Chapter 9

Uncertainty Relations on Homogeneous Groups

In this chapter we discuss relations between main operators of quantum mechanics, that is, relations between momentum and position operators as well as Euler and Coulomb potential operators on homogeneous groups as well as their consequences. Since in most uncertainty relations and in these operators the appearing weights are radially symmetric, it turns out that these relations can be extended to also hold on general homogeneous groups. In particular, we obtain both isotropic and anisotropic uncertainty principles in a refined form, where the radial derivative operators are used instead of the elliptic or hypoelliptic differential operators.

Throughout this book, most of the inequalities imply the corresponding uncertainty principles. An example of such an uncertainty principle was given, for example, in Corollary 2.1.3 as a consequence of the Hardy inequality, and also in Corollary 3.3.5. However, in this chapter we aim at presenting an independent treatment of inequalities following from certain identities involving the appearing operators. In this respect such uncertainty relations can be sometimes obtained independently from Hardy inequalities in alternative ways, see, e.g., also Ciatti, Ricci and Sundari [CRS07].

In general, the uncertainty principles in different form have attracted a lot of attention due to their physical applications. For example, a fundamental element of the quantum mechanics is the uncertainty principle of Werner Heisenberg [Hei27]. It is worth observing that his original argument, while conceptually enlightening, was experiential.

Then Wolfgang Pauli and Hermann Weyl provided the mathematical aspects of uncertainty relations involving position and momentum operators, but the first rigorous proof was given by Earle Kennard [Ken27]. Charles Fefferman's work [Fef83] and [FP81] was a starting point of studies to widely present the interpretation of uncertainty inequalities as spectral properties of differential operators. Nowadays there is vast literature on uncertainty relations and their applications. Since we do not aim here at presenting a survey of the uncertainty relations on the

M. Ruzhansky, D. Suragan, *Hardy Inequalities on Homogeneous Groups*,
Progress in Mathematics 327, https://doi.org/10.1007/978-3-030-02895-4_10

Euclidean space $\mathbb{R}^n$, we only refer to relevant works. In general, we can refer to a recent survey [CBTW15] for further discussions and references on this subject in the Euclidean setting, as well as to [FS97] for an overview of the history and the relevance of this type of inequalities from a purely mathematical point of view. The link between the uncertainty principles and the adapted Fourier analysis has been explored in [Tha04]. The exposition of the present chapter is based on [RS17f].

9.1 Abstract position and momentum operators

The idea for our presentation is to introduce abstract position and momentum operators $\mathcal{P}$ and $\mathcal{M}$ that satisfy certain relations. In particular, the classical position and momentum operators of quantum physics satisfy these assumptions. However, this abstract point of view allows one to take different versions of position-momentum pairs depending on the setting. We will exemplify such a possibility of different choices in the case of the Heisenberg group, see Example 9.1.4.

9.1.1 Definition and assumptions

Throughout this chapter, the abstract position and momentum operators $\mathcal{P}$ and $\mathcal{M}$ will be assumed to satisfy the following properties.

Definition 9.1.1 (Abstract position and momentum operators)**.** Let $\mathcal{P}$ and $\mathcal{M}$ be linear operators which are densely defined from $L^2(\mathbb{G})$ to $L^2(\mathbb{G})$, with their domains containing $C_0^\infty(\mathbb{G})$, and such that $C_0^\infty(\mathbb{G})$ is an invariant subspace for them, that is,

$$\mathcal{P}(C_0^\infty(\mathbb{G})) \subset C_0^\infty(\mathbb{G}) \text{ and } \mathcal{M}(C_0^\infty(\mathbb{G})) \subset C_0^\infty(\mathbb{G}).$$

We will say that such operators $\mathcal{P}$ and $\mathcal{M}$ are *abstract position and momentum operators* if they satisfy the relations

$$2\,\mathrm{Re}\left(\mathcal{P}f\overline{(i\mathcal{M})f}\right) = (\mathcal{P}\circ(i\mathcal{M}))|f|^2 = \mathbb{E}|f|^2 \tag{9.1}$$

for all $f \in C_0^\infty(\mathbb{G})$. We will denote by $D(\mathcal{P})$ and $D(\mathcal{M})$ the domains of operators $\mathcal{P}$ and $\mathcal{M}$, respectively.

Before giving examples, let us make some remarks concerning the meaning of the equalities in (9.1).

Remark 9.1.2.

1. The first equality in (9.1) gives a relation between the position and momentum operator; it will be clear from Example 9.1.3 that it is satisfied by the classical position and momentum operators of the Euclidean quantum mechanics. This condition is instrumental in establishing several further properties of these operators.

2. The second equality in (9.1) relates to position and momentum operators to the Euler operator $\mathbb{E}$ on $\mathbb{G}$, which was defined by

$$\mathbb{E} := |x|\mathcal{R}, \tag{9.2}$$

where $\mathcal{R}$ is the radial derivative operator, see Section 1.3.2 for a discussion of its properties. The main characterizing feature of the Euler operator is its responsibility for the homogeneity property on $\mathbb{G}$ given in Proposition 1.3.1, namely, that

$$f(\lambda x) = \lambda^{\nu} f(x) \text{ for all } \lambda > 0 \text{ if and only if } \mathbb{E}f = \nu f,$$

for any differentiable function f on $\mathbb{G}$. Thus, the assumption (9.1) for the abstract position and momentum operators says that they have to give the factorisation of the Euler operator $\mathbb{E}$ as in the second equality in (9.1). In this sense, the second equality in (9.1) relates position and momentum operators to the homogeneous structure of the group $\mathbb{G}$.

