

Basic properties of SLE

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Dedicated to Christian Pommerenke on the occasion of his 70th birthday

Abstract

SLE_κ is a random growth process based on Loewner's equation with driving parameter a one-dimensional Brownian motion running with speed κ . This process is intimately connected with scaling limits of percolation clusters and with the outer boundary of Brownian motion, and is conjectured to correspond to scaling limits of several other discrete processes in two dimensions.

The present paper attempts a first systematic study of SLE. It is proved that for all $\kappa \neq 8$ the SLE trace is a path; for $\kappa \in [0, 4]$ it is a simple path; for $\kappa \in (4, 8)$ it is a self-intersecting path; and for $\kappa > 8$ it is space-filling.

It is also shown that the Hausdorff dimension of the SLE_κ trace is almost surely (a.s.) at most $1 + \kappa/8$ and that the expected number of disks of size ε needed to cover it inside a bounded set is at least $\varepsilon^{-(1+\kappa/8)+o(1)}$ for $\kappa \in [0, 8)$ along some sequence $\varepsilon \searrow 0$. Similarly, for $\kappa \geq 4$, the Hausdorff dimension of the outer boundary of the SLE_κ hull is a.s. at most $1 + 2/\kappa$, and the expected number of disks of radius ε needed to cover it is at least $\varepsilon^{-(1+2/\kappa)+o(1)}$ for a sequence $\varepsilon \searrow 0$.

1. Introduction

Stochastic Loewner Evolution (SLE) is a random process of growth of a set K_t . The evolution of the set over time is described through the normalized conformal map $g_t = g_t(z)$ from the complement of K_t . The map g_t is the solution of Loewner's differential equation with driving parameter a one-dimensional Brownian motion. SLE, or SLE_κ , has one parameter $\kappa \geq 0$, which is the speed of the Brownian motion. A more complete definition appears in Section 2 below.

The SLE process was introduced in [Sch00]. There, it was shown that under the assumption of the existence and conformal invariance of the scaling limit of loop-erased random walk, the scaling limit is SLE_2 . (See Figure 9.1.) It was also stated there without proof that SLE_6 is the scaling limit of the

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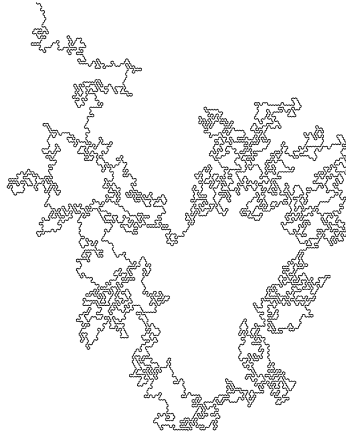


Figure 1.1: The boundary of a percolation cluster in the upper half plane, with appropriate boundary conditions. It converges to the chordal SLE_6 trace.

boundaries of critical percolation clusters, assuming their conformal invariance. Smirnov [Smi01] has recently proved the conformal invariance conjecture for critical percolation on the triangular grid and the claim that SLE_6 describes the limit. (See Figure 1.1.) With the proper setup, the outer boundary of SLE_6 is the same as the outer boundary of planar Brownian motion [LSW03] (see also [Wer01]). SLE_8 has been conjectured [Sch00] to be the scaling limit of the uniform spanning tree Peano curve (see Figure 9.2), and there are various further conjectures for other parameters. Most of these conjectures are described in Section 9 below. Also related is the work of Carleson and Makarov [CM01], which studies growth processes motivated by DLA via Loewner's equation.

SLE is amenable to computations. In [Sch00] a few properties of SLE have been derived; in particular, the winding number variance. In the series of papers [LSW01a], [LSW01b], [LSW02], a number of other properties of SLE have been studied. The goal there was not to investigate SLE for its own sake, but rather to use SLE_6 as a means for the determination of the Brownian motion intersection exponents.

As the title suggests, the goal of the present paper is to study the fundamental properties of SLE. There are two main variants of SLE, *chordal* and *radial*. For simplicity, we concentrate on chordal SLE; however, all the main results of the paper carry over to radial SLE as well. In chordal SLE, the set K_t , $t \geq 0$, called the SLE *hull*, is a subset of the closed upper half plane $\overline{\mathbb{H}}$ and $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$ is the conformal uniformizing map, suitably normalized at infinity.

