j \langle X_j, d\nu\rangle.$$

Therefore, we must have

$$\int_\Omega \sum_{k=1}^{N_1} |X_k u|^2 d\nu = 0 \qquad \text{and} \qquad \int_{\partial\Omega} u^2 \sum_{j=1}^{N_1} a_j \langle X_j, d\nu\rangle = 0.$$

As before the first equality implies that u is a constant and the first part of Proposition 11.1.5 is proved.

On the other hand, by the assumption (11.10) the second equality implies that $u = 0$ on $\partial\Omega$, and this implies that $u \equiv 0$ on Ω. □

The problems where Dirichlet or Robin conditions are imposed on different parts of the boundary can be also considered:

Proposition 11.1.6 (Dirichlet and Robin boundary value problem for sub-Laplacian)**.** *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. Let $a_k : \partial\Omega \to \mathbb{R}$, $k = 1, \ldots, N_1$, be bounded functions such that the measure*

$$\sum_{j=1}^{N_1} a_j \langle X_j, d\nu\rangle \geq 0 \tag{11.11}$$

is non-negative on $\partial\Omega$. Let $\partial\Omega_1 \subset \partial\Omega$, $\partial\Omega_1 \neq \{\emptyset\}$ and $\partial\Omega_2 := \partial\Omega\backslash\partial\Omega_1$. Then the boundary value problem

$$\mathcal{L}u(x) = 0, \;\; x \in \Omega \subset \mathbb{G}, \tag{11.12}$$

$$u = 0 \;\; on \;\; \partial\Omega_1, \tag{11.13}$$

$$\sum_{j=1}^{N_1} (a_j u + X_j u)\langle X_j, d\nu\rangle = 0 \;\; on \;\; \partial\Omega_2, \tag{11.14}$$

has the unique trivial solution $u \equiv 0$ in the class of functions $C^2(\Omega)\bigcap C^1(\overline{\Omega})$.

Proposition 11.1.6 can be proved in the same way as Proposition 11.1.5.

Remark 11.1.7 (More general boundary value problems)**.** In principle, one is certainly also interested in boundary value problems for non-zero boundary data or, more generally, for example in the Dirichlet case, for an admissible domain Ω, in

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{11.15}$$

In the Euclidean case, when $\mathcal{L}$ is the Laplacian, which is elliptic, the boundary value problem (11.15) has a classical solution in $C^2(\Omega)\cap C^1(\overline{\Omega})$ for reasonably good functions f, ϕ, for example, for $f \in C^\alpha(\Omega)$ with $\alpha > 0$, and $\phi \in C(\partial\Omega)$.

However, in general, this fact fails completely for the hypoelliptic boundary value problem (11.15). Thus, already on the Heisenberg group $\mathbb{H}^n$, even if the domain Ω and the boundary datum ϕ are real analytic and $f \equiv 0$, D. Jerison [Jer81] gave an example when the solution of the Dirichlet problem (11.15) is not better than Hölder continuous near a characteristic boundary point, that is, the solution is not classical.

We recall that the *characteristic set of* Ω (related to vector fields $\{X_1, \ldots, X_{N_1}\}$ giving the first stratum of a stratified group $\mathbb{G}$) is the set

$$\{x \in \partial\Omega : \, X_k(x) \in T_x(\partial\Omega), \, k = 1, \ldots, N_1\},$$

with $T_x(\partial\Omega)$ being the tangent space to $\partial\Omega$ at the point x. We refer to Section 11.2 more discussions on characteristic points, and also to [CGN08, Section 4] for an extensive discussion on boundary value problems and additional conditions allowing one to partially handle the appearing characteristic points.

The appearance of characteristic points is hard to avoid, see the next section.

11.2 Layer potentials of the sub-Laplacian

Let $D \subset \mathbb{R}^N$ be an open set with boundary ∂D. Let D be a *domain of class* C^1, that is, a domain such that for every $x_0 \in \partial D$ there exist a neighbourhood U_{x_0} of x_0, and a function $\phi_{x_0} \in C^1(U_{x_0})$, with

$$|\nabla \phi_{x_0}| \geq \alpha > 0$$

in U_{x_0}, where ∇ is the standard gradient in $\mathbb{R}^N$, such that

$$D \cap U_{x_0} = \{x \in U_{x_0} \mid \phi_{x_0}(x) < 0\},$$
$$\partial D \cap U_{x_0} = \{x \in U_{x_0} \mid \phi_{x_0}(x) = 0\}.$$

A point $x_0 \in \partial D$ is called *characteristic* with respect to vector fields $\{X_1, \ldots, X_{N_1}\}$, if for U_{x_0}, ϕ_{x_0} as above we have

$$X_1\phi_{x_0}(x_0) = 0, \ldots, X_{N_1}\phi_{x_0}(x_0) = 0.$$

Usually, bounded domains have a non-empty collection of characteristic points. For instance, any bounded domain of class C^1 in the Heisenberg group $\mathbb{H}^n$ with the boundary homeomorphic to the $2n$-dimensional sphere $\mathbb{S}^{2n}$, has a non-empty characteristic set, see, e.g., [DGN06].

11.2.1 Single layer potentials

We start by defining single layer potentials for the sub-Laplacian and then analyse their basic properties.

Definition 11.2.1 (Single layer potentials)**.** Let the function $\varepsilon(y,x)=\varepsilon(x,y)$ be the fundamental solution of the sub-Laplacian as in (1.89). We define *single layer potentials* for an admissible domain Ω as the functionals

$$\mathcal{S}_j u(x) := \int_{\partial\Omega} u(y)\varepsilon(y,x)\langle X_j, d\nu(y)\rangle, \quad j=1,\ldots,N_1, \tag{11.16}$$

where $\langle X_j, d\nu\rangle$ is the canonical pairing between vector fields and differential forms, and N_1 is the dimension of the first stratum of $\mathbb{G}$.

Remark 11.2.2. In [Jer81] Jerison used the single layer potential defined by

$$\mathcal{S}_0 u(x) = \int_{\partial\Omega} u(y)\varepsilon(y,x)dS(y).$$

However, it is not integrable over characteristic points of $\partial\Omega$, see [Rom91] for examples. On the contrary, as we will show in Lemma 11.2.3, the single layer potentials (11.16) are integrable over the whole boundary $\partial\Omega$ including the set of characteristic points.

We recall from (1.89) that

$$\varepsilon(x,y) = [d(x,y)]^{2-Q} = [d(x^{-1}y)]^{2-Q}, \tag{11.17}$$

with d being the $\mathcal{L}$-gauge as in (1.75).

Lemma 11.2.3 (Single layer potentials are well defined)**.** *Let $\partial\Omega$ be the boundary of an admissible domain Ω in a stratified group $\mathbb{G}$ of homogeneous dimension $Q\geq 3$. Then*

$$\int_{\partial\Omega} \varepsilon(x,y)\langle X_j, d\nu(y)\rangle$$

is a convergent integral for any $x\in\mathbb{G}$ such that $x\notin\partial\Omega$.

Proof of Lemma 11.2.3. Let $B_R := \{y : d(x,y) < R\}$ be a ball such that $\Omega\subset B_R$. In view of (11.17) we can estimate

$$\int_{\partial\Omega} [d(x,y)]^{2-Q}\langle X_j, d\nu(y)\rangle = \int_{\Omega} X_j[d(x,y)]^{2-Q} d\nu(y)$$

$$\leq \int_{\Omega} | X_j[d(x,y)]^{2-Q} | d\nu(y) \leq \int_{B_R} | X_j[d(x,y)]^{2-Q} | d\nu(y)$$
$$= C \int_0^R r^{1-Q} r^{Q-1} dr < \infty,$$

where we have used the polar decomposition from Proposition 1.2.10 with respect to the $\mathcal{L}$-gauge d. This proves Lemma 11.2.3. ☐

We now establish the main basic properties of the single layer potentials.

Theorem 11.2.4 (Continuity of single layer potentials)**.** *Let $\partial\Omega$ be the boundary of an admissible domain Ω in a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$. Let u be bounded on $\partial\Omega$, that is, $u \in L^\infty(\partial\Omega)$. Then the single layer potential $\mathcal{S}_j u$ is continuous on $\mathbb{G}$ for all $j = 1, \ldots, N_1$.*

Proof of Theorem 11.2.4. Let first $x_0 \in \mathbb{G}$ be such that $x_0 \notin \partial\Omega$. Then

$$|\mathcal{S}_j u(x) - \mathcal{S}_j u(x_0)| = \left| \int_{\partial\Omega} u(y)(\varepsilon(y,x) - \varepsilon(y,x_0)) \langle X_j, d\nu(y)\rangle \right|$$
$$\leq \sup_{y\in\partial\Omega} |u(y)| \left| \int_{\partial\Omega} \varepsilon(y,x) - \varepsilon(y,x_0) \langle X_j, d\nu(y)\rangle \right|.$$

This means that

$$\lim_{x\to x_0} \mathcal{S}_j u(x) = \mathcal{S}_j u(x_0),$$

that is, the single layer potential $\mathcal{S}_j u$ is continuous on $\mathbb{G}\backslash\partial\Omega$.

Now let $x_0 \in \mathbb{G}$ be such that $x_0 \in \partial\Omega$. Let us denote

$$\Omega_\epsilon := \{y \in \Omega : d(x_0, y) < \epsilon\}.$$

Then we have

$$|\mathcal{S}_j u(x) - \mathcal{S}_j u(x_0)| = \left| \int_{\partial\Omega} u(y)(\varepsilon(y,x) - \varepsilon(y,x_0)) \langle X_j, d\nu(y)\rangle \right|$$
$$= \sup_{y\in\partial\Omega} |u(y)| \left| \int_{\partial\Omega} (\varepsilon(y,x) - \varepsilon(y,x_0)) \langle X_j, d\nu(y)\rangle \right|$$
$$= \sup_{y\in\partial\Omega} |u(y)| \left| \int_{\Omega} X_j(\varepsilon(y,x) - \varepsilon(y,x_0)) d\nu(y) \right|$$
$$\leq \sup_{y\in\partial\Omega} |u(y)| \lim_{\epsilon\to 0} \Bigg(\left| \int_{\Omega\backslash\Omega_\epsilon} X_j(\varepsilon(y,x) - \varepsilon(y,x_0)) d\nu(y) \right|$$
$$+ \left| \int_{\Omega_\epsilon} X_j(\varepsilon(y,x) - \varepsilon(y,x_0)) d\nu(y) \right| \Bigg),$$

where the first term tends to zero when $x \to x_0$. Now what is left is to show that the second term tends to zero. This follows since

$$\begin{aligned} &\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} X_j(\varepsilon(y,x)-\varepsilon(y,x_0))d\nu(y) \\ &= \lim_{\epsilon\to 0}\left(\int_{\Omega_\epsilon} X_j\varepsilon(y,x)d\nu(y) - \int_{\Omega_\epsilon} X_j\varepsilon(y,x_0)d\nu(y)\right) \\ &= \lim_{\epsilon\to 0}(C\int_0^\epsilon r^{1-Q}r^{Q-1}dr) = C\lim_{\epsilon\to 0}\epsilon = 0. \end{aligned}$$

This completes the proof. □

11.2.2 Double layer potential

In this section, we discuss double layer potentials and establish Plemelj type jump relations for them.

Definition 11.2.5 (Double layer potential)**.** We define the *double layer potential* as the operator

$$\mathcal{D}u(x) := \int_{\partial\Omega} u(y)\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle, \tag{11.18}$$

where

$$\widetilde{\nabla}\varepsilon = \sum_{k=1}^{N_1}(X_k\varepsilon)X_k,$$

with vector fileds X_k acting on the y-variable.

We now describe the Plemelj type jump relations for the double layer potential $\mathcal{D}$.

Theorem 11.2.6 (Plemelj jump relations for double layer potential)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$. Let $u \in C^1(\Omega)\bigcap C(\overline{\Omega})$. For $x_0 \in \partial\Omega$ define*

$$\mathcal{D}^0u(x_0) := \int_{\partial\Omega} u(y)\langle\widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle,$$

$$\mathcal{D}^+u(x_0) := \lim_{x\to x_0,\, x\in\Omega}\int_{\partial\Omega} u(y)\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle,$$

and

$$\mathcal{D}^-u(x_0) := \lim_{x\to x_0,\, x\notin\overline{\Omega}}\int_{\partial\Omega} u(y)\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle.$$

Then $\mathcal{D}^+u(x_0), \mathcal{D}^-u(x_0)$ and $\mathcal{D}^0u(x_0)$ exist and verify the following jump relations:

$$\mathcal{D}^+u(x_0) - \mathcal{D}^-u(x_0) = u(x_0),$$

$$\mathcal{D}^0 u(x_0) - \mathcal{D}^- u(x_0) = \mathcal{J}(x_0) u(x_0),$$
$$\mathcal{D}^+ u(x_0) - \mathcal{D}^0 u(x_0) = (1 - \mathcal{J}(x_0)) u(x_0),$$

where the jump value $\mathcal{J}(x_0)$ *is given by the formula*

$$\mathcal{J}(x_0) = \int_{\partial\Omega} \langle \widetilde{\nabla} \varepsilon(y, x_0), d\nu(y) \rangle,$$

in the sense of the (Cauchy) principal value, and where

$$\widetilde{\nabla} \varepsilon = \sum_{k=1}^{N_1} (X_k \varepsilon) \, X_k.$$

In order to prove Theorem 11.2.6 we will need the following property.

Lemma 11.2.7. *Let* $\Omega \subset \mathbb{G}$ *be an admissible domain with the boundary* $\partial\Omega$ *and let* $x_0 \in \partial\Omega$. *Let* $u \in C^1(\Omega) \bigcap C(\overline{\Omega})$. *Then*

$$\lim_{x \to x_0} \int_{\partial\Omega} [u(y) - u(x)] \langle \widetilde{\nabla} \varepsilon(y, x), d\nu(y) \rangle = \int_{\partial\Omega} [u(y) - u(x_0)] \langle \widetilde{\nabla} \varepsilon(y, x_0), d\nu(y) \rangle.$$

Proof of Lemma 11.2.7. To simplify the notation in this proof we will use the following Einstein type convention: if the index k is repeated in an integrand, it means that it is a sum over k from 1 to N_1. For example, we abbreviate it as follows:

$$\int_{\partial\Omega} [u(y) - u(x)] X_k \varepsilon(y, x) \langle X_k, d\nu(y) \rangle := \sum_{k=1}^{N_1} \int_{\partial\Omega} [u(y) - u(x)] X_k \varepsilon(y, x) \langle X_k, d\nu(y) \rangle.$$

First, let us show that

$$\lim_{\epsilon \to 0} \int_{d(x,y) < \epsilon} X_k \left\{ (u(y) - u(x)) X_k \varepsilon(y, x) \right\} d\nu(y) = 0.$$

By using the divergence formula in Theorem 1.4.5, we can estimate

$$\begin{aligned}
&\lim_{\epsilon \to 0} \left| \int_{d(x,y) < \epsilon} X_k \left\{ (u(y) - u(x)) X_k \varepsilon(y, x) \right\} d\nu(y) \right| \\
&\leq C_1 \lim_{\epsilon \to 0} \int_{d(x,y) < \epsilon} |X_k \varepsilon(y, x)| d\nu(y) \\
&\quad + \lim_{\epsilon \to 0} \int_{d(x,y) < \epsilon} |(u(y) - u(x)) X_k X_k \varepsilon(y, x)| d\nu(y) \\
&\leq C \lim_{\epsilon \to 0} \int_0^{\epsilon} dr = 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int_{\Omega} X_k \{[u(y)-u(x)]X_k\varepsilon(y,x)\}\,d\nu(y)\\&=\int_{\Omega} X_k \{[u(y)-u(x)]X_k\varepsilon(y,x)\}\,d\nu(y)\\&\quad+\lim_{\epsilon\to 0}\int_{d(x,y)<\epsilon} X_k \{(u(y)-u(x))X_k\varepsilon(y,x)\}\,d\nu(y).\end{aligned}$$

If we take $\Omega_\epsilon = \{y \in \mathbb{G} : d(x,y) < \epsilon\}$, then by the divergence formula in Theorem 1.4.5 we have

$$\begin{aligned}&\lim_{x\to x_0}\int_{\partial\Omega}[u(y)-u(x)]X_k\varepsilon(y,x)\langle X_k, d\nu(y)\rangle\\&=\lim_{x\to x_0}\int_{\Omega} X_k \{[u(y)-u(x)]X_k\varepsilon(y,x)\}\,d\nu(y)\\&=\lim_{x\to x_0}\lim_{\epsilon\to 0}\Bigg\{\int_{\Omega\setminus\Omega_\epsilon} X_k \{[u(y)-u(x)]X_k\varepsilon(y,x)\}\,d\nu(y)\\&\quad+\int_{\Omega_\epsilon} X_k \{[u(y)-u(x)]X_k\varepsilon(y,x)\}\,d\nu(y)\Bigg\}\\&=\int_{\Omega} X_k \{[u(y)-u(x_0)]X_k\varepsilon(y,x_0)\}\,d\nu(y).\end{aligned}$$