3. We can note that already in the anisotropic and even isotropic $\mathbb{R}^n$ the results of this chapter give some new insights in view of an arbitrary choice of a homogeneous quasi-norm $|\cdot|$ and the abstract nature of these operators.

4. It is rather curious that equalities (9.1) already imply uncertainty relations of several types, such as the Heisenberg–Kennard and Heisenberg–Pauli–Weyl type uncertainty inequalities. Moreover, the property that the operators $\mathcal{P}$ and $i\mathcal{M}$ factorise the Euler operator allows one to establish further relations between them and other operators such as the radial operator, the dilations generating operator, and the Coulomb potential operator, and prove some equalities and inequalities among them. Such relations are presented in this chapter.

9.1.2 Examples

If the group $\mathbb{G}$ is the Euclidean $\mathbb{R}^n$ with isotropic (standard) dilations and the usual Abelian structure, then the operators

$$\mathcal{P} := x \quad \text{and} \quad \mathcal{M} := -i\nabla, \tag{9.3}$$

i.e., the multiplication and the gradient (multiplied by $-i$), satisfy assumptions (9.1). The same will hold on general homogeneous groups, as we show in Example 9.1.3. Thus, we now give several other examples extending this to general homogeneous groups. In particular, we give an example of a choice of abstract position and momentum operators on general homogeneous groups. Furthermore, we show that other choices are possible, which we exemplify in the case of the Heisenberg group.

Example 9.1.3 (Position and momentum on general homogeneous groups)**.** Let $\mathbb{G}$ be a homogeneous group. Let us define the operators

$$\mathcal{P} := x,\ x \in \mathbb{G}, \quad \text{and} \quad \mathcal{M} := -i\nabla_E, \tag{9.4}$$

where

$$\nabla_E = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

is an anisotropic gradient on $\mathbb{G}$ consisting of partial derivatives with respect to coordinate functions. We understand the operator $\mathcal{P}$ as the scalar multiplication operator by the coordinates of the variable x, i.e.,

$$\mathcal{P}v = \sum x_j v_j,$$

where x_j are the coordinate functions of $x \in \mathbb{G}$, see Section 1.2.4 for a discussion of these functions on homogeneous groups.

These operators $\mathcal{P}$ and $\mathcal{M}$ are the position and momentum operators in the sense of Definition 9.1.1. Indeed, first we observe that by elementary properties of derivatives the first equality in the following relations is satisfied:

$$2\,\mathrm{Re}\,(xf \cdot \nabla_E f) = x \cdot \nabla_E |f|^2 = \mathbb{E}|f|^2. \tag{9.5}$$

The second equality in (9.5) follows if we recall that $\mathbb{E}$ is the Euler operator from (1.37), that is, we have the relations

$$\mathbb{E} = x \cdot \nabla_E \text{ and } \mathcal{R} = \frac{x \cdot \nabla_E}{|x|} = \frac{d}{d|x|},$$

see (1.35). In the notation (9.4) the relations (9.5) can be expressed as

$$2\,\mathrm{Re}\left(\mathcal{P}f\overline{i\mathcal{M}f}\right) = (\mathcal{P} \circ (i\mathcal{M}))|f|^2 = \mathbb{E}|f|^2, \tag{9.6}$$

showing that (9.1) is satisfied.

We note that the left invariant gradient $\nabla = \nabla_X = (X_1, \ldots, X_n)$ and the anisotropic (Euclidean) gradient ∇_E are related and can be expressed in terms of each other. For example, we can recall the relations

$$\frac{\partial}{\partial x_j} = X_j + \sum_{\substack{1 \le k \le n \\ \nu_j < \nu_k}} p_{j,k} X_k,$$

for some homogeneous polynomials $p_{j,k}$ on $\mathbb{G}$ of homogeneous degree $\nu_k - \nu_j > 0$, see Section 1.2.4.

Example 9.1.4 (Another choice of position and momentum operators on the Heisenberg group)**.** Let us consider the Heisenberg group $\mathbb{H} = \mathbb{H}^1$, topologically equivalent to $\mathbb{R}^3$. As for general nilpotent Lie groups (see Proposition 1.1.1), the exponential map of $\mathbb{H}$ is globally invertible and its inverse map is given by the formula

$$\exp_{\mathbb{H}}^{-1}(x) = e(x) \cdot \nabla_X \equiv \sum_{j=1}^{3} e_j(x) X_j, \tag{9.7}$$

where $\nabla_X = (X_1, X_2, X_3)$ is the full gradient of $\mathbb{H}$ with

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}, \\ X_2 &= \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}, \\ X_3 &= -4 \frac{\partial}{\partial x_3}, \end{aligned}$$

as well as

$$e(x) = (e_1(x), e_2(x), e_3(x)),$$

where

$$\begin{aligned} e_1(x) &= x_1, \\ e_2(x) &= x_2, \\ e_3(x) &= -\frac{1}{4} x_3. \end{aligned}$$

We define the position and momentum operators for this case to be

$$\mathcal{P} := e(x), \ x \in \mathbb{G}, \ \text{ and } \mathcal{M} := -i\nabla_X. \tag{9.8}$$