We show that with the possible exception of $\kappa = 8$, a.s. there is a (unique) continuous path $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ such that for each $t > 0$ the set K_t is the union of $\gamma[0, t]$ and the bounded connected components of $\overline{\mathbb{H}} \setminus \gamma[0, t]$. The path γ is called the SLE trace. It is shown that $\lim_{t \rightarrow \infty} |\gamma(t)| = \infty$ a.s. We also

describe two phase transitions for the SLE process. In the range $\kappa \in [0, 4]$, a.s. $K_t = \gamma[0, t]$ for every $t \geq 0$ and γ is a simple path. For $\kappa \in (4, 8)$ the path γ is not a simple path and for every $z \in \mathbb{H}$ a.s. $z \notin \gamma[0, \infty)$ but $z \in \bigcup_{t>0} K_t$. Finally, for $\kappa > 8$ we have $\overline{\mathbb{H}} = \gamma[0, \infty)$ a.s. The reader may wish to examine Figures 9.1, 1.1 and 9.2, to get an idea of what the SLE_κ trace looks like for $\kappa = 2, 6$ and 8 , respectively.

We also discuss the expected number of disks needed to cover the SLE_κ trace and the outer boundary of K_t . It is proved that the Hausdorff dimension of the trace is a.s. at most $1 + \kappa/8$, and that the Hausdorff dimension of the outer boundary ∂K_t is a.s. at most $1 + 2/\kappa$ if $\kappa \geq 4$. For $\kappa \in [0, 8)$, we also show that the expected number of disks of size ε needed to cover the trace inside a bounded set is at least $\varepsilon^{-(1+\kappa/8)+o(1)}$ along some sequence $\varepsilon \searrow 0$. Similarly, for $\kappa \geq 4$, the expected number of disks of radius ε needed to cover the outer boundary is at least $\varepsilon^{-(1+2/\kappa)+o(1)}$ for a sequence of $\varepsilon \searrow 0$. Richard Kenyon has earlier made the conjecture that the Hausdorff dimension of the outer boundary is a.s. $1 + 2/\kappa$. These results offer strong support for this conjecture.

It is interesting to compare our results to recent results for the deterministic Loewner evolution, i.e., the solutions to the Loewner equation with a deterministic driving function $\xi(t)$. In [MR] it is shown that if ξ is Hölder continuous with exponent $1/2$ and small norm, then K_t is a simple path. On the other hand, there is a function ξ , Hölder continuous with exponent $1/2$ and having large norm, such that K_t is not even locally connected, and therefore there is no continuous path γ generating K_t . In this example, K_t spirals infinitely often around a disk D , accumulating on ∂D , and then spirals out again. It is easy to see that the disk D can be replaced by any compact connected subset of \mathbb{H} . Notice that according to the law of the iterated logarithm, a.s. Brownian motion is not Hölder continuous with exponent $1/2$. Therefore, it seems unlikely that the results of the present paper can be obtained from deterministic results.

Our results are based on the computation and estimates of the distribution of $|g'_t(z)|$ where $z \in \mathbb{H}$. Note that in [LSW01b] the derivatives $g'_t(x)$ are studied for $x \in \mathbb{R}$.

The organization of the paper is as follows. Section 2 introduces the basic definitions and some fundamental properties. The goal of Section 3 is to obtain estimates for quantities related to $\mathbf{E}[|g'_t(z)|^a]$, for various constants a (another result of this nature is Lemma 6.3), and to derive some resulting continuity properties of g_t^{-1} . Section 4 proves a general criterion for the existence of a continuous trace, which does not involve randomness. The proof that the SLE_κ trace is continuous for $\kappa \neq 8$ is then completed in Section 5. There, it is also proved that g_t^{-1} is a.s. Hölder continuous when $\kappa \neq 4$. Section 6 discusses the two phase transitions $\kappa = 4$ and $\kappa = 8$ for SLE_κ . Besides some quantitative

properties, it is shown there that the trace is a.s. a simple path if and only if $\kappa \in [0, 4]$, and that the trace is space-filling for $\kappa > 8$. The trace is proved to be transient when $\kappa \neq 8$ in Section 7. Estimates for the dimensions of the trace and the boundary of the hull are established in Section 8. Finally, a collection of open problems is presented in Section 9.