That is, we have

$$\begin{aligned}&\lim_{x\to x_0}\int_{\partial\Omega}[u(y)-u(x)]X_k\varepsilon(y,x)\langle X_k, d\nu(y)\rangle\\&=\int_{\Omega} X_k \{[u(y)-u(x_0)]X_k\varepsilon(y,x_0)\}\,d\nu(y)\\&=\int_{\partial\Omega}[u(y)-u(x_0)]X_k\varepsilon(y,x_0)\langle X_k, d\nu(y)\rangle.\end{aligned}$$

Recalling the summation convention over the repeated index k, we get

$$\lim_{x\to x_0}\int_{\partial\Omega}[u(y)-u(x)]\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle=\int_{\partial\Omega}[u(y)-u(x_0)]\langle\widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle.$$

This proves the statement of Lemma 11.2.7. □

Proof of Theorem 11.2.6. We have

$$\begin{aligned}&\lim_{\substack{x\to x_0,\\ x\notin\partial\Omega}}\int_{\partial\Omega}u(y)\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle\\&=\lim_{\substack{x\to x_0,\\ x\notin\partial\Omega}}\left(\int_{\partial\Omega}[u(y)-u(x)]\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle+u(x)\int_{\partial\Omega}\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle\right).\end{aligned}$$

Choosing $u = \varepsilon$ and $v = 1$ in Green's first formula (1.86) we get

$$\int_{\partial\Omega} \langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \bar{\Omega}, \end{cases}$$

see Corollary 1.4.9. Therefore, using Lemma 11.2.7 we obtain

$$\mathcal{D}^{+}u(x_0) = \int_{\partial\Omega} [u(y) - u(x_0)] \langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle + u(x_0) \tag{11.19}$$

and

$$\mathcal{D}^{-}u(x_0) = \int_{\partial\Omega} [u(y) - u(x_0)] \langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle. \tag{11.20}$$

This gives the first jump relation

$$\mathcal{D}^{+}u(x_0) - \mathcal{D}^{-}u(x_0) = u(x_0).$$

We also have

$$\begin{aligned} \mathcal{D}^{0}u(x_0) &= \int_{\partial\Omega} u(y) \langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle \\ &= \int_{\partial\Omega} [u(y) - u(x_0)] \langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle \\ &\quad + u(x_0) \int_{\partial\Omega} \langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle \\ &= \int_{\partial\Omega} [u(y) - u(x_0)] \left\langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y) \right\rangle + \mathcal{J}(x_0)u(x_0). \end{aligned}$$

So we obtain

$$\mathcal{D}^{0}u(x_0) = \int_{\partial\Omega} [u(y) - u(x_0)] \left\langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y) \right\rangle + \mathcal{J}(x_0)u(x_0). \tag{11.21}$$

Now we get the second jump relation by subtracting (11.20) from (11.21), and subtracting (11.21) from (11.19) we obtain the third one. □

11.3 Traces and Kac's problem for the sub-Laplacian

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain. Then it is well known that the solution to the Poisson equation in Ω,

$$\Delta u(x) = f(x), \quad x \in \Omega, \tag{11.22}$$

is given by the Green formula (or the Newton potential formula)

$$u(x) = \int_{\Omega} \varepsilon_d(x-y) f(y) dy, \quad x \in \Omega, \tag{11.23}$$

for appropriate functions f supported in Ω. Here ε_d is the fundamental solution to Δ in $\mathbb{R}^d$ given by

$$\varepsilon_d(x-y) = \begin{cases} \frac{1}{(2-d)s_d}\frac{1}{|x-y|^{d-2}}, & d \geq 3, \\ \frac{1}{2\pi}\log|x-y|, & d=2, \end{cases} \tag{11.24}$$

where $s_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the surface area of the unit sphere in $\mathbb{R}^d$. A question related to the so-called Kac's boundary value problem in Ω is

> what boundary condition can be put on u on the (piecewise smooth) boundary $\partial\Omega$ so that equation (11.22) complemented by this boundary condition would have the solution in Ω still given by the same formula (11.23), with the same kernel ε_d given by (11.24)?

This question is equivalent to finding the trace of the Newton potential (11.23) on the boundary surface $\partial\Omega$.

It turns out that the answer to these questions is the integral boundary condition

$$-\frac{1}{2}u(x)+\int_{\partial\Omega}\frac{\partial\varepsilon_d(x-y)}{\partial n_y}u(y)dS_y-\int_{\partial\Omega}\varepsilon_d(x-y)\frac{\partial u(y)}{\partial n_y}dS_y=0,\ x\in\partial\Omega, \tag{11.25}$$

where $\frac{\partial}{\partial n_y}$ is the outer normal derivative at a point y on $\partial\Omega$. Thus, the trace of the Newton potential (11.23) on the boundary surface $\partial\Omega$ is determined by (11.25).

The boundary condition (11.25) and the subsequent spectral analysis are often called "the principle of not feeling the boundary" and it has originally appeared in M. Kac's work [Kac51], with further extensions in Kac's book [Kac80]. Such an analysis has several applications to the spectral theory and the asymptotics of Weyl's eigenvalue counting function. Spectral problems related to the boundary value problem (11.22), (11.25) were considered in the papers [KS09], [KS11], [RS16b] and [RRS16]. In general, the boundary value problem (11.22), (11.25) has several interesting properties and applications, for example discussed by Kac [Kac51, Kac80] and Saito [Sai08].

11.3.1 Traces of Newton potential for the sub-Laplacian

The analysis of the trace of the Newton potential or of related Kac's boundary value problems will be relying to certain results of Folland and Stein established in the setting of anisotropic Hölder spaces. We now define these spaces following [FS74] and [Fol75].

Definition 11.3.1 (Hölder spaces on stratified groups)**.** Let $0 < \alpha < 1$. Define the anisotropic Hölder spaces $\Gamma_\alpha(\Omega)$ on $\Omega \subset \mathbb{G}$ by

$$\Gamma_\alpha(\Omega) := \{f : \Omega \to \mathbb{C} : \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{[d(x,y)]^\alpha} < \infty\}.$$

For $k \in \mathbb{N}$ and $0 < \alpha < 1$, one defines $\Gamma_{k+\alpha}(\Omega)$ as the space of all $f : \Omega \to \mathbb{C}$ such that all derivatives of f of order k belong to $\Gamma_\alpha(\Omega)$. A bounded function f is called *α-Hölder continuous* in $\Omega \subset \mathbb{G}$ if $f \in \Gamma_\alpha(\Omega)$.

We now define the analogue of the Newton potential in the setting of stratified groups. Here and in the sequel we adopt all the notations of Section 1.4.3 and of the subsequent sections.

Definition 11.3.2 (Newton potential for sub-Laplacian)**.** Let $\Omega \subset \mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ in the sense of Definition 1.4.4. The *sub-Laplacian Newton potential* is defined by

$$u(x) = \int_\Omega f(y)\varepsilon(y,x)d\nu(y), \quad x \in \Omega, \quad f \in \Gamma_\alpha(\Omega), \tag{11.26}$$

where

$$\varepsilon(y,x) = \varepsilon(x,y) = \varepsilon(y^{-1}x, 0) = \varepsilon(y^{-1}x)$$

is the fundamental solution (1.89) of the sub-Laplacian $\mathcal{L}$, i.e.,

$$\varepsilon(x,y) = [d(x,y)]^{2-Q},$$

where $d(x,y) = d(y^{-1}x)$ is the $\mathcal{L}$-gauge on $\mathbb{G}$.

Remark 11.3.3. It was shown by Folland in [Fol75] that if $f \in \Gamma_\alpha(\Omega)$ for $\alpha > 0$ then u defined by (11.26) is twice differentiable and satisfies the equation

$$\mathcal{L}u = f.$$

The following theorem describes the trace of the integral operator in (11.26) on $\partial\Omega$. This is equivalent to finding a boundary condition for u such that the equation $\mathcal{L}u = f$ has a unique solution in $C^2(\Omega)$ and this solution is the Newton potential (11.26).

Theorem 11.3.4 (Trace of Newton potential and Kac's boundary value problem)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$. Let $\varepsilon(y,x) = \varepsilon(y^{-1}x)$ be the fundamental solution to $\mathcal{L}$, so that*

$$\mathcal{L}\varepsilon = \delta \quad \text{on } \mathbb{G}. \tag{11.27}$$

Then for all $f \in \Gamma_\alpha(\Omega)$, $0 < \alpha < 1$, supp$f \subset \Omega$*, the Newton potential (11.26) is the unique solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$ of the equation*

$$\mathcal{L}u = f \quad \text{in } \Omega, \tag{11.28}$$

with the boundary condition

$$\begin{aligned}(1 - \mathcal{J}(x))u(x) &+ \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle \\ &- \int_{\partial\Omega} \varepsilon(y,x)\langle \widetilde{\nabla}u(y), d\nu(y)\rangle = 0 \quad \text{for} \quad x \in \partial\Omega,\end{aligned} \tag{11.29}$$

where the jump value is given by the formula

$$\mathcal{J}(x) = \int_{\partial\Omega} \langle \widetilde{\nabla} \varepsilon(y,x), d\nu(y)\rangle, \tag{11.30}$$

with $\widetilde{\nabla} = \widetilde{\nabla}_y$ *defined by*

$$\widetilde{\nabla} g = \sum_{k=1}^{N_1} (X_k g) X_k.$$

Proof of Theorem 11.3.4. Since the Newton potential (11.26) is a solution of (11.28) it follows from Remark 11.3.3 that u is locally in $\Gamma_{\alpha+2}(\Omega, \mathrm{loc})$ and that it is twice differentiable in Ω. In particular, it follows that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

We will use the following representation formula for functions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ shown in Corollary 1.4.10, Part 1, for all $x \in \Omega$:

$$\begin{aligned} u(x) = \int_{\Omega} f(y)\varepsilon(y,x)d\nu(y) + \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle \\ - \int_{\partial\Omega} \varepsilon(y,x)\langle \widetilde{\nabla} u(y), d\nu(y)\rangle. \end{aligned} \tag{11.31}$$

Since u given by the Newton potential formula (11.26) is a solution of (11.28), using it in (11.31) we get, for all $x \in \Omega$,

$$\int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle - \int_{\partial\Omega} \varepsilon(y,x)\langle \widetilde{\nabla} u(y), d\nu(y)\rangle = 0. \tag{11.32}$$

When $x \in \Omega$ approaches the boundary $\partial\Omega$ from the interior we can use Theorem 11.2.4 and Theorem 11.2.6 yielding the identity

$$\begin{aligned} (1 - \mathcal{J}(x))u(x) + \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle \\ - \int_{\partial\Omega} \varepsilon(y,x)\langle \widetilde{\nabla} u(y), d\nu(y)\rangle = 0 \quad \text{for any } x \in \partial\Omega. \end{aligned} \tag{11.33}$$

It follows from (11.26) that u is then a solution of the boundary value problem (11.28) with the boundary condition (11.29).

Let us now prove its uniqueness. Assuming that u and u_1 are two solutions of the boundary value problem (11.28) and (11.29), their difference

$$w := u - u_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$$

satisfies the homogeneous equation

$$\mathcal{L} w = 0 \quad \text{in } \Omega, \tag{11.34}$$

and the boundary condition (11.29), i.e.,

$$\begin{aligned}(1-\mathcal{J}(x))w(x)+\int_{\partial\Omega} w(y)\langle\widetilde{\nabla}\varepsilon(y,x),d\nu(y)\rangle \\ -\int_{\partial\Omega}\varepsilon(y,x)\langle\widetilde{\nabla}w(y),d\nu(y)\rangle=0,\quad x\in\partial\Omega.\end{aligned} \tag{11.35}$$

The representation formula from Corollary 1.4.10, Part 2, yields the formula for w as

$$w(x)=\int_{\partial\Omega} w(y)\langle\widetilde{\nabla}\varepsilon(y,x),d\nu(y)\rangle-\int_{\partial\Omega}\varepsilon(y,x)\langle\widetilde{\nabla}w(y),d\nu(y)\rangle, \tag{11.36}$$

for any $x\in\Omega$. As above, when $x\in\Omega$ approaches the boundary $\partial\Omega$ from the interior we can use Theorem 11.2.4 and Theorem 11.2.6 by using the properties of the double and single layer potentials as $x\to\partial\Omega$ from interior, formula (11.36) implies that for every $x\in\partial\Omega$ we have

$$w(x)=(1-\mathcal{J}(x))w(x)+\int_{\partial\Omega} w(y)\langle\widetilde{\nabla}\varepsilon(y,x),d\nu(y)\rangle-\int_{\partial\Omega}\varepsilon(y,x)\langle\widetilde{\nabla}w,d\nu(y)\rangle.$$

Comparing this with (11.35) we obtain that w has to satisfy

$$w(x)=0,\ x\in\partial\Omega. \tag{11.37}$$

By Proposition 11.1.1 the boundary value problem (11.28) with the boundary condition (11.34) has a unique trivial solution $w\equiv 0$ in $C^2(\Omega)\cap C^1(\overline{\Omega})$ which shows that $u=u_1$ in Ω. □

11.3.2 Powers of the sub-Laplacian

We now extend the analysis of Section 11.3.1 to traces and boundary value problems for powers of the sub-Laplacian on stratified groups. Here we naturally understand the powers by

$$\mathcal{L}^m:=\mathcal{L}\mathcal{L}^{m-1}, \tag{11.38}$$

for all $m\in\mathbb{N}$.

Let $\Omega\subset\mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q\geq 3$. Then for $m=1,2,\ldots$, for a given $f\in\Gamma_\alpha(\Omega)$, we consider the equations

$$\mathcal{L}^m u(x)=f(x),\ x\in\Omega, \tag{11.39}$$

Definition 11.3.5 (Generalized Newton potentials)**.** Let $\varepsilon(y,x)=\varepsilon(y^{-1}x)$ be the fundamental solution of the sub-Laplacian $\mathcal{L}$ as in (1.89). We define the *generalized Newton potentials* by

$$u(x)=\int_\Omega f(y)\varepsilon_m(y,x)d\nu(y), \tag{11.40}$$

where $\varepsilon_m(y,x)$ is the fundamental solution of (11.39) such that

$$\mathcal{L}^{m-1}\varepsilon_m = \varepsilon, \quad m = 1, 2, \ldots.$$

Namely, with a proper distributional interpretation for $m = 2, 3, \ldots$, we take

$$\varepsilon_m(y,x) := \int_\Omega \varepsilon_{m-1}(y,\zeta)\varepsilon(\zeta,x)d\nu(\zeta), \qquad y,x \in \Omega, \tag{11.41}$$

with

$$\varepsilon_1(y,x) = \varepsilon(y,x).$$

We note that in general higher-order hypoelliptic operators on stratified groups may not have unique fundamental solutions, see Geller [Gel83], or [FR16, Section 3.2.7] for a detailed discussion. However, in the case of the iterated sub-Laplacian $\mathcal{L}^m$ we have uniqueness in the sense of the next theorem and of the uniqueness argument in its proof.

Theorem 11.3.6 (Traces of generalized Newton potentials and Kac's boundary value problem)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$. For any $f \in \Gamma_\alpha(\Omega)$, $0 < \alpha < 1$,* supp$f \subset \Omega$*, the generalized Newton potential* (11.40) *is a unique solution of the equation* (11.39) *in $C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ with m boundary conditions*

$$\begin{aligned}(1-\mathcal{J}(x))\mathcal{L}^i u(x) + \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{j+i}u(y)\langle\widetilde{\nabla}\mathcal{L}^{m-1-j}\varepsilon_m(y,x), d\nu(y)\rangle \\ - \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{j+i}u(y)d\nu(y)\rangle = 0, \quad x \in \partial\Omega,\end{aligned} \tag{11.42}$$

for all $i = 0, 1, \ldots, m-1$, where $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla} g = \sum_{k=1}^{N_1}(X_k g)X_k$$

and $\mathcal{J}(x)$ is the jump function given by the formula (11.30)*.*

Remark 11.3.7. From Theorem 11.3.6 we have that the kernel (11.41), which is a fundamental solution of the equation (11.39), is the Green function of the boundary value problem (11.39), (11.42) in Ω. This presents an example of an explicitly solvable boundary value problem for the iterated sub-Laplacian in any (admissible) domain Ω, which can be regarded as the higher order version of the original Kac's boundary value problem.