One can readily see that these operators satisfy the relations (9.6). Now let us check the relation (1.37) between the Euler operator $E_{\mathbb{H}} := e(x) \cdot \nabla_X$ and the radial operator $\mathcal{R}_{\mathbb{H}} = \frac{d}{d|x|}$:

$$\begin{aligned} E_{\mathbb{H}} &= e(x) \cdot \nabla_X \\ &= x_1 \left(\frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3} \right) + x_2 \left(\frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3} \right) - \frac{1}{4} x_3 \left(-4 \frac{\partial}{\partial x_3} \right) \\ &= x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \\ &= |x| \left(\frac{x_1}{|x|} \frac{\partial}{\partial x_1} + \frac{x_2}{|x|} \frac{\partial}{\partial x_2} + \frac{x_3}{|x|} \frac{\partial}{\partial x_3} \right) \\ &= |x| \frac{d}{d|x|} = |x| \mathcal{R}_{\mathbb{H}}. \end{aligned}$$

9.2 Position-momentum relations

In this section, we show further relations between abstract position $\mathcal{P}$ and momentum $\mathcal{M}$ operators on homogeneous groups. The obtained relations are the consequences of the homogeneous group's structure and the equalities (9.1).

9.2.1 Further position-momentum identities

We start with certain further identities involving the abstract position and momentum operators as a consequence of equalities (9.1).

Theorem 9.2.1 (Position-momentum identities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$. Then for every $f \in D(\mathcal{P})\bigcap D(\mathcal{M})$ with $\mathcal{P}f \not\equiv 0$ and $\mathcal{M}f \not\equiv 0$, we have the identity*

$$\begin{aligned}\|\mathcal{P}f\|^2_{L^2(\mathbb{G})} + \|\mathcal{M}f\|^2_{L^2(\mathbb{G})} &= Q\|f\|^2_{L^2(\mathbb{G})} + \|\mathcal{P}f - i\mathcal{M}f\|^2_{L^2(\mathbb{G})} \\ &= \|\mathcal{P}f\|_{L^2(\mathbb{G})}\|\mathcal{M}f\|_{L^2(\mathbb{G})}\left(2 - \left\|\frac{\mathcal{P}f}{\|\mathcal{P}f\|_{L^2(\mathbb{G})}} + \frac{i\mathcal{M}f}{\|\mathcal{M}f\|_{L^2(\mathbb{G})}}\right\|^2_{L^2(\mathbb{G})}\right) \\ &\quad + \|\mathcal{P}f + i\mathcal{M}f\|^2_{L^2(\mathbb{G})}. \end{aligned} \tag{9.9}$$

Proof of Theorem 9.2.1. It is enough to show (9.9) for functions $f \in C_0^\infty(\mathbb{G})$. Indeed, in this case, because $C_0^\infty(\mathbb{G})$ is dense in $L^2(\mathbb{G})$, it is also true on $D(\mathcal{P})\bigcap D(\mathcal{M})$ by density. Using the polar decomposition from Proposition 1.2.10, the definition (1.30) of the radial operator, and equality (9.1), we calculate

$$\begin{aligned}-2\,\mathrm{Re}\int_{\mathbb{G}} \mathcal{P}f\overline{i\mathcal{M}f}dx &= -\int_{\mathbb{G}} \mathcal{P}i\mathcal{M}|f|^2dx \\ &= -\int_0^\infty\int_{\wp} r^Q\frac{1}{r}\mathbb{E}|f|^2d\sigma(y)dr \\ &= -\int_0^\infty\int_{\wp} r^Q\frac{d|f|^2}{dr}d\sigma(y)dr \\ &= Q\int_0^\infty\int_{\wp} r^{Q-1}|f|^2d\sigma(y)dr \\ &= Q\int_{\mathbb{G}}|f|^2dx \\ &= Q\|f\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

Combining this with the equality

$$\|\mathcal{P}f\|^2_{L^2(\mathbb{G})} + \|\mathcal{M}f\|^2_{L^2(\mathbb{G})} = \|\mathcal{P}f + i\mathcal{M}f\|^2_{L^2(\mathbb{G})} - 2\,\mathrm{Re}\int_{\mathbb{G}}\mathcal{P}f\overline{i\mathcal{M}f}dx$$

we obtain the first equality in (9.9). On the other hand, we have

$$\begin{aligned}&-2\,\mathrm{Re}\int_{\mathbb{G}}\mathcal{P}f\overline{i\mathcal{M}f}dx\\&=\|\mathcal{M}f\|_{L^2(\mathbb{G})}\|\mathcal{P}f\|_{L^2(\mathbb{G})}\left(2-\left\|\frac{\mathcal{P}f}{\|\mathcal{P}f\|_{L^2(\mathbb{G})}}+\frac{i\mathcal{M}f}{\|\mathcal{M}f\|_{L^2(\mathbb{G})}}\right\|^2_{L^2(\mathbb{G})}\right),\end{aligned}$$

yielding the second equality in (9.9). □

9.2.2 Heisenberg–Kennard and Pythagorean inequalities

Immediately from (9.9) we observe the Heisenberg–Kennard type inequality as its consequence. In the Abelian case (see, e.g., [SZ97] and [WM08]) it is also sometimes called the Kennard uncertainty inequality.