Update. Since the completion and distribution of the first version of this paper, there has been some further progress. In [LSW] it was proven that the scaling limit of loop-erased random walk is SLE_2 and the scaling limit of the UST Peano path is SLE_8 . As a corollary of the convergence of the UST Peano path to SLE_8 , it was also established there that SLE_8 is generated by a continuous transient path, thus answering some of the issues left open in the current paper. However, it is quite natural to ask for a more direct analytic proof of these properties of SLE_8 .

Recently, Vincent Beffara [Bef] has announced a proof that the Hausdorff dimension of the SLE_κ trace is $1 + \kappa/8$ when $4 \neq \kappa \leq 8$.

The paper [SS] proves the convergence of the harmonic explorer to SLE_4 .

2. Definitions and background

2.1. *Chordal SLE.* Let B_t be Brownian motion on \mathbb{R} , started from $B_0 = 0$. For $\kappa \geq 0$ let $\xi(t) := \sqrt{\kappa}B_t$ and for each $z \in \overline{\mathbb{H}} \setminus \{0\}$ let $g_t(z)$ be the solution of the ordinary differential equation

$$(2.1) \quad \partial_t g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z.$$

The solution exists as long as $g_t(z) - \xi(t)$ is bounded away from zero. We denote by $\tau(z)$ the first time τ such that 0 is a limit point of $g_t(z) - \xi(t)$ as $t \nearrow \tau$. Set

$$H_t := \{z \in \mathbb{H} : \tau(z) > t\}, \quad K_t := \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}.$$

It is immediate to verify that K_t is compact and H_t is open for all t . The parametrized collection of maps $(g_t : t \geq 0)$ is called *chordal SLE $_\kappa$* . The sets K_t are the *hulls* of the SLE. It is easy to verify that for every $t \geq 0$ the map $g_t : H_t \rightarrow \mathbb{H}$ is a conformal homeomorphism and that H_t is the unbounded component of $\mathbb{H} \setminus K_t$. The inverse of g_t is obtained by flowing backwards from any point $w \in \mathbb{H}$ according to the equation (2.1). (That is, the fact that g_t is invertible is a particular case of a general result on solutions of ODE's.) One only needs to note that in this backward flow, the imaginary part increases, hence the point cannot hit the singularity. It also cannot escape to infinity in finite time. The fact that $g_t(z)$ is analytic in z is clear, since the right-hand side of (2.1) is analytic in $g_t(z)$.

The map g_t satisfies the so-called *hydrodynamic* normalization at infinity:

$$(2.2) \quad \lim_{z \rightarrow \infty} g_t(z) - z = 0.$$

Note that this uniquely determines g_t among conformal maps from H_t onto \mathbb{H} . In fact, (2.1) implies that g_t has the power series expansion

$$(2.3) \quad g_t(z) = z + \frac{2t}{z} + \dots, \quad z \rightarrow \infty.$$

The fact that $g_t \neq g_{t'}$ when $t' > t$ implies that $K_{t'} \neq K_t$. The relation $K_{t'} \supset K_t$ is clear.

Two important properties of chordal SLE are scale-invariance and a sort of stationarity. These are summarized in the following proposition. (A similar statement appeared in [LSW01a].)

PROPOSITION 2.1. (i) *SLE $_{\kappa}$ is scale-invariant in the following sense. Let K_t be the hull of SLE $_{\kappa}$, and let $\alpha > 0$. Then the process $t \mapsto \alpha^{-1/2}K_{\alpha t}$ has the same law as $t \mapsto K_t$. The process $(t, z) \mapsto \alpha^{-1/2}g_{\alpha t}(\sqrt{\alpha}z)$ has the same law as the process $(t, z) \mapsto g_t(z)$.*

(ii) *Let $t_0 > 0$. Then the map $(t, z) \mapsto \tilde{g}_t(z) := g_{t+t_0} \circ g_{t_0}^{-1}(z + \xi(t_0)) - \xi(t_0)$ has the same law as $(t, z) \mapsto g_t(z)$; moreover, $(\tilde{g}_t)_{t \geq 0}$ is independent from $(g_t)_{0 \leq t \leq t_0}$.*

The scaling property easily follows from the scaling property of Brownian motion, and the second property follows from the Markov property and translation invariance of Brownian motion. One just needs to write down the expression for $\partial_t \tilde{g}_t$. We leave the details as an exercise to the reader.