Proof of Theorem 11.3.6*.* For each $x \in \Omega$, applying Green's second formula from Theorem 1.4.6 we obtain

$$u(x) = \int_\Omega f(y)\varepsilon_m(y,x)d\nu(y) = \int_\Omega \mathcal{L}^m u(y)\varepsilon_m(y,x)d\nu(y)$$

$$
\begin{aligned}
&= \int_\Omega \mathcal{L}^{m-1}u(y)\mathcal{L}\varepsilon_m(y,x)d\nu(y) - \int_{\partial\Omega} \mathcal{L}^{m-1}u(y)\langle\widetilde{\nabla}\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-1}u(y), d\nu(y)\rangle \\
&= \int_\Omega \mathcal{L}^{m-2}u(y)\mathcal{L}^2\varepsilon_m(y,x)d\nu(y) - \int_{\partial\Omega} \mathcal{L}^{m-2}u(y)\langle\widetilde{\nabla}\mathcal{L}\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-2}u(y), d\nu(y)\rangle - \int_{\partial\Omega} \mathcal{L}^{m-1}u(y)\langle\widetilde{\nabla}\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-1}u(y), d\nu(y)\rangle \\
&= \cdots \\
&= u(x) - \sum_{j=0}^{m-1}\int_{\partial\Omega} \mathcal{L}^j u(y)\langle\widetilde{\nabla}\mathcal{L}^{m-1-j}\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \sum_{j=0}^{m-1}\int_{\partial\Omega} \mathcal{L}^{m-1-j}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^j u(y), d\nu(y)\rangle, \quad x \in \Omega.
\end{aligned}
$$

This implies the identity

$$
\begin{aligned}
&\sum_{j=0}^{m-1}\int_{\partial\Omega} \mathcal{L}^j u(y)\langle\widetilde{\nabla}\mathcal{L}^{m-1-j}\varepsilon_m(y,x), d\nu(y)\rangle \\
&- \sum_{j=0}^{m-1}\int_{\partial\Omega} \mathcal{L}^{m-1-j}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^j u(y), d\nu(y)\rangle = 0, \quad x \in \Omega.
\end{aligned}
\tag{11.43}
$$

We note that the first term of the first summand, i.e., the term with $j = 0$ given by

$$
\int_{\partial\Omega} u(y)\langle\widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle
$$

is the double layer potential analysed in Theorem 11.2.6. The other terms in the above sum are single layer type potentials. In particular, by Theorem 11.2.4 they are continuous functions on $\mathbb{G}$. By using the properties of the double and single layer potentials as x approaches the boundary $\partial\Omega$ from the interior, from (11.43) we obtain the equality

$$
\begin{aligned}
(1-\mathcal{J}(x))u(x) &+ \sum_{j=0}^{m-1}\int_{\partial\Omega} \mathcal{L}^j u(y)\langle\widetilde{\nabla}\mathcal{L}^{m-1-j}\varepsilon_m(y,x), d\nu(y)\rangle \\
&- \sum_{j=0}^{m-1}\int_{\partial\Omega} \mathcal{L}^{m-1-j}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^j u(y), d\nu(y)\rangle = 0, \quad x \in \partial\Omega.
\end{aligned}
$$

From this relation we obtain the first boundary conditions for (11.40). The remaining boundary conditions can be derived by writing

$$\mathcal{L}^{m-i}\mathcal{L}^i u = f, \quad i = 0, 1, \ldots, m-1, \quad m = 1, 2, \ldots, \tag{11.44}$$

and carrying out similar arguments as above. Indeed, we have

$$\begin{aligned}
\mathcal{L}^i u(x) &= \int_\Omega f(y)\mathcal{L}^i \varepsilon_m(y,x)d\nu(y) = \int_\Omega \mathcal{L}^{m-i}\mathcal{L}^i u(y)\mathcal{L}^i\varepsilon_m(y,x)d\nu(y) \\
&= \int_\Omega \mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\mathcal{L}\mathcal{L}^i\varepsilon_m(y,x)d\nu(y) \\
&\quad - \int_{\partial\Omega} \mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\langle\widetilde{\nabla}\mathcal{L}^i\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}^i\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-i-1}\mathcal{L}^i u(y), d\nu(y)\rangle \\
&= \int_\Omega \mathcal{L}^{m-i-2}\mathcal{L}^i u(y)\mathcal{L}^2\mathcal{L}^i\varepsilon_m(y,x)d\nu(y) \\
&\quad - \int_{\partial\Omega} \mathcal{L}^{m-i-2}\mathcal{L}^i u(y)\langle\widetilde{\nabla}\mathcal{L}\mathcal{L}^i\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}\mathcal{L}^i\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-i-2}\mathcal{L}^i u(y), d\nu(y)\rangle \\
&\quad - \int_{\partial\Omega} \mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\langle\widetilde{\nabla}\mathcal{L}^i\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}^i\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-i-1}\mathcal{L}^i u(y), d\nu(y)\rangle \\
&= \cdots \\
&= \int_\Omega \mathcal{L}^i u(y)\mathcal{L}^{m-i}\mathcal{L}^i\varepsilon_m(y,x)d\nu(y) \\
&\quad - \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}^j\mathcal{L}^i u(y)\langle\widetilde{\nabla}\mathcal{L}^{m-i-1-j}\mathcal{L}^i\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}^{m-i-1-j}\mathcal{L}^i\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^j\mathcal{L}^i u(y), d\nu(y)\rangle \\
&= \mathcal{L}^i u(x) - \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}^{j+i}u(y)\langle\widetilde{\nabla}\mathcal{L}^{m-1-j}\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}^{m-1-j}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{j+i}u(y), d\nu(y)\rangle, \quad x \in \Omega,
\end{aligned}$$

where $\mathcal{L}^i\varepsilon_m$ is a fundamental solution of the equation (11.44), that is,

$$\mathcal{L}^{m-i}\mathcal{L}^i\varepsilon_m = \delta, \qquad i = 0, 1, \ldots, m-1.$$

From the previous relations, for all $x \in \Omega$ and $i = 0, 1, \ldots, m-1$, we get the identities

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{j+i} u(y) \langle \widetilde{\nabla} \mathcal{L}^{m-1-j} \varepsilon_m(y,x), d\nu(y) \rangle - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{m-1-j} \varepsilon_m(y,x) \langle \widetilde{\nabla} \mathcal{L}^{j+i} u(y), d\nu(y) \rangle = 0.$$

By using the properties of the double and single layer potentials in Theorem 11.2.6 and Theorem 11.2.4 as x approaches the boundary $\partial\Omega$ from the interior of Ω, we find that

$$(1 - \mathcal{J}(x)) \mathcal{L}^i u(x) + \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{j+i} u(y) \langle \widetilde{\nabla} \mathcal{L}^{m-1-j} \varepsilon_m(y,x), d\nu(y) \rangle - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{m-1-j} \varepsilon_m(y,x) \langle \widetilde{\nabla} \mathcal{L}^{j+i} u(y), d\nu(y) \rangle = 0, \quad x \in \partial\Omega,$$

are all the boundary conditions of (11.40) for each $i = 0, 1, \ldots, m-1$.

Let us now show the uniqueness: if a function $w \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ satisfies the equation $\mathcal{L}^m w = f$ and the boundary conditions (11.42), then it must coincide with the solution (11.40). Indeed, the function

$$v := u - w \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega}),$$

where u is the generalized Newton potential (11.40), satisfies the homogeneous equation

$$\mathcal{L}^m v = 0 \tag{11.45}$$

and the boundary conditions (11.42), i.e.,

$$I_i(v)(x) := (1 - \mathcal{J}(x)) \mathcal{L}^i v(x) + \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{j+i} v(y) \langle \widetilde{\nabla} \mathcal{L}^{m-1-j} \varepsilon_m(y,x), d\nu(y) \rangle - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{m-1-j} \varepsilon_m(y,x) \langle \widetilde{\nabla} \mathcal{L}^{j+i} v(y), d\nu(y) \rangle = 0, \quad i = 0, 1, \ldots, m-1,$$

for $x \in \partial\Omega$. By applying the Green formula from Theorem 1.4.6 to the function $v \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ and by following the lines of the previous calculation we obtain

$$0 = \int_{\Omega} \mathcal{L}^m v(x) \mathcal{L}^i \varepsilon_m(y,x) d\nu(y) = \int_{\Omega} \mathcal{L}^{m-i} \mathcal{L}^i v(x) \mathcal{L}^i \varepsilon_m(y,x) d\nu(y)$$

$$
\begin{aligned}
&= \int_\Omega \mathcal{L}^{m-1}v(x)\mathcal{L}\mathcal{L}^i\varepsilon_m(y,x)d\nu(y) - \int_{\partial\Omega} \mathcal{L}^{m-1}v(x)\langle\widetilde{\nabla}\mathcal{L}^i\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}^i\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-1}v(x), d\nu(y)\rangle \\
&= \int_\Omega \mathcal{L}^{m-2}v(x)\mathcal{L}^2\mathcal{L}^i\varepsilon_m(y,x)d\nu(y) - \int_{\partial\Omega} \mathcal{L}^{m-2}v(x)\langle\widetilde{\nabla}\mathcal{L}^{i+1}\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}^{i+1}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-2}v(x), d\nu(y)\rangle - \int_{\partial\Omega} \mathcal{L}^{m-1}v(x)\langle\widetilde{\nabla}\mathcal{L}^i\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}^i\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{m-1}v(x), d\nu(y)\rangle \\
&= \cdots \\
&= \mathcal{L}^i v(x) - \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}^{j+i}v(y)\langle\widetilde{\nabla}\mathcal{L}^{m-1-j}\varepsilon_m(y,x), d\nu(y)\rangle \\
&\quad + \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}^{m-1-j}\varepsilon_m(y,x)\langle\widetilde{\nabla}\mathcal{L}^{j+i}v(y), d\nu(y)\rangle, \quad i = 0,1,\ldots,m-1.
\end{aligned}
$$

By passing to the limit as $x \to \partial\Omega$ from interior, we obtain the relations

$$
\mathcal{L}^i v(x)\ |_{x\in\partial\Omega} = I_i(v)(x)\ |_{x\in\partial\Omega} = 0, \quad i = 0,1,\ldots,m-1. \tag{11.46}
$$

We claim that the boundary value problem

$$
\begin{gathered}
\mathcal{L}^m v = 0, \\
\mathcal{L}^i v\ |_{\partial\Omega} = 0, \quad i = 0,1,\ldots,m-1,
\end{gathered} \tag{11.47}
$$

has uniqueness trivial solution $v \equiv 0$ in $C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$.

Assuming this claim for the moment we get that $v = u - w \equiv 0$, for all $x \in \Omega$, i.e., w coincides with u in Ω. Thus (11.40) is the unique solution of the boundary value problem (11.39), (11.42) in Ω.

Let us now show the above claim, that is, that the boundary value problem (11.47) has a unique solution in $C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$. Denoting

$$
\tilde{v} := \mathcal{L}^{m-1}v,
$$

the claim follows by induction from the uniqueness in $C^2(\Omega)\cap C^1(\overline{\Omega})$ of the boundary value problem

$$
\begin{gathered}
\mathcal{L}\tilde{v} = 0, \\
\tilde{v}\ |_{\partial\Omega} = 0.
\end{gathered}
$$

The proof of Theorem 11.3.6 is complete. □

11.3.3 Extended Kohn Laplacians on the Heisenberg group

In this section we describe more explicit formulae for the traces of the Newton potential presented in Section 11.3.1 in the setting of the Heisenberg group. In this analysis we adapt the complex description of the Heisenberg group as described in Section 1.4.8, so we follow all the notation introduced there. The presentation in this and the following sections follows [RS16c].

In particular, we will use the coordinates on the Heisenberg group $\mathbb{H}^n$ given by variables $z = (\zeta, t)$, so that a basis for the complex tangent space of $\mathbb{H}^n$ at the point z is given by the left invariant vector fields

$$X_j = \frac{\partial}{\partial z_j} + i\overline{z}\frac{\partial}{\partial t}, \quad j = 1, \ldots, n.$$

Denoting their conjugates by

$$X_{\overline{j}} := \overline{X}_j = \frac{\partial}{\partial \overline{z}_j} - iz\frac{\partial}{\partial t},$$

we will be dealing with the extended Kohn Laplacian

$$\mathcal{L}_{a,b} = \sum_{j=1}^{n}(aX_jX_{\overline{j}} + bX_{\overline{j}}X_j), \quad a + b = n. \tag{11.48}$$

As described in Section 1.4.8, this operator has a fundamental solution given by a constant multiple of (1.110). Moreover, in [FS74], Folland and Stein defined the Newton potential (volume potential) for a function f with compact support contained in a set $\Omega \subset \mathbb{H}_n$ by

$$u(z) = \int_{\Omega} f(\xi)\varepsilon(\xi^{-1}z)d\nu(\xi), \tag{11.49}$$

with $d\nu$ being the volume element: this is given by the Haar measure on $\mathbb{H}_n$ which in turn coincides with the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$. More precisely, they proved that

$$\mathcal{L}_{a,b}u = c_{a,b}f,$$

where the constant $c_{a,b}$ is zero if a and $b = -1, -2, \ldots, n, n+1, \ldots,$ and $c_{a,b} \neq 0$ if a or $b \neq -1, -2, \ldots, n, n+1, \ldots,$ given by (1.112) for $a \notin \mathbb{Z}$. For different approaches to the fundamental solutions of $\mathcal{L}_{a,b}$ we can also refer to Greiner and Stein [GS77].

With the notation for $\varepsilon_{a,b} = \varepsilon$ as in (1.110) the distribution $\frac{1}{c_{a,b}}\varepsilon$ is the fundamental solution of $\mathcal{L}_{a,b}$, while ε satisfies the equation

$$\mathcal{L}_{a,b}\varepsilon = c_{a,b}\delta. \tag{11.50}$$

In order to make our presentation compatible with the accepted ones in [FS74] and [Rom91], in this section we will also work with the function ε although there is a constant $c_{a,b}$ appearing in the identity (11.50).

Therefore, throughout this and next sections we assume that $c_{a,b} \neq 0$, which means that a and b satisfy the condition

$$a \text{ and } b \neq -1, -2, \ldots, n, n+1, \ldots.$$

In addition, without loss of generality we may also assume that $a, b \geq 0$.

We recall the anisotropic Hölder space $\Gamma_\alpha(\Omega)$ from Definition 11.3.1. Since all homogeneous quasi-norms on homogeneous groups are equivalent for such a definition we can also use an explicitly defined distance on $\mathbb{H}^n$ given by

$$|z| := (|\zeta|^4 + |t|^2)^{1/4}.$$

Thus, for $0 < \alpha < 1$, the anisotropic Hölder spaces $\Gamma_\alpha(\Omega)$ from Definition 11.3.1 can be also described by

$$\Gamma_\alpha(\Omega) = \left\{ f : \Omega \to \mathbb{C} : \sup_{\substack{z_1, z_2 \in \Omega \\ z_1 \neq z_2}} \frac{|f(z_2) - f(z_1)|}{|z_2^{-1} z_1|^\alpha} < \infty \right\}.$$

Remark 11.3.8. In addition to Remark 11.3.3, in the setting of the Heisenberg groups we have that if $f \in \Gamma_\alpha(\Omega)$ for $\alpha > 0$ then u defined by (12.43) is twice differentiable in the complex directions and satisfies the equation

$$\mathcal{L}_{a,b} u = c_{a,b} f.$$

There are several different approaches to show this result by Folland and Stein [FS74], Greiner and Stein [GS77], and Romero [Rom91]. Moreover, Folland and Stein have shown that if $f \in \Gamma_\alpha(\Omega, \mathrm{loc})$ and $\mathcal{L}_{a,b} u = c_{a,b} f$, then $f \in \Gamma_{\alpha+2}(\Omega, \mathrm{loc})$, for naturally defined localisations $\Gamma_\alpha(\Omega, \mathrm{loc})$ of the anisotropic Hölder spaces $\Gamma_\alpha(\Omega)$.

We will be using the single layer potentials introduced in Definition 11.2.1, namely, the operators

$$S_j g(z) = \int_{\partial\Omega} g(\xi) \varepsilon(\xi, z) \langle X_j, d\nu(\xi) \rangle,$$

where $\langle X, d\nu \rangle$ is the canonical pairing between vector fields and differential forms, which are well defined by Theorem 11.2.3. Moreover, specifically in the setting of the Heisenberg group, it was shown in [Rom91, Theorem 2.3] that if the density of $g(\xi)\langle X_j, d\nu \rangle$ in the operator S_j is bounded then $Sg \in \Gamma_\alpha(\mathbb{H}_n)$ for all $\alpha < 1$.