Corollary 9.2.2 (Heisenberg–Kennard uncertainty principle)**.** *We have*

$$\frac{Q}{2}\|f\|^2_{L^2(\mathbb{G})}\leq\|\mathcal{P}f\|_{L^2(\mathbb{G})}\|\mathcal{M}f\|_{L^2(\mathbb{G})}. \tag{9.10}$$

The first equality in (9.9) also implies the following Pythagorean type inequality:

Corollary 9.2.3 (Pythagorean type inequality)**.** *We have*

$$\|\sqrt{Q}f\|^2_{L^2(\mathbb{G})}\leq\|\mathcal{P}f\|^2_{L^2(\mathbb{G})}+\|\mathcal{M}f\|^2_{L^2(\mathbb{G})}. \tag{9.11}$$

Equalities (9.9) also imply the following conditions for reaching the equalities in Heisenberg–Kennard and Pythagorean inequalities:

Corollary 9.2.4 (Equalities in Heisenberg–Kennard and Pythagorean inequalities)**.** *Let $f\in D(\mathcal{P})\bigcap D(\mathcal{M})$ be such that $\mathcal{P}f\not\equiv 0$ and $\mathcal{M}f\not\equiv 0$.*

(i) *The equality case in the Heisenberg–Kennard uncertainty inequality* (9.10) *holds, that is,*

$$\frac{Q}{2}\|f\|^2_{L^2(\mathbb{G})}=\|\mathcal{P}f\|_{L^2(\mathbb{G})}\|\mathcal{M}f\|_{L^2(\mathbb{G})}$$

if and only if

$$\|\mathcal{P}f\|_{L^2(\mathbb{G})}i\mathcal{M}f=\|\mathcal{M}f\|_{L^2(\mathbb{G})}\mathcal{P}f.$$

(ii) *For $f\in D(\mathcal{P})\bigcap D(\mathcal{M})$ we have the Pythagorean equality*

$$\|\sqrt{Q}f\|^2_{L^2(\mathbb{G})}=\|\mathcal{P}f\|^2_{L^2(\mathbb{G})}+\|\mathcal{M}f\|^2_{L^2(\mathbb{G})}$$

if and only if

$$\mathcal{P}f=i\mathcal{M}f.$$

9.3 Euler–Coulomb relations

In addition to the Euler operator that was defined by

$$\mathbb{E}f := |x|\mathcal{R}f, \tag{9.12}$$

we also define the *Coulomb potential operator* as

$$\mathcal{C}f := \frac{1}{|x|}f. \tag{9.13}$$

The domains of these operators are given, respectively, by

$$D(\mathbb{E}) = \{f \in L^2(\mathbb{G}) : \ \ \mathbb{E}f \in L^2(\mathbb{G})\} \tag{9.14}$$

and

$$D(\mathcal{C}) = \{f \in L^2(\mathbb{G}) : \frac{1}{|x|}f \in L^2(\mathbb{G})\}. \tag{9.15}$$

It is also immediate to observe from (1.30) that the composition of the Euler operator and Coulomb operators gives the radial derivative operator $\mathcal{R}$:

$$\mathcal{R} := \mathcal{C}\mathbb{E}. \tag{9.16}$$

Recall that in Theorem 2.1.5 it was shown that for each $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ one has the identity

$$\left\|\frac{1}{|x|^\alpha}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})} = \left(\frac{Q-2}{2}-\alpha\right)^2\left\|\frac{f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})} + \left\|\frac{1}{|x|^\alpha}\mathcal{R}f + \frac{Q-2-2\alpha}{2|x|^{\alpha+1}}f\right\|^2_{L^2(\mathbb{G})}, \tag{9.17}$$

for all $\alpha \in \mathbb{R}$. If $\alpha = 0$ from (9.17) we obtain the equality

$$\|\mathcal{R}f\|^2_{L^2(\mathbb{G})} = \left(\frac{Q-2}{2}\right)^2\left\|\frac{1}{|x|}f\right\|^2_{L^2(\mathbb{G})} + \left\|\mathcal{R}f + \frac{Q-2}{2|x|}f\right\|^2_{L^2(\mathbb{G})}. \tag{9.18}$$

As it was already shown before, by dropping the non-negative last term in (9.18) we immediately obtain a version of L^2-Hardy's inequality on $\mathbb{G}$:

$$\left\|\frac{f}{|x|}\right\|_{L^2(\mathbb{G})} \leq \frac{2}{Q-2}\|\mathcal{R}f\|_{L^2(\mathbb{G})}, \quad Q \geq 3, \tag{9.19}$$

with the constant being sharp for any quasi-norm $|\cdot|$.

9.3.1 Heisenberg–Pauli–Weyl uncertainty principle

An L^p-version of the Heisenberg–Pauli–Weyl uncertainty principle was given in Corollary 3.3.5 for a particular choice of position and momentum operators. In this section we show its abstract version for abstract position and momentum operators, although restricting the consideration, as usual in this chapter, to the case of L^2-spaces.

Thus, by a standard argument the inequality (9.19) implies the following Heisenberg–Pauli–Weyl type uncertainty principle on homogeneous groups:

Proposition 9.3.1 (Heisenberg–Pauli–Weyl type uncertainty principle)**.** *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q \geq 3$*. Then for each* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *and any homogeneous quasi-norm* $|\cdot|$ *on* $\mathbb{G}$ *we have*

$$\|f\|^2_{L^2(\mathbb{G})} \leq \frac{2}{Q-2} \|\mathcal{R}f\|_{L^2(\mathbb{G})} \||x|f\|_{L^2(\mathbb{G})} \,. \tag{9.20}$$

Proof of Corollary 9.3.1*.* From the inequality (9.19) we get

$$\begin{aligned}\left(\int_{\mathbb{G}} |\mathcal{R}f|^2\, dx\right)^{1/2} &\left(\int_{\mathbb{G}} |x|^2|f|^2 dx\right)^{1/2} \\ &\geq \frac{Q-2}{2}\left(\int_{\mathbb{G}} \frac{|f|^2}{|x|^2}\, dx\right)^{1/2} \left(\int_{\mathbb{G}} |x|^2|f|^2 dx\right)^{1/2} \geq \frac{Q-2}{2}\int_{\mathbb{G}} |f|^2 dx,\end{aligned}$$

where we have used the Hölder inequality in the last line. This shows (9.20). □

Remark 9.3.2.

1. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $Q = n$, so that (9.20) implies the uncertainty principle with any homogeneous quasi-norm $|\cdot|$:

$$\left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^2 \leq \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} \left|\frac{x}{|x|}\cdot\nabla f(x)\right|^2 dx \int_{\mathbb{R}^n} |x|^2|f(x)|^2 dx, \tag{9.21}$$

which in turn implies the classical uncertainty principle for $\mathbb{G} \equiv \mathbb{R}^n$ with the standard Euclidean distance $|x|_E$:

$$\left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^2 \leq \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |\nabla f(x)|^2 dx \int_{\mathbb{R}^n} |x|_E^2|f(x)|^2 dx, \tag{9.22}$$

which is the classical *Heisenberg–Pauli–Weyl uncertainty principle* on $\mathbb{R}^n$. For the improved constant in (9.22) see (9.39).

2. Different versions of this uncertainty principle have been considered in different settings, for example in those of stratified groups. We can refer to [GL90], [CRS07], [CCR15] for some results, and further estimates will be shown in Section 12.4.

Moreover, we have the following Pythagorean relation for the Euler operator:

Proposition 9.3.3 (Pythagorean relation for Euler operator)**.** *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q \geq 3$. *Then we have*

$$\|\mathbb{E}f\|^2_{L^2(\mathbb{G})} = \left\|\frac{Q}{2}f\right\|^2_{L^2(\mathbb{G})} + \left\|\mathbb{E}f + \frac{Q}{2}f\right\|^2_{L^2(\mathbb{G})} \tag{9.23}$$

for any $f \in D(\mathbb{E})$.

Proof of Proposition 9.3.3. Taking $\alpha = -1$, from (9.17) we obtain (9.23) for any $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Since $D(\mathbb{E}) \subset L^2(\mathbb{G})$ and $C_0^\infty(\mathbb{G}\backslash\{0\})$ is dense in $L^2(\mathbb{G})$, this implies that (9.23) is also true on $D(\mathbb{E})$ by density. □

Simply by dropping the positive term in the right-hand side, (9.23) implies

Corollary 9.3.4 (Lower bound for Euler operator)**.** *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q \geq 3$. *Then we have*

$$\|f\|_{L^2(\mathbb{G})} \leq \frac{2}{Q}\|\mathbb{E}f\|_{L^2(\mathbb{G})}, \tag{9.24}$$

for any $f \in D(\mathbb{E})$.

9.4 Radial dilations – Coulomb relations

Using the radial derivative and Coulomb operators we can define the generator of dilations operator by

$$\mathcal{R}_g := -i\left(\mathcal{R} + \frac{Q-1}{2}\mathcal{C}\right) \tag{9.25}$$

with the domain

$$D(\mathcal{R}_g) = \{f \in L^2(\mathbb{G}) : \mathcal{R}f \in L^2(\mathbb{G}), \mathcal{C}f \in L^2(\mathbb{G})\}. \tag{9.26}$$

First we record a commutator relation between this generator of dilations operator $\mathcal{R}_g$ and the Coulomb potential operator:

Lemma 9.4.1 (Commutator relation between generator of dilations and Coulomb operators)**.** *Let* $\mathbb{G}$ *be a homogeneous group of dimension* $Q \geq 1$. *Then for any* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have*

$$[\mathcal{R}_g, \mathcal{C}]f = i\mathcal{C}^2 f, \tag{9.27}$$

where $[\mathcal{R}_g, \mathcal{C}] = \mathcal{R}_g\mathcal{C} - \mathcal{C}\mathcal{R}_g$.

Proof of Lemma 9.4.1. Denoting $r := |x|$ we have $\mathcal{C} = \frac{1}{r}$, and from (1.30) it follows that $\mathcal{R}_g = -i\left(\frac{d}{dr} + \frac{Q-1}{2r}\right)$. Thus, a direct calculation shows

$$\begin{aligned}\mathcal{R}_g, \mathcal{C}tf &= \mathcal{R}_g\mathcal{C}f - \mathcal{C}\mathcal{R}_g f \\ &= -i\left(-\frac{1}{r^2} + \frac{1}{r}\frac{d}{dr} + \frac{Q-1}{2r^2} - \frac{1}{r}\frac{d}{dr} - \frac{Q-1}{2r^2}\right)f = i\frac{1}{r^2}f = i\mathcal{C}^2 f,\end{aligned}$$

establishing (9.27). □

We now analyse further properties of these operators.