The following notations will be used throughout the paper.

$$f_t := g_t^{-1}, \quad \hat{f}_t(z) := f_t(z + \xi(t)).$$

The *trace* γ of SLE is defined by

$$\gamma(t) := \lim_{z \rightarrow 0} \hat{f}_t(z),$$

where z tends to 0 within \mathbb{H} . If the limit does not exist, let $\gamma(t)$ denote the set of all limit points. We say that the SLE trace is a continuous path if the limit exists for every t and $\gamma(t)$ is a continuous function of t .

2.2. Radial SLE. Another version of SLE $_{\kappa}$ is called *radial* SLE $_{\kappa}$. It is similar to chordal SLE but appropriate for the situation where there is a distinguished point in the interior of the domain. Radial SLE $_{\kappa}$ is defined as follows. Let $B(t)$ be Brownian motion on the unit circle $\partial\mathbb{U}$, started from a uniform-random point $B(0)$, and set $\xi(t) := B(\kappa t)$. The conformal maps g_t are defined as the solution of

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + \xi(t)}{g_t(z) - \xi(t)}, \quad g_0(z) = z,$$

for $z \in \mathbb{U}$. The sets K_t and H_t are defined as for chordal SLE. Note that the scaling property 2.1.(i) fails for radial SLE. Mainly due to this reason, chordal SLE is easier to work with. However, appropriately stated, all the main results of this paper are valid also for radial SLE. This follows from [LSW01b, Prop. 4.2], which says in a precise way that chordal and radial SLE are equivalent.

2.3. *Local martingales and martingales.* The purpose of this subsection is to present a slightly technical lemma giving a sufficient condition for a local martingale to be a martingale. Although we have not been able to find an appropriate reference, the lemma must be known (and is rather obvious to the experts).

See, for example, [RY99, §IV.1] for a discussion of the distinction between a local martingale and a martingale.

While the stochastic calculus needed for the rest of the paper is not much more than familiarity with Itô’s formula, this subsection does assume a bit more.

LEMMA 2.2. *Let B_t be standard one dimensional Brownian motion, and let a_t be a progressive real valued locally bounded process. Suppose that X_t satisfies*

$$X_t = \int_0^t a_s dB_s,$$

and that for every $t > 0$ there is a finite constant $c(t)$ such that

$$(2.4) \quad a_s^2 \leq c(t)X_s^2 + c(t)$$

for all $s \in [0, t]$ a.s. Then X is a martingale.

Proof. We know that X is a local martingale. Let $M > 0$ be large, and let $T := \inf\{t : |X_t| \geq M\}$. Then $Y_t := X_{t \wedge T}$ is a martingale (where $t \wedge T = \min\{t, T\}$). Let $f(t) := \mathbf{E}[Y_t^2]$. Itô’s formula then gives

$$f(t') = \mathbf{E}\left[\int_0^{t'} a_s^2 1_{s < T} ds\right].$$

Our assumption $a_s^2 \leq c(t)X_s^2 + c(t)$ therefore implies that for $t' \in [0, t]$

$$(2.5) \quad f(t') \leq c(t)t' + c(t) \int_0^{t'} f(s) ds.$$

This implies $f(s) < \exp(2c(t)s)$ for all $s \in [0, t]$, since (2.5) shows that t' cannot be the least $s \in [0, t]$ where $f(s) \geq \exp(2c(t)s)$. Thus,

$$\mathbf{E}[\langle X, X \rangle_{t \wedge T}] = \mathbf{E}[\langle Y, Y \rangle_t] = \mathbf{E}[Y_t^2] = f(t) < \exp(2c(t)t).$$

Taking $M \rightarrow \infty$, we get by monotone convergence $\mathbf{E}[\langle X, X \rangle_t] \leq \exp(2c(t)t) < \infty$. Thus X is a martingale (for example, by [RY99, IV.1.25]). \square

3. Derivative expectation

In this section, g_t is the SLE_κ flow; that is, the solution of (2.1) where $\xi(t) := B(\kappa t)$, and B is standard Brownian motion on \mathbb{R} starting from 0. Our only assumption on κ is $\kappa > 0$. The goal of the section is to derive bounds on quantities related to $\mathbf{E}[|g'_t(z)|^a]$. Another result of this nature is Lemma 6.3, which is deferred to a later section.