The double layer potentials have been introduced in Definition 11.2.5 on general stratified groups. However, it will be now convenient to adapt its definition slightly to the extended Kohn Laplacians $\mathcal{L}_{a,b}$. Thus, for every differentiable function g we define the vector field

$$\nabla^{a,b} g = \sum_{j=1}^{n-1} (a X_j g X_{\bar{j}} + b X_{\bar{j}} g X_j). \tag{11.51}$$

Consequently, we will be working with the double layer potential

$$Wu(z) = \int_{\partial\Omega} u(\xi)\langle \nabla^{a,b}\varepsilon(\xi, z), d\nu(\xi)\rangle. \tag{11.52}$$

Let us now record the Plemelj jump relations for double layer potential which follow from Theorem 11.2.6.

Corollary 11.3.9. *The double layer potential Wu in (11.52) has two limits*

$$W^{+}u(z) = \lim_{\substack{z_0\to z\\ z_0\in\Omega}} \int_{\partial\Omega} u(\xi)\langle \nabla^{a,b}\varepsilon(\xi, z_0), d\nu(\xi)\rangle$$

and

$$W^{-}u(z) = \lim_{\substack{z_0\to z\\ z_0\notin\Omega}} \int_{\partial\Omega} u(\xi)\langle \nabla^{a,b}\varepsilon(\xi, z_0), d\nu(\xi)\rangle,$$

and the principal value

$$W^{0}u(z) = \text{p.v. } Wu(z) = \lim_{\delta\to 0} \int_{\partial\Omega\setminus\{|\xi^{-1}z|<\delta\}} u(\xi)\langle \nabla^{a,b}\varepsilon(\xi, z), d\nu(\xi)\rangle.$$

For $u \in \Gamma_\alpha(\Omega)$ and $z \in \partial\Omega$ the above limits exist and satisfy the jump relations

$$\begin{aligned} W^{+}u(z) - W^{-}u(z) &= c_{a,b}u(z),\\ W^{0}u(z) - W^{-}u(z) &= H.R.(z)u(z),\\ W^{+}u(z) - W^{0}u(z) &= (c_{a,b} - H.R.(z))u(z), \end{aligned} \tag{11.53}$$

where $H.R(z)$ is the so-called half-residue given by the formula

$$H.R(z) = \lim_{\delta\to 0} \int_{\partial\Omega\setminus\{|\xi^{-1}z|<\delta\}} \langle \nabla^{a,b}\varepsilon(\xi, z), d\nu(\xi)\rangle. \tag{11.54}$$

The first two jump relations in (11.53) follow by an adaptation of the argument in Theorem 11.2.6 keeping in mind the appearing constant $c_{a,b}$. The third jump relation in (11.53) follows by subtraction of the first two. The statement of Corollary 11.3.9 was shown in [Rom91, Theorem 2.4].

Then we have the following analogue of Theorem 11.3.4.

Theorem 11.3.10 (Trace of Newton potential and Kac's boundary value problem for $\mathcal{L}_{a,b}$)**.** *Let $\varepsilon(\xi, z) = \varepsilon(\xi^{-1}z)$ be the rescaled fundamental solution to $\mathcal{L}_{a,b}$, so that*

$$\mathcal{L}_{a,b}\varepsilon = c_{a,b}\delta \quad \text{on } \mathbb{H}_n. \tag{11.55}$$

For any $f \in \Gamma_\alpha(\Omega)$, $0 < \alpha < 1$, $\operatorname{supp} f \subset \Omega$, the Newton potential (12.43) *is the unique solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$ of the equation*

$$\mathcal{L}_{a,b}u = c_{a,b}f \tag{11.56}$$

with the boundary condition

$$(c_{a,b} - H.R(z))u(z) + \lim_{\delta\to 0}\int_{\partial\Omega\setminus\{|\xi^{-1}z|<\delta\}} u(\xi)\langle\nabla^{a,b}\varepsilon(\xi,z), d\nu(\xi)\rangle \\ - \int_{\partial\Omega}\varepsilon(\xi,z)\langle\nabla^{b,a}u(\xi), d\nu(\xi)\rangle = 0, \qquad \text{for } z\in\partial\Omega, \tag{11.57}$$

where $H.R(z)$ is the so-called half-residue given by (11.54).

Proof of Theorem 11.3.10. Since the Newton potential

$$u(z) = \int_{\Omega} f(\xi)\varepsilon(\xi,z)d\nu(\xi) \tag{11.58}$$

is a solution of (11.56), from Remark 11.3.8 it follows that u is locally in $\Gamma_{\alpha+2}(\Omega,\text{loc})$ and that it is twice complex differentiable in Ω. In particular, it follows that $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$.

From Green's second formula in Theorem 1.4.6, similarly to Corollary 1.4.10, Part 1, by taking into account constants $c_{a,b}$, for every $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ and every $z\in\Omega$ we have the representation formula

$$c_{a,b}u(z) = c_{a,b}\int_{\Omega} f(\xi)\varepsilon(\xi,z)d\nu(\xi) + \int_{\partial\Omega} u(\xi)\langle\nabla^{a,b}\varepsilon(\xi,z), d\nu(\xi)\rangle \\ - \int_{\partial\Omega}\varepsilon(\xi,z)\langle\nabla^{b,a}u(\xi), d\nu(\xi)\rangle. \tag{11.59}$$

Since $u(z)$ given by (11.58) is a solution of (11.56), using it in (11.59), for every $z\in\Omega$ we have

$$\int_{\partial\Omega} u(\xi)\langle\nabla^{a,b}\varepsilon(\xi,z), d\nu(\xi)\rangle - \int_{\partial\Omega}\varepsilon(\xi,z)\langle\nabla^{b,a}u(\xi), d\nu(\xi)\rangle = 0. \tag{11.60}$$

The fundamental solution $\varepsilon(z)$ is homogeneous of degree $2-Q=-2n$; this is a general fact (see, e.g., [FR16, Theorem 3.2.40]), or can be seen directly from formula (1.110) since

$$\varepsilon(\lambda z) = \lambda^{-2a-2b}\varepsilon(z) = \lambda^{-2n}\varepsilon(z) \quad \text{for any } \lambda>0,$$

since $a+b=n$. It follows that ε and its first-order complex derivatives are locally integrable. Since $\varepsilon(\xi,z)=\varepsilon(\xi^{-1}z)$, we obtain that as z approaches the boundary, we can pass to the limit in the second term in (11.60).

By using this and the relation (11.53) as $z\in\Omega$ approaches the boundary $\partial\Omega$ from the interior, by passing to the limit, for every $z\in\partial\Omega$ we get

$$(c_{a,b} - H.R(z))u(z) + \lim_{\delta\to 0}\int_{\partial\Omega\setminus\{|\xi^{-1}z|<\delta\}} u(\xi)\langle\nabla^{a,b}\varepsilon(\xi,z), d\nu(\xi)\rangle \\ - \int_{\partial\Omega}\varepsilon(\xi,z)\langle\nabla^{b,a}u(\xi), d\nu(\xi)\rangle = 0. \tag{11.61}$$

This shows that (11.58) is a solution of the boundary value problem (11.56) with the boundary condition (11.57).

Now let us show the uniqueness which can be done similarly to the proof of the uniqueness in Theorem 11.3.4. If the boundary value problem (11.56)–(11.57) has two solutions u and u_1 then the function $w := u - u_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies the homogeneous equation

$$\mathcal{L}_{a,b} w = 0 \quad \text{in } \Omega, \tag{11.62}$$

and the boundary condition (11.57), that is,

$$\begin{aligned}(c_{a,b} - H.R(z))w(z) + \lim_{\delta \to 0} \int_{\partial\Omega \setminus \{|\xi^{-1} z| < \delta\}} w(\xi) \langle \nabla^{a,b} \varepsilon(\xi, z), d\nu(\xi) \rangle \\ - \int_{\partial\Omega} \varepsilon(\xi, z) \langle \nabla^{b,a} w(\xi), d\nu(\xi) \rangle = 0,\end{aligned} \tag{11.63}$$

for all $z \in \partial\Omega$.

Using the representation formula (11.59) for solutions with $f \equiv 0$ we have the following representation formula for w:

$$c_{a,b} w(z) = \int_{\partial\Omega} w(\xi) \langle \nabla^{a,b} \varepsilon(\xi, z), d\nu(\xi) \rangle - \int_{\partial\Omega} \varepsilon(\xi, z) \langle \nabla^{b,a} w(\xi), d\nu(\xi) \rangle$$

for all $z \in \Omega$. As in the existence proof, by using the properties of the double and single layer potentials as $z \to \partial\Omega$, we obtain that

$$\begin{aligned}c_{a,b} w(z) = (c_{a,b} - H.R(z))w(z) + \lim_{\delta \to 0} \int_{\partial\Omega \setminus \{|\xi^{-1} z| < \delta\}} w(\xi) \langle \nabla^{a,b} \varepsilon(\xi, z), d\nu(\xi) \rangle \\ - \int_{\partial\Omega} \varepsilon(\xi, z) \langle \nabla^{b,a} w, d\nu(\xi) \rangle\end{aligned}$$

for all $z \in \partial\Omega$. Comparing this with (11.63) we see that w must satisfy

$$w(z) = 0, \; z \in \partial\Omega. \tag{11.64}$$

By an argument similar to that in Proposition 11.1.1, the homogeneous equation (11.62) with the Dirichlet boundary condition (11.64) has a unique trivial solution $w \equiv 0$ in Ω. Therefore, we must have $u = u_1$ in Ω. This completes the proof of Theorem 11.3.10. □

11.3.4 Powers of the Kohn Laplacian

In this section, we carry out the analysis similar to that in Section 11.3.2 for the powers of the extended Kohn Laplacian $\mathcal{L}_{a,b}$.

As before, let Ω be an admissible domain in $\mathbb{H}^n$. For $m \in \mathbb{N}$, we denote

$$\mathcal{L}_{a,b}^m := \mathcal{L}_{a,b} \mathcal{L}_{a,b}^{m-1}.$$

Then for $m \in \mathbb{N}$, we consider the equation

$$\mathcal{L}_{a,b}^m u(z) = c_{a,b} f(z), \ z \in \Omega. \tag{11.65}$$

Let $\varepsilon(\xi, z) = \varepsilon(\xi^{-1} z)$ be the rescaled fundamental solution of the Kohn Laplacian as in (11.55). Let us now define the Newton potential for the operator $\mathcal{L}_{a,b}^m$ by

$$u(z) = \int_\Omega f(\xi)\varepsilon_m(\xi, z)d\nu(\xi), \tag{11.66}$$

where $\varepsilon_m(\xi, z)$ is a rescaled fundamental solution of (11.65) satisfying

$$\mathcal{L}_{a,b}^{m-1}\varepsilon_m = \varepsilon.$$

With a suitable distributional interpretation, for $m = 2, 3, \ldots$, we can write

$$\varepsilon_m(\xi, z) = \int_\Omega \varepsilon_{m-1}(\xi, \zeta)\varepsilon(\zeta, z)d\nu(\zeta), \qquad \xi, z \in \Omega, \tag{11.67}$$

with

$$\varepsilon_1(\xi, z) = \varepsilon(\xi, z).$$

Theorem 11.3.11 (Traces and Kac's boundary value problem for powers of Kohn Laplacians $\mathcal{L}_{a,b}$)**.** *For any $f \in \Gamma_\alpha(\Omega)$, $0 < \alpha < 1$,* supp$f$ $\subset \Omega$, *the generalized Newton potential* (11.66) *is a unique solution of the equation* (11.65) *in* $C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ *with m boundary conditions*

$$\begin{aligned} &(c_{a,b} - H.R(z))\mathcal{L}_{a,b}^i u(z) \\ &\quad + \sum_{j=0}^{m-i-1} \lim_{\delta \to 0} \int_{\partial\Omega \setminus \{|\xi^{-1}z| < \delta\}} \mathcal{L}_{a,b}^{j+i} u(\xi)\langle \nabla^{a,b}\mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi, z), d\nu(\xi)\rangle \\ &\quad - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi, z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{j+i} u(\xi) d\nu(\xi)\rangle = 0, \quad z \in \partial\Omega, \end{aligned} \tag{11.68}$$

for $i = 0, 1, \ldots, m-1$, where $H.R(z)$ is the half-residue given by (11.54).

Proof of Theorem 11.3.11*.* Applying Green's second formula from Theorem 1.4.6 for each $z \in \Omega$, similarly to (11.59) we get

$$\begin{aligned} c_{a,b}u(z) &= c_{a,b}\int_\Omega f(\xi)\varepsilon_m(\xi, z)d\nu(\xi) \\ &= \int_\Omega \mathcal{L}_{a,b}^m u(\xi)\varepsilon_m(\xi, z)d\nu(\xi) = \int_\Omega \mathcal{L}_{a,b}^{m-1} u(\xi)\mathcal{L}_{a,b}\varepsilon_m(\xi, z)d\nu(\xi) \\ &\quad - \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1} u(\xi)\langle \nabla^{a,b}\varepsilon_m(\xi, z), d\nu(\xi)\rangle + \int_{\partial\Omega} \varepsilon_m(\xi, z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{m-1} u(\xi), d\nu(\xi)\rangle \end{aligned}$$

$$= \int_\Omega \mathcal{L}_{a,b}^{m-2} u(\xi) \mathcal{L}_{a,b}^2 \varepsilon_m(\xi,z) d\nu(\xi) - \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-2} u(\xi) \langle \nabla^{a,b} \mathcal{L}_{a,b} \varepsilon_m(\xi,z), d\nu(\xi)\rangle$$
$$+ \int_{\partial\Omega} \mathcal{L}_{a,b} \varepsilon_m(\xi,z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{m-2} u(\xi), d\nu(\xi)\rangle - \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1} u(\xi) \langle \nabla^{a,b} \varepsilon_m(\xi,z), d\nu(\xi)\rangle$$
$$+ \int_{\partial\Omega} \varepsilon_m(\xi,z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{m-1} u(\xi), d\nu(\xi)\rangle = \cdots$$
$$= c_{a,b} u(z) - \sum_{j=0}^{m-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{j} u(\xi) \langle \nabla^{a,b} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi,z), d\nu(\xi)\rangle$$
$$+ \sum_{j=0}^{m-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi,z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{j} u(\xi), d\nu(\xi)\rangle, \quad z \in \Omega.$$

It follows that

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{j} u(\xi) \langle \nabla^{a,b} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi,z), d\nu(\xi)\rangle$$
$$- \sum_{j=0}^{m-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi,z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{j} u(\xi), d\nu(\xi)\rangle = 0, \quad z \in \Omega. \tag{11.69}$$

By using the continuity of the single layer potential and Corollary 11.3.9 for the double layer potential as z approaches the boundary $\partial\Omega$ from the interior, from (11.69) we obtain

$$(c_{a,b} - H.R(z))u(z) + \sum_{j=0}^{m-1} \lim_{\delta\to 0} \int_{\partial\Omega \setminus \{|\xi^{-1}z|<\delta\}} \mathcal{L}_{a,b}^{j} u(\xi) \langle \nabla^{a,b} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi,z), d\nu(\xi)\rangle$$
$$- \sum_{j=0}^{m-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi,z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{j} u(\xi), d\nu(\xi)\rangle = 0, \; z \in \partial\Omega.$$

This relation is one of the boundary conditions (11.68). Let us now derive the remaining boundary conditions. For this, we will rewrite the equation as

$$\mathcal{L}_{a,b}^{m-i} \mathcal{L}_{a,b}^{i} u = c_{a,b} f, \quad i = 0, 1, \ldots, m-1, \quad m = 1, 2, \ldots, \tag{11.70}$$

and then use a similar argument. Thus,

$$c_{a,b} \mathcal{L}_{a,b}^{i} u(z) = c_{a,b} \int_\Omega f(\xi) \mathcal{L}_{a,b}^{i} \varepsilon_m(\xi,z) d\nu(\xi) = \int_\Omega \mathcal{L}_{a,b}^{m-i} \mathcal{L}_{a,b}^{i} u(\xi) \mathcal{L}_{a,b}^{i} \varepsilon_m(\xi,z) d\nu(\xi)$$
$$= \int_\Omega \mathcal{L}_{a,b}^{m-i-1} \mathcal{L}_{a,b}^{i} u(\xi) \mathcal{L}_{a,b} \mathcal{L}_{a,b}^{i} \varepsilon_m(\xi,z) d\nu(\xi)$$
$$- \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-i-1} \mathcal{L}_{a,b}^{i} u(\xi) \langle \nabla^{a,b} \mathcal{L}_{a,b}^{i} \varepsilon_m(\xi,z), d\nu(\xi)\rangle$$
$$+ \int_{\partial\Omega} \mathcal{L}_{a,b}^{i} \varepsilon_m(\xi,z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{m-i-1} \mathcal{L}_{a,b}^{i} u(\xi), d\nu(\xi)\rangle$$

$$
\begin{aligned}
&= \int_\Omega \mathcal{L}_{a,b}^{m-i-2}\mathcal{L}_{a,b}^{i}u(\xi)\mathcal{L}_{a,b}^{2}\mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z)d\nu(\xi) \\
&\quad - \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-i-2}\mathcal{L}_{a,b}^{i}u(\xi)\langle \nabla^{a,b}\mathcal{L}_{a,b}\mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z), d\nu(\xi)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}_{a,b}\mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{m-i-2}\mathcal{L}_{a,b}^{i}u(\xi), d\nu(\xi)\rangle \\
&\quad - \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-i-1}\mathcal{L}_{a,b}^{i}u(\xi)\langle \nabla^{a,b}\mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z), d\nu(\xi)\rangle \\
&\quad + \int_{\partial\Omega} \mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{m-i-1}\mathcal{L}_{a,b}^{i}u(\xi), d\nu(\xi)\rangle \\
&= \cdots \\
&= \int_\Omega \mathcal{L}_{a,b}^{i}u(\xi)\mathcal{L}_{a,b}^{m-i}\mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z)d\nu(\xi) \\
&\quad - \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}_{a,b}^{j}\mathcal{L}_{a,b}^{i}u(\xi)\langle \nabla^{a,b}\mathcal{L}_{a,b}^{m-i-1-j}\mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z), d\nu(\xi)\rangle \\
&\quad + \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}_{a,b}^{m-i-1-j}\mathcal{L}_{a,b}^{i}\varepsilon_m(\xi,z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{j}\mathcal{L}_{a,b}^{i}u(\xi), d\nu(\xi)\rangle \\
&= c_{a,b}\mathcal{L}_{a,b}^{i}u(z) - \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}_{a,b}^{j+i}u(\xi)\langle \nabla^{a,b}\mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi,z), d\nu(\xi)\rangle \\
&\quad + \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi,z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{j+i}u(\xi), d\nu(\xi)\rangle, \quad z\in\Omega,
\end{aligned}
$$

where we used that $\mathcal{L}_{a,b}^{i}\varepsilon_m$ is a rescaled fundamental solution of $\mathcal{L}_{a,b}^{m-i}$ as in (11.70).