Lemma 9.4.2 (Operators $\mathcal{R}_g$ and $\mathcal{C}$ are symmetric)**.** *Operators $\mathcal{R}_g$ and $\mathcal{C}$ are symmetric.*

Proof of Lemma 9.4.2. It is a straightforward to see that $\mathcal{C}$ is symmetric, that is,

$$\int_{\mathbb{G}} (\mathcal{C}f)\overline{f}dx = \int_{\mathbb{G}} f(\overline{\mathcal{C}f})dx.$$

Now we need to show that

$$\int_{\mathbb{G}} (\mathcal{R}_g f)\overline{f}dx = \int_{\mathbb{G}} f(\overline{\mathcal{R}_g f})dx \tag{9.28}$$

for any $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Since $D(\mathcal{R}_g) \subset L^2(\mathbb{G})$ and $C_0^\infty(\mathbb{G}\backslash\{0\})$ are dense in $L^2(\mathbb{G})$ it follows that it is enough to show (9.28) on $C_0^\infty(\mathbb{G}\backslash\{0\})$ since it then follows also on $D(\mathcal{R}_g)$ by density. Using the polar decomposition from Proposition 1.2.10 and the expression $\mathcal{R}_g = -i\left(\frac{d}{dr} + \frac{Q-1}{2r}\right)$ we can calculate

$$\begin{aligned}
\int_{\mathbb{G}} (\mathcal{R}_g f)\overline{f}dx &= -i\int_0^\infty \int_{\wp} r^{Q-1}\left(\frac{df}{dr} + \frac{Q-1}{2r}f\right)\overline{f}d\sigma(y)dr \\
&= -i\int_0^\infty \int_{\wp} \frac{df}{dr}\overline{f}r^{Q-1}d\sigma(y)dr - i\frac{Q-1}{2}\int_0^\infty \int_{\wp} r^{Q-1}\frac{f}{r}\overline{f}d\sigma(y)dr \\
&= i\int_0^\infty \int_{\wp} f\frac{d\overline{f}}{dr}r^{Q-1}d\sigma(y)dr + i(Q-1)\int_0^\infty \int_{\wp} r^{Q-1}\frac{f}{r}\overline{f}d\sigma(y)dr \\
&\quad - i\frac{Q-1}{2}\int_0^\infty \int_{\wp} r^{Q-1}\frac{f}{r}\overline{f}d\sigma(y)dr \\
&= \int_0^\infty \int_{\wp} r^{Q-1}f\overline{\left(-i\frac{df}{dr} - i\frac{Q-1}{2r}f\right)}d\sigma(y)dr = \int_{\mathbb{G}} f\overline{\mathcal{R}_g f}d\nu,
\end{aligned}$$

proving that $\mathcal{R}_g$ is also symmetric. $\square$

For any symmetric operators A and B in L^2 with domains $D(A)$ and $D(B)$, respectively, a straightforward calculation (see, e.g., [OY17, Theorem 2.1]) shows the equality

$$\begin{aligned}
&-i\int_{\mathbb{G}} ([A,B]f)\overline{f}d\nu \\
&\quad = \|Af\|_{L^2(\mathbb{G})}\|Bf\|_{L^2(\mathbb{G})}\left(2 - \left\|\frac{Af}{\|Af\|_{L^2(\mathbb{G})}} + i\frac{Bf}{\|Bf\|_{L^2(\mathbb{G})}}\right\|^2_{L^2(\mathbb{G})}\right),
\end{aligned} \tag{9.29}$$

for $f \in D(A) \cap D(B)$ with $Af \not\equiv 0$ and $Bf \not\equiv 0$, which will be useful in our next proof.

Theorem 9.4.3 (Identities involving $\mathcal{R}$, $\mathcal{R}_g$, and $\mathcal{C}$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$. Then for every $f \in D(\mathcal{R}) \cap D(\mathcal{C})$ such that $f \not\equiv 0$ and $\mathcal{R}_g f \not\equiv 0$ we have*

$$\|\mathcal{R}f\|^2_{L^2(\mathbb{G})} = \|\mathcal{R}_g f\|^2_{L^2(\mathbb{G})} + \frac{(Q-1)(Q-3)}{4}\|\mathcal{C}f\|^2_{L^2(\mathbb{G})}, \tag{9.30}$$

and

$$\|\mathcal{C}f\|_{L^2(\mathbb{G})} = \|\mathcal{R}_g f\|_{L^2(\mathbb{G})} \left(2 - \left\| \frac{\mathcal{R}_g f}{\|\mathcal{R}_g f\|_{L^2(\mathbb{G})}} + i\frac{\mathcal{C}f}{\|\mathcal{C}f\|_{L^2(\mathbb{G})}} \right\|^2_{L^2(\mathbb{G})}\right). \tag{9.31}$$

Proof of Theorem 9.4.3. As in the proof of Theorem 9.2.1 we can calculate

$$\begin{aligned}
&\|\mathcal{R}_g f\|^2_{L^2(\mathbb{G})} \\
&= \left\| \mathcal{R}f + \frac{Q-1}{2|x|} f \right\|^2_{L^2(\mathbb{G})} \\
&= \|\mathcal{R}f\|^2_{L^2(\mathbb{G})} + (Q-1)\,\mathrm{Re} \int_{\mathbb{G}} (\mathcal{R}f)\, \overline{\frac{1}{|x|} f} dx + \left\| \frac{Q-1}{2|x|} f \right\|^2_{L^2(\mathbb{G})} \\
&= \|\mathcal{R}f\|^2_{L^2(\mathbb{G})} + (Q-1)\,\mathrm{Re} \int_0^\infty \int_{\wp} r^{Q-1} \left(\frac{d}{dr} f\right) \overline{\frac{1}{r} f} d\sigma(y) dr + \left\| \frac{Q-1}{2|x|} f \right\|^2_{L^2(\mathbb{G})} \\
&= \|\mathcal{R}f\|^2_{L^2(\mathbb{G})} + \frac{Q-1}{2} \int_0^\infty \int_{\wp} r^{Q-2} \frac{d}{dr} |f|^2 d\sigma(y) dr + \frac{(Q-1)^2}{4} \|\mathcal{C}f\|^2_{L^2(\mathbb{G})} \\
&= \|\mathcal{R}f\|^2_{L^2(\mathbb{G})} - \frac{(Q-1)(Q-2)}{2} \int_0^\infty \int_{\wp} r^{Q-1} \frac{1}{r^2} |f|^2 d\sigma(y) dr + \frac{(Q-1)^2}{4} \|\mathcal{C}f\|^2_{L^2(\mathbb{G})} \\
&= \|\mathcal{R}f\|^2_{L^2(\mathbb{G})} - \frac{(Q-1)(Q-2)}{2} \int_{\mathbb{G}} |\mathcal{C}f|^2 dx + \frac{(Q-1)^2}{4} \|\mathcal{C}f\|^2_{L^2(\mathbb{G})} \\
&= \|\mathcal{R}f\|^2_{L^2(\mathbb{G})} - \frac{(Q-1)(Q-3)}{4} \|\mathcal{C}f\|^2_{L^2(\mathbb{G})}.
\end{aligned}$$