3.1. *Basic derivative expectation.* We will need estimates for the moments of $|\hat{f}'_t|$. In this subsection, we will describe a change of time and obtain derivative estimates for the changed time.

For convenience, we take B to be two-sided Brownian motion. The equation (2.1) can also be solved for negative t , and g_t is a conformal map from \mathbb{H} into a subset of \mathbb{H} when $t < 0$. Notice that the scale invariance (Proposition 2.1.(i)) also holds for $t < 0$.

LEMMA 3.1. *For all fixed $t \in \mathbb{R}$ the map $z \mapsto g_{-t}(z)$ has the same distribution as the map $z \mapsto \hat{f}_t(z) - \xi(t)$.*

Proof. Fix $t_1 \in \mathbb{R}$, and let

$$(3.1) \quad \hat{\xi}_{t_1}(t) = \xi(t_1 + t) - \xi(t_1).$$

Then $\hat{\xi}_{t_1} : \mathbb{R} \rightarrow \mathbb{R}$ has the same law as $\xi : \mathbb{R} \rightarrow \mathbb{R}$. Let

$$\hat{g}_t(z) := g_{t_1+t} \circ g_{t_1}^{-1}(z + \xi(t_1)) - \xi(t_1),$$

and note that $\hat{g}_0(z) = z$ and $\hat{g}_{-t_1}(z) = \hat{f}_{t_1}(z) - \xi(t_1)$. Since

$$\partial_t \hat{g}_t = \frac{2}{\hat{g}_t + \xi(t_1) - \xi(t_1 + t)} = \frac{2}{\hat{g}_t - \hat{\xi}_{t_1}(t)},$$

the lemma follows from (2.1). □

Note that (2.1) implies that $\text{Im}(g_t(z))$ is monotone decreasing in t for every $z \in \mathbb{H}$. For $z \in \mathbb{H}$ and $u \in \mathbb{R}$ set

$$(3.2) \quad T_u = T_u(z) := \sup \left\{ t \in \mathbb{R} : \text{Im}(g_t(z)) \geq e^u \right\}.$$

We claim that for all $z \in \mathbb{H}$ a.s. $T_u \neq \pm\infty$. By (2.1),

$$|\partial_t g_t(z)| = 2|g_t(z) - \xi(t)|^{-1}.$$

Setting $\bar{\xi}(t) := \sup\{|\xi(s)| : s \in [0, t]\}$, the above implies that $|g_t(z)| \leq |z| + \bar{\xi}(t) + 2\sqrt{t}$ for $t < \tau(z)$, since $|\partial_t |g_t(z)|| \leq 2/(|g_t(z)| - \bar{\xi}(t))$ whenever $|g_t(z)| > \bar{\xi}(t)$. Setting $y_t := \text{Im}g_t(z)$, we get from (2.1),

$$-\partial_t \log y_t \geq 2(|z| + 2\bar{\xi}(t) + 2\sqrt{t})^{-2}.$$

The law of iterated logarithms implies that the right-hand side is not integrable over $[0, \infty)$ nor over $(-\infty, 0]$. Thus, $|T_u| < \infty$ a.s.

We will need the formula

$$(3.3) \quad \begin{aligned} \partial_t \log |g'_t(z)| &= \operatorname{Re} \left(\frac{\partial_z \partial_t g_t(z)}{g'_t(z)} \right) = \operatorname{Re} \left(g'_t(z)^{-1} \partial_z \frac{2}{g_t(z) - \xi(t)} \right) \\ &= -2 \operatorname{Re} \left((g_t(z) - \xi(t))^{-2} \right). \end{aligned}$$

Set $u = u(z, t) := \log \operatorname{Im} g_t(z)$. Observe that (2.1) gives

$$(3.4) \quad \partial_t u = -2 |g_t(z) - \xi(t)|^{-2},$$

and (3.3) gives

$$(3.5) \quad \partial_u \log |g'_t(z)| = \frac{\operatorname{Re}((g_t(z) - \xi(t))^2)}{|g_t(z) - \xi(t)|^2}.$$