This implies the identities

$$
\begin{aligned}
&\sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}_{a,b}^{j+i}u(\xi)\langle \nabla^{a,b}\mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi,z), d\nu(\xi)\rangle \\
&- \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi,z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{j+i}u(\xi), d\nu(\xi)\rangle = 0,
\end{aligned}
$$

for all $z\in\Omega$ and $i=0,1,\ldots,m-1$. As before, by using the continuity of the single layer potential and Corollary 11.3.9 for the double layer potential as z approaches the boundary $\partial\Omega$ from the interior, we find that for all $z\in\partial\Omega$ we have

$$
\begin{aligned}
&(c_{a,b} - H.R(z))\mathcal{L}_{a,b}^{i}u(z) \\
&\quad + \sum_{j=0}^{m-i-1}\lim_{\delta\to 0}\int_{\partial\Omega\setminus\{|\xi^{-1}z|<\delta\}} \mathcal{L}_{a,b}^{j+i}u(\xi)\langle \nabla^{a,b}\mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi,z), d\nu(\xi)\rangle \\
&\quad - \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j}\varepsilon_m(\xi,z)\langle \nabla^{b,a}\mathcal{L}_{a,b}^{j+i}u(\xi), d\nu(\xi)\rangle = 0,
\end{aligned}
$$

which are the boundary conditions (11.68) for all $i=0,1,\ldots,m-1$.

Let us now show the uniqueness, that is, if a function $w \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ satisfies the equation $\mathcal{L}_{a,b}^m w = f$ and the boundary conditions (11.68), then it must be given by (11.66). Indeed, with u given by (11.66), the function

$$v := u - w \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega}),$$

satisfies the homogeneous equation

$$\mathcal{L}_{a,b}^m v = 0$$

and the boundary conditions for all $i = 0, 1, \ldots, m-1$,

$$\begin{aligned} I_i(v)(z) &:= (c_{a,b} - H.R(z))\mathcal{L}_{a,b}^i v(z) \\ &\quad + \sum_{j=0}^{m-i-1} \lim_{\delta \to 0} \int_{\partial\Omega \setminus \{|\xi^{-1}z| < \delta\}} \mathcal{L}_{a,b}^{j+i} v(\xi) \langle \nabla^{a,b} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi, z), d\nu(\xi)\rangle \\ &\quad - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi, z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{j+i} v(\xi), d\nu(\xi)\rangle = 0, \end{aligned}$$

for $z \in \partial\Omega$. Applying the Green formula from Theorem 1.4.6 to the function $v \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ we obtain

$$\begin{aligned} 0 &= \int_\Omega \mathcal{L}_{a,b}^m v(z) \mathcal{L}_{a,b}^i \varepsilon_m(\xi, z) d\nu(\xi) = \int_\Omega \mathcal{L}_{a,b}^{m-i} \mathcal{L}_{a,b}^i v(z) \mathcal{L}_{a,b}^i \varepsilon_m(\xi, z) d\nu(\xi) \\ &= \int_\Omega \mathcal{L}_{a,b}^{m-1} v(z) \mathcal{L}_{a,b} \mathcal{L}_{a,b}^i \varepsilon_m(\xi, z) d\nu(\xi) - \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1} v(z) \langle \nabla^{a,b} \mathcal{L}_{a,b}^i \varepsilon_m(\xi, z), d\nu(\xi)\rangle \\ &\quad + \int_{\partial\Omega} \mathcal{L}_{a,b}^i \varepsilon_m(\xi, z) \langle \nabla^{a,b} \mathcal{L}_{a,b}^{m-1} v(z), d\nu(\xi)\rangle \\ &= \cdots \\ &= c_{a,b} \mathcal{L}_{a,b}^i v(z) - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{j+i} v(\xi) \langle \nabla^{a,b} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi, z), d\nu(\xi)\rangle \\ &\quad + \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}_{a,b}^{m-1-j} \varepsilon_m(\xi, z) \langle \nabla^{b,a} \mathcal{L}_{a,b}^{j+i} v(\xi), d\nu(\xi)\rangle, \quad i = 0, 1, \ldots, m-1. \end{aligned}$$

By passing to the limit as $z \to \partial\Omega$, we obtain the additional relations

$$\mathcal{L}_{a,b}^i v(z) \,|_{z\in\partial\Omega} = I_i(v)(z) \,|_{z\in\partial\Omega} = 0, \quad i = 0, 1, \ldots, m-1.$$

We claim that the boundary value problem

$$\begin{aligned} &\mathcal{L}_{a,b}^m v = 0, \\ &\mathcal{L}_{a,b}^i v \,|_{\partial\Omega} = 0, \quad i = 0, 1, \ldots, m-1, \end{aligned} \tag{11.71}$$

has a unique trivial solution. Assuming this for a moment, we would get that $v = u - w \equiv 0$, which means that w coincides with u in Ω, so that (11.66) is the unique solution of the boundary value problem (11.65), (11.68) in Ω.

To show that the boundary value problem (11.71) has a unique solution in $C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$, we denote $\tilde{v} := \mathcal{L}_{a,b}^{m-1} v$, and then the statement follows by induction from the uniqueness in $C^2(\Omega) \cap C^1(\overline{\Omega})$ of the problem

$$\mathcal{L}_{a,b}\tilde{v} = 0, \quad \tilde{v}\mid_{\partial\Omega} = 0.$$

The proof of Theorem 11.3.11 is complete. □

11.4 Hardy inequalities with boundary terms on stratified groups

In this section we demonstrate how the developed potential theory for the sub-Laplacian can be used to derive a refinement to Hardy inequalities. The basic inequality that we are referring to here is inequality (7.4) which says that on a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$, for all $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ and $\alpha > 2 - Q$ we have

$$\int_{\mathbb{G}} d^\alpha |\nabla_{\mathbb{G}} u|^2 \, d\nu \geq \left(\frac{Q+\alpha-2}{2}\right)^2 \int_{\mathbb{G}} d^\alpha \frac{|\nabla_{\mathbb{G}} d|^2}{d^2} |u|^2 \, d\nu, \tag{11.72}$$

and the constant $\left(\frac{Q+\alpha-2}{2}\right)^2$ is sharp. We refer to Remark 7.1.2 for a historic discussion of this and other related inequalities.

We will now derive a version of (11.72) over domains in $\mathbb{G}$ in such a way that the boundary term appears as well, in the case the function does not vanish on the boundary. Certainly, the following inequality implies (11.72) if we take the domain Ω in such a way that it contains the support of a compactly supported function u. As before, d stands for the $\mathcal{L}$-gauge and we also write

$$\nabla_H = (X_1, \ldots, X_{N_1})$$

for the horizontal gradient. A discussion from a point of view of more general weights was also done in Section 7.7.

Theorem 11.4.1 (Hardy inequality with boundary term)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain in a stratified group of homogeneous dimension $Q \geq 3$ with $0 \notin \partial\Omega$. Let $\alpha \in \mathbb{R}$ and $\alpha > 2 - Q$. Let $u \in C^1(\Omega) \bigcap C(\overline{\Omega})$. Then we have*

$$\begin{aligned} \int_\Omega d^\alpha |\nabla_H u|^2 \, d\nu \geq & \left(\frac{Q+\alpha-2}{2}\right)^2 \int_\Omega d^\alpha \frac{|\nabla_H d|^2}{d^2} |u|^2 \, d\nu \\ & + \frac{Q+\alpha-2}{2(Q-2)} \int_{\partial\Omega} d^{Q+\alpha-2} |u|^2 \langle \widetilde{\nabla} d^{2-Q}, d\nu \rangle. \end{aligned} \tag{11.73}$$

Remark 11.4.2.

1. If $u = 0$ on $\partial\Omega$ then (11.73) reduces to (11.72).
2. The boundary term in (11.73) sometimes can be of different sign. To show this, take $u = e^{-\frac{R}{2}d}$ for $R > 0$. Green's first formula in Theorem 1.4.6 allows us to calculate

$$\begin{aligned}
&\int_{\partial\Omega} d^{Q+\alpha-2}e^{-Rd}\langle\widetilde{\nabla}d^{2-Q}, d\nu\rangle \\
&\quad = \int_{\Omega} \widetilde{\nabla}(d^{Q+\alpha-2}e^{-Rd})d^{2-Q}d\nu + \frac{1}{\beta_d}\int_{\Omega} d^{Q+\alpha-2}e^{-Rd}\mathcal{L}\beta_d d^{2-Q}d\nu \\
&\quad = \int_{\Omega} \widetilde{\nabla}(d^{Q+\alpha-2}e^{-Rd})d^{2-Q}d\nu \\
&\quad = \sum_{k=1}^{N_1}\int_{\Omega} X_k(d^{Q+\alpha-2}e^{-Rd})X_k d^{2-Q}d\nu \\
&\quad = \sum_{k=1}^{N_1}\int_{\Omega} \big((Q+\alpha-2)d^{Q+\alpha-2-1}e^{-Rd}X_k d - Rd^{Q+\alpha-2}e^{-Rd}X_k d\big) \\
&\qquad \times (2-Q)d^{2-Q-1}X_k d\, d\nu.
\end{aligned}$$

Let $\alpha = 0$, $Q = 3$, and $\Omega \bigcap B_{\frac{1}{R}} = \{\emptyset\}$, where $B_{\frac{1}{R}} = \{x \in \mathbb{G} : d(x) < \frac{1}{R}\}$. Then we get

$$\int_{\partial\Omega} de^{-Rd}\langle\widetilde{\nabla}d^{-1}, d\nu\rangle = \sum_{k=1}^{N_1}\int_{\Omega}(Rd^{-1} - d^{-2})e^{-Rd}(X_k d)^2\, d\nu > 0$$

is positive. On the other hand, if $u := C = \text{const}$, then we get

$$\begin{aligned}
&\int_{\partial\Omega} d^{Q+\alpha-2}C^2\langle\widetilde{\nabla}d^{2-Q}, d\nu\rangle \\
&\quad = C^2\sum_{k=1}^{N_1}\int_{\Omega}\big((Q+\alpha-2)d^{Q+\alpha-2-1}X_k d\big)\,(2-Q)d^{2-Q-1}X_k d\, d\nu \\
&\quad = -C^2(Q+\alpha-2)(Q-2)\sum_{k=1}^{N_1}\int_{\Omega} d^{\alpha-2}(X_k d)^2 d\nu < 0,
\end{aligned}$$

which shows that the boundary term can also be negative.
3. Since it is known that the constant $\left(\frac{Q+\alpha-2}{2}\right)^2$ in (11.73) (or rather in (11.72)) is sharp, the local inequality (11.73) gives a refinement to (11.72), and even more so if the boundary term is positive.
4. Note that in comparison to (11.72), we do not assume in Theorem 11.4.1 that 0 is not in the support of the function u since for $\alpha > 2 - Q$ all the integrals in (11.73) are convergent.

5. Even if $0 \in \partial\Omega$, the statement of Theorem 11.4.1 remains true if $0 \notin \partial\Omega \cap \operatorname{supp} u$.
6. Without the second (boundary) term inequality (11.73) was studied on the Heisenberg group in [GL90] for a particular choice of $d(x)$, or on Carnot groups [GK08] for $\mathcal{L}$-gauges $d(x)$. We can refer to these papers as well as to [GZ01] for other references on this subject. From the point of view of the boundary term the inequality (11.73) can be thought of as a refinement of the usual Hardy because, also as it was pointed out in Part 2 of this Remark, this boundary term in (11.73) can be positive. We call these inequalities local due to the presence of a contribution from the boundary.

Proof of Theorem 11.4.1. By an argument similar to that in Remark 2.6, Part 3, without loss of generality we can assume that u is real-valued. In this case, recalling that

$$(\widetilde{\nabla} u)u = \sum_{k=1}^{N_1} (X_k u) X_k u = |\nabla_H u|^2,$$

the estimate (11.73) reduces to

$$\begin{aligned}\int_\Omega d^\alpha (\widetilde{\nabla} u) u \, d\nu \geq \left(\frac{Q+\alpha-2}{2}\right)^2 \int_\Omega d^\alpha \frac{(\widetilde{\nabla} d) d}{d^2} u^2 \, d\nu \\ + \frac{Q+\alpha-2}{2(Q-2)} \int_{\partial\Omega} d^{Q+\alpha-2} u^2 \langle \widetilde{\nabla} d^{2-Q}, d\nu \rangle,\end{aligned} \tag{11.74}$$

which we will now prove. Let us set $u = d^\gamma q$ for some $\gamma \neq 0$ to be chosen later. Then we can calculate

$$\begin{aligned}(\widetilde{\nabla} u) u = (\widetilde{\nabla} d^\gamma q) d^\gamma q &= \sum_{k=1}^{N_1} X_k (d^\gamma q) X_k (d^\gamma q) \\ &= \gamma^2 d^{2\gamma-2} \sum_{k=1}^{N_1} (X_k d)^2 q^2 + 2\gamma d^{2\gamma-1} q \sum_{k=1}^{N_1} X_k d \, X_k q + d^{2\gamma} \sum_{k=1}^{N_1} (X_k q)^2 \\ &= \gamma^2 d^{2\gamma-2} ((\widetilde{\nabla} d) d) q^2 + 2\gamma d^{2\gamma-1} q (\widetilde{\nabla} d) q + d^{2\gamma} (\widetilde{\nabla} q) q.\end{aligned}$$

Multiplying both sides of this equality by d^α and applying Green's first formula from Theorem 1.4.6, we get

$$\begin{aligned}\int_\Omega d^\alpha (\widetilde{\nabla} u) u d\nu &= \gamma^2 \int_\Omega d^{\alpha+2\gamma-2} ((\widetilde{\nabla} d) d) \, q^2 d\nu + \frac{\gamma}{\alpha+2\gamma} \int_{\partial\Omega} q^2 \langle \widetilde{\nabla} d^{\alpha+2\gamma}, d\nu \rangle \\ &\quad - \frac{\gamma}{\alpha+2\gamma} \int_\Omega (\mathcal{L} d^{\alpha+2\gamma}) q^2 d\nu + \int_\Omega d^{\alpha+2\gamma} (\widetilde{\nabla} q) q d\nu \\ &\geq \int_\Omega \gamma^2 d^{\alpha+2\gamma-2} ((\widetilde{\nabla} d) d) \, q^2 d\nu + \frac{\gamma}{\alpha+2\gamma} \int_{\partial\Omega} q^2 \langle \widetilde{\nabla} d^{\alpha+2\gamma}, d\nu \rangle \\ &\quad - \frac{\gamma}{\alpha+2\gamma} \int_\Omega (\mathcal{L} d^{\alpha+2\gamma}) q^2 d\nu.\end{aligned} \tag{11.75}$$