This proves (9.30). Using (9.27) and Lemma 9.4.2, in view of (9.29) we obtain

$$\begin{aligned}
\|\mathcal{C}f\|^2_{L^2(\mathbb{G})} &= -i \int_{\mathbb{G}} [\mathcal{R}_g, \mathcal{C}] f \overline{f} dx \\
&= \|\mathcal{R}_g f\|_{L^2(\mathbb{G})} \|\mathcal{C}f\|_{L^2(\mathbb{G})} \left(2 - \left\| \frac{\mathcal{R}_g f}{\|\mathcal{R}_g f\|_{L^2(\mathbb{G})}} + i\frac{\mathcal{C}f}{\|\mathcal{C}f\|_{L^2(\mathbb{G})}} \right\|^2_{L^2(\mathbb{G})}\right).
\end{aligned}$$

Since $C_0^\infty(\mathbb{G})$ is dense in $L^2(\mathbb{G})$, it implies that this equality is also true on $D(\mathcal{R}) \cap D(\mathcal{C})$ by density. $\square$

The equality (9.30) implies the following estimates:

Corollary 9.4.4 (Estimates for $\mathcal{R}_g$ and $\mathcal{C}$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$. The generator of dilations and Coulomb potential operator are bounded by the radial operator $\mathcal{R}$, that is, we have the estimates*

$$\|\mathcal{R}_g f\|_{L^2(\mathbb{G})} \leq \|\mathcal{R} f\|_{L^2(\mathbb{G})}, \tag{9.32}$$

and

$$\frac{\sqrt{(Q-1)(Q-3)}}{2}\|\mathcal{C} f\|_{L^2(\mathbb{G})} \leq \|\mathcal{R} f\|_{L^2(\mathbb{G})}, \tag{9.33}$$

for all $f \in D(\mathcal{R}) \cap D(\mathcal{C})$.

The equality (9.31) implies the following bound with an explicit constant, independent on the choice of a homogeneous norm on $\mathbb{G}$.

Corollary 9.4.5 (Bound of Coulomb operator by generator of dilations)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$. The Coulomb potential operator is bounded by the generator of dilations operator with relative bound 2, that is,*

$$\|\mathcal{C} f\|_{L^2(\mathbb{G})} \leq 2\|\mathcal{R}_g f\|_{L^2(\mathbb{G})}, \tag{9.34}$$

for all $f \in D(\mathcal{R}) \cap D(\mathcal{C})$ such that $\mathcal{R}_g f \not\equiv 0$.

9.5 Further weighted uncertainty type inequalities

In this section, we give an overview of a number of further uncertainty type inequalities.

Theorem 9.5.1. *For any quasi-norm $|\cdot|$, all differentiable $|\cdot|$-radial functions ϕ, all $p > 1$, $Q \geq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$, and all $f \in C_0^1(\mathbb{G})$ we have*

$$\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}}|f|^p dx \leq \int_{\mathbb{G}} |\mathcal{R} f|^p \, dx + \frac{p}{q}\int_{\mathbb{G}} \frac{|\phi(|x|)|^q}{|x|^{q(Q-1)}}|f|^p dx, \tag{9.35}$$

and

$$\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}}|f|^p dx \leq p\left(\int_{\mathbb{G}} |\mathcal{R} f|^p \, dx\right)^{1/p} \left(\int_{\mathbb{G}} \frac{|\phi(|x|)|^q}{|x|^{q(Q-1)}}|f|^p dx\right)^{1/q}. \tag{9.36}$$

Before proving this theorem, let us point out several of its consequences.

Remark 9.5.2.

1. In (9.35) taking $\phi = \log|x|$ in the Euclidean (Abelian) case $\mathbb{G} = (\mathbb{R}^n, +)$, $n \geq 2$, we have $Q = n$, and taking $p = n \geq 2$, for any quasi-norm $|\cdot|$ on $\mathbb{R}^n$, it implies the new inequality

$$\int_{\mathbb{R}^n} \frac{|f|^n}{|x|^n} dx \leq \int_{\mathbb{R}^n} |\mathcal{R} f|^n \, dx + (n-1)\int_{\mathbb{R}^n} \frac{\left|\log \frac{1}{|x|}\right|^{\frac{n}{n-1}}}{|x|^n} |f|^n \, dx.$$