On the other hand, it can be readily checked that we have

$$-\frac{\gamma}{\alpha+2\gamma}\mathcal{L}d^{\alpha+2\gamma}=-\gamma(\alpha+2\gamma+Q-2)d^{\alpha+2\gamma-2}(\widetilde{\nabla}d)d-\frac{\gamma}{2-Q}d^{\alpha+2\gamma+Q-2}\mathcal{L}d^{2-Q}.$$

Since $q^2=d^{-2\gamma}u^2$, substituting this into (11.75) we obtain

$$\int_\Omega d^\alpha(\widetilde{\nabla}u)ud\nu\geq(-\gamma^2-\gamma(\alpha+Q-2))\int_\Omega d^\alpha\frac{(\widetilde{\nabla}d)d}{d^2}u^2d\nu \\ -\frac{\gamma}{2-Q}\int_\Omega(\mathcal{L}d^{2-Q})d^{\alpha+Q-2}u^2dx+\frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega}d^{-2\gamma}u^2\langle\widetilde{\nabla}d^{\alpha+2\gamma},d\nu\rangle.$$

Recalling that for $Q\geq 3$, $\varepsilon=\beta_d d^{2-Q}$ is the fundamental solution of the sub-Laplacian $\mathcal{L}$, we have

$$\int_\Omega(\mathcal{L}d^{2-Q})d^{\alpha+Q-2}u^2dx=0,\ \alpha>2-Q,$$

independent of whether 0 belongs to Ω or not, since $\mathcal{L}d^{2-Q}=\frac{1}{\beta_d}\delta$ in $\mathbb{G}$. It follows that

$$\int_\Omega d^\alpha(\widetilde{\nabla}u)ud\nu \\ \geq(-\gamma^2-\gamma(\alpha+Q-2))\int_\Omega d^\alpha\frac{(\widetilde{\nabla}d)d}{d^2}u^2d\nu+\frac{\gamma}{\alpha+2\gamma}\int_{\partial\Omega}d^{-2\gamma}u^2\langle\widetilde{\nabla}d^{\alpha+2\gamma},d\nu\rangle.$$

Taking $\gamma=\frac{2-Q-\alpha}{2}$, we obtain (11.74). □

As usual, a Hardy inequality implies several uncertainty principles.

Corollary 11.4.3 (Local uncertainty principle with boundary terms)**.** *Let $\Omega\subset\mathbb{G}$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q\geq 3$, with $0\notin\partial\Omega$. Then for all $u\in C^1(\Omega)\bigcap C(\overline{\Omega})$ we have*

$$\left(\int_\Omega d^2|\nabla_H d|^2|u|^2d\nu\right)\left(\int_\Omega|\nabla_H u|^2d\nu\right) \\ \geq\left(\frac{Q-2}{2}\right)^2\left(\int_\Omega|\nabla_H d|^2|u|^2d\nu\right)^2 \\ +\frac{1}{2}\int_{\partial\Omega}d^{Q-2}|u|^2\langle\widetilde{\nabla}d^{2-Q},d\nu\rangle\left(\int_\Omega d^2|\nabla_H d|^2|u|^2d\nu\right) \tag{11.76}$$

and

$$\left(\int_\Omega\frac{d^2}{|\nabla_H d|^2}|u|^2d\nu\right)\left(\int_\Omega|\nabla_H u|^2d\nu\right) \\ \geq\left(\frac{Q-2}{2}\right)^2\left(\int_\Omega|u|^2d\nu\right)^2\frac{1}{2}\int_{\partial\Omega}d^{Q-2}|u|^2\langle\widetilde{\nabla}d^{2-Q},d\nu\rangle\left(\int_\Omega\frac{d^2}{|\nabla_H d|^2}|u|^2d\nu\right). \tag{11.77}$$

Proof of Corollary 11.4.3. Again, assuming u is real-valued, and taking $\alpha = 0$ in the inequality (11.74) we get

$$\begin{aligned}
&\left(\int_\Omega d^2((\widetilde{\nabla} d)d)|u|^2 d\nu\right)\left(\int_\Omega (\widetilde{\nabla} u)u d\nu\right) \\
&\geq \left(\frac{Q-2}{2}\right)^2 \left(\int_\Omega d^2((\widetilde{\nabla} d)d)|u|^2 d\nu\right) \int_\Omega \frac{(\widetilde{\nabla} d)d}{d^2}|u|^2\, d\nu \\
&\quad + \frac{1}{2}\int_{\partial\Omega} d^{Q-2}|u|^2 \langle \widetilde{\nabla} d^{2-Q}, d\nu\rangle \left(\int_\Omega d^2 |\nabla_H d|^2 |u|^2 d\nu\right) \\
&\geq \left(\frac{Q-2}{2}\right)^2 \left(\int_\Omega ((\widetilde{\nabla} d)d)|u|^2 d\nu\right)^2 \\
&\quad + \frac{1}{2}\int_{\partial\Omega} d^{Q-2}|u|^2 \langle \widetilde{\nabla} d^{2-Q}, d\nu\rangle \left(\int_\Omega d^2 |\nabla_H d|^2 |u|^2 d\nu\right),
\end{aligned}$$

where we have used the Hölder inequality in the last line. This shows (11.76). The proof of (11.77) is similar. □

11.5 Green functions on *H*-type groups

In this section we discuss applications of the described potential theory to the construction of solutions of boundary value problems. We restrict our attention to the prototype H-type groups discussed in Section 1.4.10. The examples of the prototype H-type groups are the Abelian group $(\mathbb{R}^d; +)$ and the Heisenberg group $\mathbb{H}^d$. The fact that will be important for our purposes in this section is that the fundamental solution to the sub-Laplacian in this setting is explicitly known, see Theorem 1.4.20. Our presentation in this section follows [GRS17].

Thus, in this section, let $\mathbb{G} \simeq \mathbb{R}^m \times \mathbb{R}^n$ be a prototype H-type group and let $\mathcal{L}$ be the sub-Laplacian on $\mathbb{G}$ as defined in (1.119). We consider an open set $\Omega \subset \mathbb{G}$ with piecewise smooth boundary $\partial\Omega$, and study the Dirichlet problem for the sub-Laplacian

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{11.78}$$

As discussed in Remark 11.1.7, there are several difficulties in solving such boundary problems in view of the usually present characteristic points.

However, it turns out that there are still a number of cases of interest when the Dirichlet boundary value problem (11.78) can be solved in the classical sense. This is achieved by constructing a well-behaved Green function for the problem (11.78). In view of Jerison's example discussed in Remark 11.1.7 one wants to avoid the characteristic points which is not always possible, since even ball-like bounded domains on (non-Abelian) H-type groups have non-empty collection of characteristic points. However, domains that are unbounded may have no characteristic points. Thus, in this section we present such an analysis for so-called l-wedge and l-strip domains. This includes domains such as half-spaces, quadrant-spaces and so on.

An important tool for this analysis will be the fundamental solution for the sub-Laplacian $\mathcal{L}$. More precisely, recalling Theorem 1.4.20, the function

$$\Gamma(\xi) := c\left(|x|^4 + 16|t|^2\right)^{(2-Q)/4} \tag{11.79}$$

is the fundamental solution of the sub-Laplacian, that is,

$$\mathcal{L}\Gamma_\zeta = -\delta_\zeta, \tag{11.80}$$

where $\Gamma_\zeta(\xi) = \Gamma(\zeta^{-1} \circ \xi)$ and δ_ζ is the Dirac distribution at $\zeta \equiv (y, \tau) \in \mathbb{G}$.

Definition 11.5.1 (Green function for Dirichlet sub-Laplacian)**.** We define the *Green function for the Dirichlet sub-Laplacian* in Ω by the formula

$$G_\Omega(\xi, \zeta) := \Gamma(\zeta^{-1} \circ \xi) - h_\zeta(\xi),$$

where $h_\zeta(\xi)$ is a harmonic function, that is,

$$\mathcal{L}h_\zeta(\xi) = 0 \quad \text{in } \Omega, \tag{11.81}$$

having the same boundary values on $\partial\Omega$ as the fundamental solution Γ_ζ with pole at $\zeta \in \Omega$.

In particular, we have

$$G_\Omega(\xi, \zeta) = 0, \quad \xi \in \partial\Omega. \tag{11.82}$$

11.5.1 Green functions and Dirichlet problem in wedge domains

We first define the class of domains that we will be working in.

Definition 11.5.2 (l-wedge domains)**.** Let $1 \le l \le m$ and let $\mathbb{G}^\ddagger$ be the *l-wedge space*

$$\mathbb{G}^\ddagger := \{\xi = (x_1, \ldots, x_m, t_1, \ldots, t_n) : \ x_1, \ldots, x_l > 0\}.$$

Let the point $\zeta = (y, \tau) = (y_1, y_2, \ldots, y_m, \tau_1, \ldots, \tau_n)$ lie in this l-wedge space, so that $y_1 > 0, \ldots, y_l > 0$. The point ζ_{x_k} defined by

$$\zeta_{x_k} := (y_1, \ldots, -y_k, \ldots, y_m, \tau_1, \ldots, \tau_n)$$

is said to be *symmetric for the point ζ with respect to the hyperplane $x_k = 0$*. Consecutively, the point

$$\zeta_{x_k x_s} := (y_1, \ldots, -y_k, \ldots, -y_s, \ldots, y_m, \tau_1, \ldots, \tau_n)$$

is said to be symmetric for the point ζ_{x_k} with respect to the hyperplane $x_s = 0$, and this process can be continued further.

It is clear that the symmetry indices are invariant under permutations. We will also use the notation

$$\Gamma((\zeta_{(j,l)})^{-1} \circ \xi) \quad \text{for } j \leq l,$$

meaning that we take the sum of the functions $\Gamma((\zeta^{-1}_{x_{k_1} \cdots x_{k_j}} \circ \xi)$, $j \leq l$, over all possible combinations of the symmetry arguments $x_{k_1} \cdots x_{k_j}$: here in order to reduce the number of subindices we write $(\zeta_{(j,l)})^{-1} \circ \xi$ for $\zeta^{-1}_{x_{k_1} \cdots x_{k_j}} \circ \xi$. For example, if $l = 3$, $j = 2$, then

$$\Gamma((\zeta_{(2,3)})^{-1} \circ \xi) = \Gamma((\zeta_{x_1 x_2})^{-1} \circ \xi) + \Gamma((\zeta_{x_1 x_3})^{-1} \circ \xi) + \Gamma((\zeta_{x_3 x_2})^{-1} \circ \xi),$$

and if $l = 3$, $j = 3$, then

$$\Gamma((\zeta_{(3,3)})^{-1} \circ \xi) = \Gamma((\zeta_{x_1 x_2 x_3})^{-1} \circ \xi).$$

In this notation the Green function for the Dirichlet sub-Laplacian in wedge domains takes the following form, for any $1 \leq l \leq m$.

Proposition 11.5.3 (Green function for Dirichlet sub-Laplacian in wedge domains)**.** *Let $\mathbb{G}^{\ddagger}$ be an l-wedge domain for $1 \leq l \leq m$. Then the function*

$$G_{\mathbb{G}^{\ddagger}}(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) + \sum_{j=1}^{l} (-1)^j \Gamma((\zeta_{(j,l)})^{-1} \circ \xi) \tag{11.83}$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^{\ddagger}$.

Proof of Proposition 11.5.3. Since any symmetric point $\zeta_{(j,l)}$ is not in $\mathbb{G}^{\ddagger}$, for any $j = 1, \ldots, l$, it follows from (11.80) that

$$\mathcal{L}\Gamma((\zeta_{(j,l)})^{-1} \circ \xi) = -\delta_{\zeta_{(j,l)}} = 0 \text{ in } \mathbb{G}^{\ddagger},$$

for any $\xi \in \mathbb{G}^{\ddagger}$ and $j = 1, \ldots, l$. Thus, the function

$$\sum_{j=1}^{l} (-1)^j \Gamma((\zeta_{(j,l)})^{-1} \circ \xi)$$

satisfies condition (11.81), i.e., it is harmonic in $\mathbb{G}^{\ddagger}$. Now let us verify boundary condition for the domain $\mathbb{G}^{\ddagger}$, that is, the function $G_{\mathbb{G}^{\ddagger}}$ should become zero at $x_1 = 0$ and at infinity. Recall that

$$d(\xi, \zeta) := \left(\Gamma(\zeta^{-1} \circ \xi)\right)^{\frac{1}{2-Q}}$$

is an actual distance on $\mathbb{G}$ defining a norm (and not only a quasi-norm, see, e.g., [Cyg81]). Now it is easy to see that the d-distance from any point of the hyperplane $x_k = 0$ to the points ζ and ζ_{x_k} is the same, that is, $G_{\mathbb{G}^\ddagger}$ satisfies the Dirichlet condition at the hyperplanes $x_1 = 0, \dots, x_l = 0$ and it is also clear (by the construction) that the function $G_{\mathbb{G}^\ddagger}$ is zero at the infinity. It proves that

$$G_{\mathbb{G}^\ddagger}(\xi, \zeta) = 0, \quad \xi \in \partial\mathbb{G}^\ddagger,$$

that is, $G_{\mathbb{G}^\ddagger}$ is a Green function according to Definition 11.5.1. □

In the next theorem we will use the anisotropic Hölder space Γ_α from Definition 11.3.1. We consider the Dirichlet problem for the sub-Laplacian

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathbb{G}^\ddagger, \\ \quad u = \phi & \text{on } \partial\mathbb{G}^\ddagger. \end{cases} \tag{11.84}$$

Theorem 11.5.4 (Dirichlet problem for sub-Laplacian in wedge domain)**.** *Let* $f \in \Gamma_\alpha(\mathbb{G}^\ddagger)$ *for* $0 < \alpha < 1$*, and assume that* $\operatorname{supp} f \subset \mathbb{G}^\ddagger$*, and that* $\phi \in C^\infty(\partial\mathbb{G}^\ddagger)$*. Then the boundary value problem* (11.84) *has a unique solution* $u \in C^2(\mathbb{G}^\ddagger) \cap C^1(\overline{\mathbb{G}^\ddagger})$ *and it can be represented by the formula*

$$u(\xi) = \int_{\mathbb{G}^\ddagger} G_{\mathbb{G}^\ddagger}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int_{\partial\mathbb{G}^\ddagger} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^\ddagger}(\xi, \zeta), d\nu(\zeta)\rangle, \quad \xi \in \mathbb{G}^\ddagger, \tag{11.85}$$

where

$$\widetilde{\nabla} G_{\mathbb{G}^\ddagger} = \sum_{k=1}^{m} (X_k G_{\mathbb{G}^\ddagger}) X_k.$$

Proof of Theorem 11.5.4*.* Let us take $u \in C^2(\mathbb{G}^\ddagger) \cap C^1(\overline{\mathbb{G}^\ddagger})$ and assume that u tends to zero at infinity. By Remark 1.4.7, Part 2, we can apply Green's identities in $\mathbb{G}^\ddagger$. Thus, Green's second formula (1.88) applied to the functions u and $v(\zeta) = G_{\mathbb{G}^\ddagger}(\xi, \zeta)$ yields

$$u(\xi) = \int_{\mathbb{G}^\ddagger} G_{\mathbb{G}^\ddagger}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int_{\partial\mathbb{G}^\ddagger} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^\ddagger}(\xi, \zeta), d\nu(\zeta)\rangle.$$

Here by using the properties of the Green function, we have

$$G_{\mathbb{G}^\ddagger}(\xi, \zeta) = 0, \quad \zeta \in \partial\mathbb{G}^\ddagger,$$

and, by construction the function $G_{\mathbb{G}^\ddagger}$ is symmetric, that is,

$$G_{\mathbb{G}^\ddagger}(\xi, \zeta) = G_{\mathbb{G}^\ddagger}(\zeta, \xi)$$

in $\mathbb{G}^\ddagger$, so

$$\mathcal{L}_\zeta G_{\mathbb{G}^\ddagger}(\xi, \zeta) = -\delta_\xi,$$

where δ_ξ is the Dirac distribution at $\xi \in \mathbb{G}^\ddagger$.

Let us now show that the function defined by (11.85) belongs to $C^2(\mathbb{G}^\ddagger) \cap C^1(\overline{\mathbb{G}^\ddagger})$. Since $f \in \Gamma_\alpha(\mathbb{G}^\ddagger)$ and $\operatorname{supp} f \subset \mathbb{G}^\ddagger$, the volume potential (i.e., the first term of the right-hand side in (11.85)) belongs to $C^2(\overline{\mathbb{G}^\ddagger})$ by Folland's theorem (see [Fol75, Theorem 6.1], see also [FS74]). Hörmander's hypoellipticity theorem (see [Hör67]) guarantees that every harmonic function is C^∞, hence the Dirichlet double layer potential (the second term of the right-hand side in (11.85)) is in $C^2(\mathbb{G}^\ddagger)$. On the other hand, since $\phi \in C^\infty(\partial\mathbb{G}^\ddagger)$, $\operatorname{supp}\phi \subset \{x_1 = 0, \dots, x_l = 0\}$ and the boundary hyperplanes $\{x_1 = 0\}, \dots, \{x_l = 0\}$ have no characteristic points, that is, recalling that the characteristic set of $\mathbb{G}^\ddagger$ is the set

$$\{x \in \partial\mathbb{G}^\ddagger : \, X_k(x) \in T_x(\partial\mathbb{G}^\ddagger), \, k = 1, \dots, m\},$$

with $T_x(\partial\mathbb{G}^\ddagger)$ being the tangent space to $\partial\mathbb{G}^\ddagger$ at the point x, so we see that $X_k(x_0) \notin T_x\{\partial\mathbb{G}^\ddagger\}$ for all $x_0 \in \{x_1 = 0\}, \dots, \{x_l = 0\}$

(see [GV00, Section 8] for more discussions on the non-characteristic hyperplanes in $\mathbb{G}$). Thus, the Dirichlet double layer potential is continuous on the boundary by the Kohn–Nirenberg theorem (see [CGN08, Theorem 3.12], which is a consequence of [KN65, Theorem 4], see also [Der71]-[Der72]). ☐

Remark 11.5.5. Let us point out some special cases and extensions of Theorem 11.5.4 and Proposition 11.5.3.

1. Let $\mathbb{G}^+$ be the *half-space*.

$$\mathbb{G}^+ = \{\xi = (x_1, \dots, x_m, t_1, \dots, t_n) : \, x_1 > 0\}.$$

Let the point $\zeta = (y, \tau) = (y_1, y_2, \dots, y_m, \tau_1, \dots, \tau_n)$ lie in this half-space, so that $y_1 > 0$. The point

$$\zeta^* = (y^*, \tau) := (-y_1, y_2, \dots, y_m, \tau_1, \dots, \tau_n)$$

is said to be symmetric for the point ζ with respect to the hyperplane $x_1 = 0$. As a direct consequence of Proposition 11.5.3 we have that the function

$$G_{\mathbb{G}^+}(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) - \Gamma((\zeta^*)^{-1} \circ \xi)$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^+$.

Let now $f \in \Gamma_\alpha(\mathbb{G}^+)$, $0 < \alpha < 1$, with $\operatorname{supp} f \subset \mathbb{G}^+$, and let $\phi \in C^\infty(\partial\mathbb{G}^+)$ with $\operatorname{supp}\phi \subset \{x_1 = 0\}$, and consider the Dirichlet sub-Laplacian problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathbb{G}^+, \\ u = \phi & \text{on } \partial\mathbb{G}^+. \end{cases} \tag{11.86}$$

As a consequence of Theorem 11.5.4 we get that the boundary value problem (11.86) has a unique solution $u \in C^2(\mathbb{G}^+) \cap C^1(\overline{\mathbb{G}^+})$ and it can be represented by the formula

$$u(\xi) = \int_{\mathbb{G}^+} G_{\mathbb{G}^+}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int_{\partial\mathbb{G}^+} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^+}(\xi, \zeta), \, d\nu(\zeta) \rangle, \quad \xi \in \mathbb{G}^+.$$

2. Let us point out another consequence for the quadrant-space in $\mathbb{G}$. Let $\mathbb{G}^{\oplus}$ be the *quadrant-space*

$$\mathbb{G}^{\oplus} := \{\xi = (x_1, x_2, \dots, x_m, t_1, \dots, t_n) : \ x_1 > 0, \, x_2 > 0\}.$$

Let the point $\zeta = (y, \tau) = (y_1, y_2, \dots, y_m, \tau_1, \dots, \tau_n)$ lie in this quadrant-space, so that $y_1 > 0$, $y_2 > 0$. Denote by

$$\zeta^* = (y^*, \tau) := (-y_1, y_2, \dots, y_m, \tau_1, \dots, \tau_n)$$

and

$$\overline{\zeta} = (\overline{y}, \tau) := (y_1, -y_2, \dots, y_m, \tau_1, \dots, \tau_n)$$

the symmetric points for ζ with respect to the hyperplanes $x_1 = 0$ and $x_2 = 0$, respectively. The point

$$\overline{\zeta}^* = (\overline{y}^*, \tau) = (-y_1, -y_2, \dots, y_m, \tau_1, \dots, \tau_n)$$

is the symmetric point for ζ^* with respect to the hyperplane $x_2 = 0$ and the symmetric point for $\overline{\zeta}$ with respect to the hyperplane $x_1 = 0$. Then as a direct consequence of Proposition 11.5.3 we have that the function

$$G_{\mathbb{G}^{\oplus}}(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) + \Gamma((\overline{\zeta}^*)^{-1} \circ \xi) - \Gamma((\zeta^*)^{-1} \circ \xi) - \Gamma((\overline{\zeta})^{-1} \circ \xi)$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^{\oplus}$.

Furthermore, if $f \in \Gamma_\alpha(\mathbb{G}^{\oplus})$, $0 < \alpha < 1$, with $\operatorname{supp} f \subset \mathbb{G}^{\oplus}$, and $\phi \in C^\infty(\partial \mathbb{G}^{\oplus})$ with $\operatorname{supp} \phi \subset \{x_1 = 0\}$, then the boundary value problem

$$\begin{cases} \mathcal{L} u = f & \text{in } \mathbb{G}^{\oplus}, \\ u = \phi & \text{on } \partial \mathbb{G}^{\oplus}, \end{cases}$$

has a unique solution $u \in C^2(\mathbb{G}^{\oplus}) \cap C^1(\overline{\mathbb{G}^{\oplus}})$ and it can be represented by the formula

$$u(\xi) = \int_{\mathbb{G}^{\oplus}} G_{\mathbb{G}^{\oplus}}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int_{\partial \mathbb{G}^{\oplus}} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^{\oplus}}(\xi, \zeta), \, d\nu(\zeta) \rangle, \quad \xi \in \mathbb{G}^{\oplus}.$$

3. The statements of Theorem 11.5.4 and Proposition 11.5.3 can be extended in a straightforward way to shifted l-wedge like spaces, for any $a = (a_1, \dots, a_l) \in \mathbb{R}^l$ defined by

$$\mathbb{G}^{\ddagger}_a := \{\xi = (x_1, \dots, x_m, t_1, \dots, t_n) : \ x_1 > a_1, \dots, x_l > a_l\}.$$

In this space the Green function $G_{\mathbb{G}^{\ddagger}_a}$ has the same formula as in (11.83) if we choose the symmetry points now with respect to the hyperplanes $\{x_1 = a_1\}, \dots, \{x_l = a_l\}$. In this case Theorem 11.5.4 remains true and can be obtained by the same argument.

11.5.2 Green functions and Dirichlet problem in strip domains

Similarly to wedge domain we can carry out a similar analysis in strip domains.

Definition 11.5.6 (l-strip domains)**.** Let $1 \leq l \leq m$. We define $\mathbb{G}^{\models}$ to be the *l-strip space* if it is given by

$$\mathbb{G}^{\models} = \{\xi = (x_1, \ldots, x_m, t_1, \ldots, t_n) : \; a > x_l > 0\}.$$

Let the point $\zeta = (y, \tau) = (y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)$ lie in this l-strip space, so that $a > y_l > 0$. We will use the notations

$$\zeta_{+,j} := (y_1, \ldots, y_l - 2aj, \ldots, y_m, \tau_1, \ldots, \tau_n),$$

and

$$\zeta_{-,j} := (y_1, \ldots, -y_l + 2aj, \ldots, y_m, \tau_1, \ldots, \tau_n),$$

for all $j = 0, 1, 2, \ldots$.

The Green function for the Dirichlet sub-Laplacian in the strip domains takes the following form, where compared to Proposition 11.5.3 in wedge domains, the formula is now given by an infinite series.

Proposition 11.5.7 (Green function for Dirichlet sub-Laplacian in strip domains)**.** *Let $\mathbb{G}^{\ddagger}$ be an l-strip domain for $1 \leq l \leq m$. The function*

$$G_{\mathbb{G}^{\models}}(\xi, \zeta) = \sum_{j=-\infty}^{\infty} \left(\Gamma(\zeta_{+,j}^{-1} \circ \xi) - \Gamma(\zeta_{-,j}^{-1} \circ \xi)\right) \tag{11.87}$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^{\models}$.

Proof of Proposition 11.5.7. The formula (11.87) consists in the $j = 0$ term, i.e., the term $\Gamma(\zeta_{+,0}^{-1} \circ \xi)$, which corresponds to the fundamental solution, and all the other terms which are subharmonic functions in $\mathbb{G}^{\models}$. Let us check that traces of (11.87) vanish on hyperplanes $x_l = 0$ and $x_l = a$. If $x_l = 0$, then (11.87) gives

$$\begin{aligned} &G_{\mathbb{G}^{\models}}(\xi,\zeta)|_{x_l=0} \\ &= c\sum_{j=-\infty}^{\infty}\Big(\left(((x_1-y_1)^2+\cdots+(-y_l+2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4} \\ &\quad -\left(((x_1-y_1)^2+\cdots+(y_l-2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4}\Big)=0. \end{aligned}$$

If $x_l = a$, then (11.87) gives

$$\begin{aligned} &G_{\mathbb{G}^{\models}}(\xi,\zeta)|_{x_l=a} \\ &= c\sum_{j=-\infty}^{\infty}\Big(\left(((x_1-y_1)^2+\cdots+(a-y_l+2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4} \\ &\quad -\left(((x_1-y_1)^2+\cdots+(a+y_l-2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4}\Big) \end{aligned}$$

$$
\begin{aligned}
&= c\sum_{j=0}^{\infty}\left(((x_1-y_1)^2+\cdots+(a-y_l+2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4}\\
&\quad - c\sum_{j=1}^{\infty}\left(((x_1-y_1)^2+\cdots+(a+y_l-2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4}\\
&\quad + c\sum_{j=-1}^{-\infty}\left(((x_1-y_1)^2+\cdots+(a-y_l+2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4}\\
&\quad - c\sum_{j=0}^{-\infty}\left(((x_1-y_1)^2+\cdots+(a+y_l-2aj)^2+\cdots+(x_m-y_m)^2)^2+16|t-\tau|^2\right)^{(2-Q)/4}\\
&= 0.
\end{aligned}
$$

Here the first term ($j = 0$ term) of the first sum is cancelled with the first term ($j = 1$ term) of the second sum and the second terms of the first sum is cancelled with the second term of the second sum and so on, that is, the first two sums give zero. Similarly, the first term of the third sum is cancelled with the first term of the last sum and the second term of the third sum is cancelled with the second term of the last sum and so on, that is, the last two sums also give zero. As a result, the trace vanishes at $x_l = a$. □

For $f \in \Gamma_\alpha(\mathbb{G}^{\models})$, $0 < \alpha < 1$, with $\operatorname{supp} f \subset \mathbb{G}^{\models}$, and for $\phi \in C^\infty(\partial\mathbb{G}^{\models})$ with $\operatorname{supp}\phi \subset \{x_l = 0\}\bigcup\{x_l = a\}$, we now consider the Dirichlet problem for the sub-Laplacian

$$
\begin{cases} \mathcal{L}u = f & \text{in } \mathbb{G}^{\models}, \\ \quad u = \phi & \text{on } \partial\mathbb{G}^{\models}. \end{cases} \tag{11.88}
$$

Theorem 11.5.8 (Dirichlet problem for sub-Laplacian in strip domain)**.** *Let* $f \in \Gamma_\alpha(\mathbb{G}^{\models})$*,* $0 < \alpha < 1$*, with* $\operatorname{supp} f \subset \mathbb{G}^{\models}$*, and let* $\phi \in C^\infty(\partial\mathbb{G}^{\models})$ *with* $\operatorname{supp}\phi \subset \{x_l = 0\}\bigcup\{x_l = a\}$*. Then the boundary value problem* (11.88) *has a unique solution* $u \in C^2(\mathbb{G}^{\models}) \cap C^1(\overline{\mathbb{G}^{\models}})$ *and it can be represented by the formula*

$$
u(\xi) = \int_{\mathbb{G}^{\models}} G_{\mathbb{G}^{\models}}(\xi,\zeta) f(\zeta)d\nu(\zeta) - \int_{\partial\mathbb{G}^{\models}} \phi(\zeta)\langle\widetilde{\nabla} G_{\mathbb{G}^{\models}}(\xi,\zeta),\, d\nu(\zeta)\rangle, \quad \xi \in \mathbb{G}^{\models},
$$

where

$$
\widetilde{\nabla} G_{\mathbb{G}^{\models}} = \sum_{k=1}^{m} (X_k G_{\mathbb{G}^{\models}})\, X_k.
$$

Proof of Theorem 11.5.8*.* The proof is almost the same as the proof of Theorem 11.5.4, so we omit it. □

11.6 p-sub-Laplacian Picone's inequality and consequences

In this section, we present Picone's identities for the p-sub-Laplacian on stratified Lie groups, which implies a generalized Díaz–Saá inequality for the p-sub-Laplacian on stratified Lie groups. As consequences, a comparison principle and uniqueness of a positive solution to nonlinear p-sub-Laplacian equations are derived. The presentation of this section follows the results of [RS17a].

First let us introduce the functional spaces $S^{1,p}(\Omega)$.

Definition 11.6.1 (Functional spaces $S^{1,p}(\Omega)$ and $\overset{\circ}{S}{}^{1,p}(\Omega)$)**.** We define

$$S^{1,p}(\Omega) := \{u : \Omega \to \mathbb{R};\ u, |\nabla_H u| \in L^p(\Omega)\}.$$

Consider the functional

$$J_p(u) := \left(\int_\Omega |\nabla_H u|^p dx\right)^{1/p},$$

then we define the functional class $\overset{\circ}{S}{}^{1,p}(\Omega)$ to be the completion of $C_0^1(\Omega)$ in the norm generated by J_p (see, e.g., [CDG93]).

In an admissible domain $\Omega \subset \mathbb{G}$ with smooth boundary $\partial\Omega$ we study the p-sub-Laplacian Dirichlet problem:

$$\begin{cases} -\mathcal{L}_p u = F(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{11.89}$$

Here we assume that

(a) The function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a positive, bounded and measurable, and there exists a positive constant $C > 0$ such that

$$F(x,\rho) \le C(\rho^{p-1} + 1)$$

holds for a.e. $x \in \Omega$.

(b) The function

$$\rho \mapsto \frac{F(x,\rho)}{\rho^{p-1}}$$

is strictly decreasing on $(0,\infty)$ for a.e. $x \in \Omega$.

As usual, if a function $u \in \overset{\circ}{S}{}^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$\int_\Omega |\nabla_H u|^{p-2}(\widetilde{\nabla} u)\phi d\nu = \int_\Omega F(x,u)\phi d\nu$$

for all $\phi \in C_0^\infty(\Omega)$, then it is called a *weak solution* of (11.89).

Keeping in mind the stratified group discussions in previous sections for any set $\Omega \subset \mathbb{G}$ we denote, for any $1 < p < \infty$,

$$L(u,v) := |\nabla_H u|^p - p\frac{|u|^{p-2}u}{f(v)}\nabla_H u \cdot \nabla_H v|\nabla_H v|^{p-2} + \frac{f'(v)|u|^p}{f^2(v)}|\nabla_H v|^p. \quad (11.90)$$

We also denote

$$R(u,v) := |\nabla_H u|^p - \nabla_H \left(\frac{|u|^p}{f(v)}\right)|\nabla_H v|^{p-2}\nabla_H v, \quad 1 < p < \infty, \quad (11.91)$$

a.e. in Ω. Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz function such that

$$(p-1)|f(t)|^{\frac{p-2}{p-1}} \leq f'(t) \quad (11.92)$$

holds a.e. in $\mathbb{R}^+$. Then we have the following Picone identity on a stratified Lie group $\mathbb{G}$.

Lemma 11.6.2 (Picone identity)**.** *Under the above assumptions we have*

$$L(u,v) = R(u,v) \geq 0$$

a.e. in Ω, where u and v are differentiable real-valued functions and $\Omega \subset \mathbb{G}$ is any set.