In turn, by using Schwarz' inequality with the standard Euclidean distance $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$, it implies the 'critical' Hardy inequality

$$\int_{\mathbb{R}^n} \frac{|f|^n}{|x|_E^n} dx \leq \int_{\mathbb{R}^n} |\nabla f|^n \, dx + (n-1) \int_{\mathbb{R}^n} \frac{\left|\log \frac{1}{|x|_E}\right|^{\frac{n}{n-1}}}{|x|_E^n} |f|^n \, dx, \tag{9.37}$$

where ∇ is the standard gradient on $\mathbb{R}^n$. It is known that there is no positive constant C such that

$$\int_{\mathbb{R}^n} \frac{|f|^n}{|x|_E^n} dx \leq C \int_{\mathbb{R}^n} |\nabla f|^n \, dx$$

for all $f \in C_0^1(\mathbb{R}^n)$. Therefore, the appearance of a positive additional term (the second term) on the right-hand side of (9.37) seems essential. Critical inequalities for different versions of critical Hardy–Sobolev type inequalities have been investigated in [RS16a].

2. Note that this type of inequalities (Hardy–Sobolev type inequalities with an additional term on the right-hand side) can be applied, for example in the Euclidean case, to establish the existence and nonexistence of positive exponentially bounded weak solutions to a parabolic type operator perturbed by a critical singular potential (see, e.g., [ST18b]).
3. In (9.36), taking $\phi = |x|^n$ in the Euclidean (Abelian) case $\mathbb{G} = (\mathbb{R}^n, +)$, $n \geq 2$, we have $Q = n$, so for any quasi-norm $|\cdot|$ on $\mathbb{R}^n$ it implies the following uncertainty principle

$$\int_{\mathbb{R}^n} |f|^p dx \leq \frac{p}{n} \left(\int_{\mathbb{R}^n} |\mathcal{R} f|^p \, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |x|^{\frac{p}{p-1}} |f|^p dx \right)^{\frac{p-1}{p}}. \tag{9.38}$$

In turn, by using Schwarz' inequality with the standard Euclidean distance $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$, it implies that

$$\int_{\mathbb{R}^n} |f|^p dx \leq \frac{p}{n} \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |x|_E^{\frac{p}{p-1}} |f|^p dx \right)^{\frac{p-1}{p}}, \tag{9.39}$$

where ∇ is the standard gradient on $\mathbb{R}^n$. In the case when $p = 2$ we have

$$\left(\int_{\mathbb{R}^n} |f|^2 dx \right)^2 \leq \left(\frac{2}{n} \right)^2 \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \int_{\mathbb{R}^n} |x|_E^2 |f|^2 dx, \quad n \geq 2, \tag{9.40}$$

for all $f \in C_0^1(\mathbb{R}^n)$. Thus, when $n = 2$ inequality (9.40) gives the critical case of the *Heisenberg–Pauli–Weyl uncertainty principle* (9.22). Moreover, since $\frac{2}{n-2} \geq \frac{2}{n}$, $n \geq 3$, inequality (9.40) is an improved version of (9.22).

Note that equality case in (9.40) holds for the family of functions $f = C \exp(-b|x|_E)$, $b > 0$.
4. Uncertainty inequalities have been extended to many settings such as more general Lie groups and manifolds; see Folland and Sitaram [FS97] for more

information about older studies. On the general topic of uncertainty principles on groups and manifolds we can refer to, e.g., [CRS07], [VSCC92], [Tao05], among many others.

Proof of Theorem 9.5.1. By applying the polar decomposition formula in Proposition 1.2.10, and integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}}|f|^p dx &= \int_0^\infty \int_{\wp} |f|^p \frac{\phi'(r)}{r^{Q-1}} r^{Q-1} d\sigma(y) dr \\ &= \int_0^\infty \int_{\wp} |f|^p \frac{d}{dr}\phi(r) d\sigma(y) dr = -\int_0^\infty \int_{\wp} \phi(r) \frac{d}{dr}|f|^p d\sigma(y) dr \\ &= -\int_{\mathbb{G}} \frac{\phi(|x|)}{|x|^{Q-1}} \mathcal{R}|f|^p dx = -p\mathrm{Re}\int_{\mathbb{G}} \frac{\phi(|x|)|f|^{p-2}f}{|x|^{Q-1}} \overline{\mathcal{R}f} dx.\end{aligned}$$

Now by using Young's inequality for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we arrive at

$$\begin{aligned}\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}}|f|^p dx &= -p\mathrm{Re}\int_{\mathbb{G}} \frac{\phi|f|^{p-2}f}{|x|^{Q-1}} \overline{\mathcal{R}f} dx \\ &\le p \int_{\mathbb{G}} \frac{|\phi||f|^{p-1}}{|x|^{Q-1}} |\mathcal{R}f| dx \\ &\le \int_{\mathbb{G}} |\mathcal{R}f|^p \, dx + \frac{p}{q}\int_{\mathbb{G}} \frac{|\phi(|x|)|^q}{|x|^{q(Q-1)}}|f|^p dx. \end{aligned} \tag{9.41}$$

This proves inequality (9.35). Furthermore, from (9.41) by using Hölder's inequality for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we establish

$$\begin{aligned}\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}}|f|^p dx &\le p \int_{\mathbb{G}} \frac{|\phi||f|^{p-1}}{|x|^{Q-1}} |\mathcal{R}f| dx \\ &\le p \left(\int_{\mathbb{G}} |\mathcal{R}f|^p \, dx \right)^{1/p} \left(\int_{\mathbb{G}} \frac{|\phi(|x|)|^q}{|x|^{q(Q-1)}}|f|^p dx \right)^{1/q}.\end{aligned}$$

This completes the proof. □