Proof of Lemma 11.6.2. We have the equality

$$\nabla_H \left(\frac{|u|^p}{f(v)}\right) = \frac{pf(v)|u|^{p-2}u\nabla_H u - f'(v)|u|^p\nabla_H v}{f^2(v)} = \frac{p|u|^{p-2}u\nabla_H u}{f(v)} - \frac{f'(v)|u|^p\nabla_H v}{f^2(v)}.$$

It implies that

$$\begin{aligned} R(u,v) &= |\nabla_H u|^p - \nabla_H \left(\frac{|u|^p}{f(v)}\right)|\nabla_H v|^{p-2}\nabla_H v \\ &= |\nabla_H u|^p - \frac{p|u|^{p-2}u}{f(v)}|\nabla_H v|^{p-2}\nabla_H u \cdot \nabla_H v + \frac{f'(v)|u|^p}{f^2(v)}|\nabla_H v|^p \\ &= L(u,v). \end{aligned}$$

Now it remains to show the non-negativity of $R(u,v)$ (that is, alternatively, of $L(u,v)$). Since

$$\frac{p|u|^{p-2}u}{f(v)}|\nabla_H v|^{p-2}\nabla_H u \cdot \nabla_H v \leq \frac{p|u|^{p-1}}{f(v)}|\nabla_H v|^{p-1}|\nabla_H u|,$$

by applying the Young inequality to the right-hand side of this inequality we obtain

$$\frac{p|u|^{p-2}u}{f(v)}|\nabla_H v|^{p-2}\nabla_H u\cdot\nabla_H v \le |\nabla_H u|^p + (p-1)\frac{|u|^p|\nabla_H v|^p}{f^{\frac{p}{p-1}}(v)}.$$

Thus, we have

$$\frac{f'(v)|u|^p|\nabla_H v|^p}{f^2(v)} - (p-1)\frac{|u|^p|\nabla_H v|^p}{f^{\frac{p}{p-1}}(v)} \le R(u,v).$$

By assumption (11.92) we have $(p-1)|f(t)|^{\frac{p-2}{p-1}} \le f'(t)$, which means that $0 \le R(u,v)$, completing the proof. □

Lemma 11.6.3. *Let $1<p<\infty$ and let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a locally Lipschitz function such that*

$$(p-1)|f(t)|^{\frac{p-2}{p-1}} \le f'(t)$$

holds a.e. in $\mathbb{R}^+$. Let $\Omega\subset\mathbb{G}$ be a bounded open set and let $v\in \overset{\circ}{S}{}^{1,p}(\Omega)$ be such that $v\ge\epsilon>0$. Then for any $u\in C_0^\infty(\Omega)$ we have

$$\int_\Omega \frac{|u|^p}{f(v)}(-\mathcal{L}_p v)dx \le \int_\Omega |\nabla_H u|^p dx. \tag{11.93}$$

Proof of Lemma 11.6.3. Let $v\in \overset{\circ}{S}{}^{1,p}(\Omega)$ be such as in the assumptions, that is, $v\ge\epsilon>0$. Then by the density argument we can choose $v_k\in C_0^1(\Omega),\ k=1,2,\ldots,$ such that $v_k>\frac{\epsilon}{2}$ in Ω and $v_k\to v$ a.e. in Ω. By using Lemma 11.6.2 we have

$$0\le\int_\Omega R(u,v_k)dx,$$

for each k. That is,

$$\int_\Omega \frac{|u|^p}{f(v_k)}(-\mathcal{L}_p v_k)dx \le \int_\Omega |\nabla_H u|^p dx.$$

Since $\mathcal{L}_p$ is a continuous operator from $\overset{\circ}{S}{}^{1,p}(\Omega)$ to $S^{-1,p'}(\Omega)$, $p'=\frac{p}{p-1}$, we have $\mathcal{L}_p v_k\to\mathcal{L}_p v$ in $S^{-1,p'}(\Omega)$ and $f(v_k)\to f(v)$ pointwise since f is a locally Lipschitz continuous function on $(0,\infty)$. Thus, by the Lebesque dominated convergence theorem and using the fact that f is an increasing function on $(0,\infty)$, for any $u\in C_0^\infty(\Omega)$ we arrive at the inequality

$$\int_\Omega \frac{|u|^p}{f(v)}(-\mathcal{L}_p v)dx \le \int_\Omega |\nabla_H u|^p dx,$$

proving (11.93). □

As a consequence of the Harnack inequality for the general hypoelliptic operator (see [CDG93, Theorem 3.1]) one has the following strong maximum principle for the p-sub-Laplacian.

Lemma 11.6.4 (Strong maximum principle for the p-sub-Laplacian)**.** *Let $1 < p \leq Q$, let $\Omega \subset \mathbb{G}$ be a bounded open set and let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$|F(x, \rho)| \leq C(\rho^{p-1} + 1)$$

for all $\rho > 0$. Let $u \in \overset{\circ}{S}{}^{1,p}(\Omega)$ be a non-negative solution of

$$\begin{cases} -\mathcal{L}_p u = F(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then $u \equiv 0$ or $u > 0$ in Ω.

Proof of Lemma 11.6.4. Since $u \in \overset{\circ}{S}{}^{1,p}(\Omega)$ by using the Harnack inequality [CDG93, Theorem 3.1] for $1 < p \leq Q$ there exists a constant C_R such that

$$\operatorname*{ess\,sup}_{B(x,R)} u \leq C_R \operatorname*{ess\,inf}_{B(x,R)} u$$

holds for any $x \in \Omega$ and quasi-ball $B(0, R)$. This means that $u \equiv 0$ or $u > 0$ in Ω. □

Combining Lemma 11.6.3 and Lemma 11.6.4 one has the following generalized Picone inequality on $\mathbb{G}$, which is a key ingredient for proofs of both a comparison principle and uniqueness of a positive solution to nonlinear p-sub-Laplacian equations.

Lemma 11.6.5 (Generalized Picone inequality)**.** *Let $\Omega \subset \mathbb{G}$ be a bounded open set and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be positive, bounded and measurable function such that*

$$g(x, \rho) \leq C(\rho^{p-1} + 1)$$

for all $\rho > 0$. Then we have

$$\int_\Omega \frac{|u|^p}{f(v)}(-\mathcal{L}_p v)dx \leq \int_\Omega |\nabla_H u|^p dx, \quad 1 < p < \infty, \tag{11.94}$$

for all v, $u \in \overset{\circ}{S}{}^{1,p}(\Omega)$ and $v(\not\equiv 0) \geq 0$ a.e. $\Omega \in \mathbb{G}$ such that

$$-\mathcal{L}_p v = g(x, v).$$

Proof of Lemma 11.6.5. By Lemma 11.6.4 we have $v > 0$ in Ω. Let

$$v_k(x) := v(x) + \frac{1}{k}, \; k = 1, 2, \ldots.$$

Then we have $\mathcal{L}_p v_k = \mathcal{L}_p v$ in $S^{-1,p'}(\Omega)$, $v_k \to v$ a.e. in Ω and also $f(v_k) \to f(v)$ pointwise in Ω. Let $u_k \in C_0^\infty(\Omega)$ be such that $u_k \to u$ in $\overset{\circ}{S}{}^{1,p}(\Omega)$. For the functions u_k and v_k Lemma 11.6.3 gives

$$\int_\Omega \frac{|u_k|^p}{f(v_k)}(-\mathcal{L}_p v_k)dx \leq \int_\Omega |\nabla_H u_k|^p dx.$$

Now since $f(v_k) \to f(v)$ pointwise, by the Fatou lemma we arrive at

$$\int_\Omega \frac{|u|^p}{f(v)}(-\mathcal{L}_p v)dx \leq \int_\Omega |\nabla_H u|^p dx.$$

This completes the proof. □

Lemma 11.6.5 implies the following comparison type principle:

Theorem 11.6.6 (Comparison principle for p-sub-Laplacian)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain. Let $0 < q < p-1$ and let F be a non-negative function such that $F \not\equiv 0$. Let $u, v \in \overset{\circ}{S}{}^{1,p}(\Omega)$ be real-valued functions such that*

$$\begin{cases} -\mathcal{L}_p u \geq F(x)u^q, & u > 0 \quad \text{in } \Omega, \\ -\mathcal{L}_p v \leq F(x)v^q, & v > 0 \quad \text{in } \Omega. \end{cases} \tag{11.95}$$

Then we have $v \leq u$ a.e. in Ω.

Proof of Theorem 11.6.6. It follows from (11.95) that

$$F(x)\left(\frac{u^q}{u^{p-1}} - \frac{v^q}{v^{p-1}}\right) \leq \frac{-\mathcal{L}_p u}{u^{p-1}} + \frac{\mathcal{L}_p v}{v^{p-1}}.$$

Multiplying both sides by $w = (v^p - u^p)_+$ and integrating over Ω we have

$$\begin{aligned} \int_{[v>u]} F(x)\left(\frac{u^q}{u^{p-1}} - \frac{v^q}{v^{p-1}}\right) w dx &= \int_\Omega F(x)\left(\frac{u^q}{u^{p-1}} - \frac{v^q}{v^{p-1}}\right) w dx \\ &\leq \int_\Omega \left(\frac{-\mathcal{L}_p u}{u^{p-1}} + \frac{\mathcal{L}_p v}{v^{p-1}}\right) w dx. \end{aligned} \tag{11.96}$$

Moreover, we have

$$\begin{aligned} &\int_\Omega \left(\frac{-\mathcal{L}_p u}{u^{p-1}} + \frac{\mathcal{L}_p v}{v^{p-1}}\right) w dx \\ &= \int_\Omega |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H \left(\frac{w}{u^{p-1}}\right) dx \\ &\quad - \int_\Omega |\nabla_H v|^{p-2}\nabla_H v \cdot \nabla_H \left(\frac{w}{v^{p-1}}\right) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega\cap[v>u]} |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H \left(\frac{v^p - u^p}{u^{p-1}}\right) dx \\
&\quad - \int_{\Omega\cap[v>u]} |\nabla_H v|^{p-2}\nabla_H v \cdot \nabla_H \left(\frac{v^p - u^p}{v^{p-1}}\right) dx \\
&= \int_{\Omega\cap[v>u]} \left(|\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H \left(\frac{v^p}{u^{p-1}}\right) - |\nabla_H v|^p\right) dx \\
&\quad + \int_{\Omega\cap[v>u]} \left(|\nabla_H v|^{p-2}\nabla_H v \cdot \nabla_H \left(\frac{u^p}{v^{p-1}}\right) - |\nabla_H u|^p\right) dx \\
&= I_1 + I_2,
\end{aligned}
$$

where

$$
I_1 := \int_{\Omega\cap[v>u]} \left(|\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H \left(\frac{v^p}{u^{p-1}}\right) - |\nabla_H v|^p\right) dx
$$

and

$$
I_2 := \int_{\Omega\cap[v>u]} \left(|\nabla_H v|^{p-2}\nabla_H v \cdot \nabla_H \left(\frac{u^p}{v^{p-1}}\right) - |\nabla_H u|^p\right) dx.
$$

We notice that

$$
\begin{aligned}
I_1 &= \int_{\Omega\cap[v>u]} |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H \left(\frac{v^p}{u^{p-1}}\right) dx - \int_{\Omega\cap[v>u]} |\nabla_H v|^p dx \\
&= -\int_{\Omega\cap[v>u]} \frac{v^p}{u^{p-1}} \mathcal{L}_p u dx - \int_{\Omega\cap[v>u]} |\nabla_H v|^p dx \leq 0.
\end{aligned}
$$

In the last line we have used Green's first identity (1.90) and the Picone inequality (11.94). Similarly, we see that $I_2 \leq 0$. Thus, we obtain

$$
\int_\Omega \left(\frac{-\mathcal{L}_p u}{u^{p-1}} + \frac{\mathcal{L}_p v}{v^{p-1}}\right) w dx \leq 0.
$$

Consequently, (11.96) implies that

$$
\int_{\Omega\cap[v>u]} F(x) \left(\frac{u^q}{u^{p-1}} + \frac{v^q}{v^{p-1}}\right)(v^p - u^p) dx \leq 0.
$$

On the other hand, we have

$$
0 \leq F(x) \left(\frac{u^q}{u^{p-1}} + \frac{v^q}{v^{p-1}}\right)
$$

on the set $[v > u]$. This means that $|[v > u]| = 0$. □

As another consequence of Lemma 11.6.5 we have the following Díaz–Saá inequality on stratified Lie groups.

Lemma 11.6.7 (Díaz–Saá inequality). *Let Ω be an admissible domain. Let functions g_1 and g_2 satisfy the assumption of Theorem 11.6.5. If functions u_1, $u_2 \in \overset{\circ}{S}{}^{1,p}(\Omega)$ with u_1, $u_2 (\not\equiv 0) \geq 0$ a.e. $\Omega \in \mathbb{G}$ are such that*

$$-\mathcal{L}_p u_1 = g_1(x, u_1) \quad \text{and} \quad -\mathcal{L}_p u_2 = g_2(x, u_2),$$

then we have

$$0 \leq \int_\Omega \left(\frac{-\mathcal{L}_p u_1}{u_1^{p-1}} + \frac{\mathcal{L}_p u_2}{u_2^{p-1}} \right) (u_1^p - u_2^p) dx.$$

Proof of Lemma 11.6.7. Let functions u_1 and u_2 satisfy the assumptions. Then by the inequality (11.94) with $f(u) = u^{p-1}$ as well as for u_1 and u_2 we have

$$\int_\Omega \frac{|u_1|^p}{u_2^{p-1}} (-\mathcal{L}_p u_2) dx \leq \int_\Omega |\nabla_H u_1|^p dx.$$

Using Green's first identity (1.90) we get

$$0 \leq \int_\Omega \left(\frac{-\mathcal{L}_p u_1}{u_1^{p-1}} + \frac{\mathcal{L}_p u_2}{u_2^{p-1}} \right) u_1^p dx. \tag{11.97}$$

Again, by the inequality (11.94) we have

$$\int_\Omega \frac{|u_2|^p}{u_1^{p-1}} (-\mathcal{L}_p u_1) dx \leq \int_\Omega |\nabla_H u_2|^p dx.$$

As above, this implies

$$0 \leq \int_\Omega \left(\frac{\mathcal{L}_p u_1}{u_1^{p-1}} - \frac{\mathcal{L}_p u_2}{u_2^{p-1}} \right) u_2^p dx. \tag{11.98}$$

Now the combination of (11.97) and (11.98) completes the proof. □

The established properties allow one to show the uniqueness of a positive solution for the equation

$$\begin{cases} -\mathcal{L}_p u = F(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{11.99}$$

where Ω is an admissible domain. For convenience, we recall once again the assumptions on $F(x, u)$:

(a) The function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a positive, bounded and measurable function and there exists a positive constant $C > 0$ such that

$$F(x, \rho) \leq C(\rho^{p-1} + 1)$$

holds for a.e. $x \in \Omega$.

(b) The function

$$\rho \mapsto \frac{F(x,\rho)}{\rho^{p-1}}$$

is strictly decreasing on $(0,\infty)$ for a.e. $x \in \Omega$.

Theorem 11.6.8 (Uniqueness of positive solutions)**.** *There exists at most one positive weak solution to* (11.99) *for* $1 < p \leq Q$.

Proof of Theorem 11.6.8*.* First, assuming that u_1 and u_2 are two different ($u_1 \not\equiv u_2$) non-negative solutions of (11.99) and using Lemma 11.6.4 for these functions we conclude that $u_1 > 0$ and $u_2 > 0$ in Ω. Then Lemma 11.6.7 implies that

$$0 \leq \int_\Omega \left(\frac{-\mathcal{L}_p u_1}{u_1^{p-1}} + \frac{\mathcal{L}_p u_2}{u_2^{p-1}} \right) (u_1^p - u_2^p) dx.$$

Moreover, from the assumption (b) it follows that

$$\int_\Omega \left(\frac{F(x,u_1)}{u_1^{p-1}} - \frac{F(x,u_2)}{u_2^{p-1}} \right) (u_1^p - u_2^p) dx < 0.$$

On the other hand, we have

$$\int_\Omega \left(\frac{-\mathcal{L}_p u_1}{u_1^{p-1}} + \frac{\mathcal{L}_p u_2}{u_2^{p-1}} \right) (u_1^p - u_2^p) dx = \int_\Omega \left(\frac{F(x,u_1)}{u_1^{p-1}} - \frac{F(x,u_2)}{u_2^{p-1}} \right) (u_1^p - u_2^p) dx,$$

and this contradicts that both u_1 and u_2 ($u_1 \not\equiv u_2$) are non-negative solutions of (11.99). ☐