Fix some $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$. For every $u \in \mathbb{R}$, let

$$z(u) := g_{T_u(\hat{z})}(\hat{z}) - \xi(T_u), \quad \psi(u) := \frac{\hat{y}}{y(u)} |g'_{T_u}(\hat{z})|,$$

and

$$x(u) := \operatorname{Re}(z(u)), \quad y(u) := \operatorname{Im}(z(u)) = \exp(u).$$

THEOREM 3.2. *Let $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$ as above. Assume that $\hat{y} \neq 1$, and set $\nu := -\operatorname{sign}(\log \hat{y})$. Let $b \in \mathbb{R}$. Define a and λ by*

$$(3.6) \quad a := 2b + \nu \kappa b(1 - b)/2, \quad \lambda := 4b + \nu \kappa b(1 - 2b)/2.$$

Set

$$F(\hat{z}) = F_b(\hat{z}) := \hat{y}^a \mathbf{E} \left[(1 + x(0)^2)^b |g'_{T_0(\hat{z})}(\hat{z})|^a \right].$$

Then

$$(3.7) \quad F(\hat{z}) = \left(1 + (\hat{x}/\hat{y})^2 \right)^b \hat{y}^\lambda.$$

Before we give the short proof of the theorem, a few remarks may be of help to motivate the formulation and the proof. Our goal was to find an expression for $\mathbf{E} \left[|g'_{T_0(\hat{z})}(\hat{z})|^a \right]$. It turns out to be more convenient to consider

$$\bar{F}(\hat{z}) := \hat{y}^a \mathbf{E} \left[|g'_{T_0(\hat{z})}(\hat{z})|^a \right] = \mathbf{E} \left[\psi(0)^a \right].$$

The obvious strategy is to find a differential equation which \bar{F} must satisfy and search for a solution. The first part is not too difficult, and proceeds as follows. Let \mathcal{F}_u denote the σ -field generated by $\langle \xi(t) : (t - T_u)\nu \geq 0 \rangle$. Note that the

strong Markov property for ξ and the chain rule imply that for u between 0 and $\hat{u} := \log \hat{y}$,

$$(3.8) \quad \mathbf{E} \left[\psi(0)^a \mid \mathcal{F}_u \right] = \psi(u)^a \bar{F}(z(u)).$$

Hence, the right-hand side is a martingale. Observe that

$$(3.9) \quad dx = \frac{2x \, dt}{x^2 + y^2} - d\xi, \quad dy = \frac{-2y \, dt}{x^2 + y^2}, \quad d \log \psi = \frac{4y^2 \, dt}{(x^2 + y^2)^2}.$$

(The latter easily follows from (3.3) and (3.4).) We assume for now that \bar{F} is smooth. Itô's formula may then be calculated for the right-hand side of (3.8). Since it is a martingale, the drift term of the Itô derivative must vanish; that is,

$$\psi^a \cdot \Lambda \bar{F} = 0,$$

where

$$(3.10) \quad \Lambda G := -\frac{4\nu a y^2}{(x^2 + y^2)^2} G - \frac{2\nu x}{x^2 + y^2} \partial_x G + \frac{2\nu y}{x^2 + y^2} \partial_y G + \frac{\kappa}{2} \partial_x^2 G.$$

(The $-\nu$ factor comes from the fact that t is monotone decreasing with respect to the filtration \mathcal{F}_u if and only if $\nu = 1$.) Guessing a solution for the equation $\Lambda G = 0$ is not too difficult (after changing to coordinates where scale invariance is more apparent). It is easy to verify that

$$\hat{F}(x + iy) = \hat{F}_{b,\lambda}(x + iy) := \left(1 + (x/y)^2 \right)^b y^\lambda,$$

satisfies $\Lambda \hat{F} = 0$. Unfortunately, \hat{F} does not satisfy the boundary values $\hat{F} = 1$ for $y = 1$, which hold for \bar{F} . Consequently, the theorem gives a formula for F , rather than for \bar{F} . (Remark 3.4 concerns the problem of determining \bar{F} .) Assuming that F is C^2 , the above derivation does apply to F , and shows that $\Lambda F = 0$. However, we have not found a clean reference to the fact that $F \in C^2$. Fortunately, the proof below does not need to assume this.

Proof of Theorem 3.2. Note that by (3.4)

$$du = -2|z|^{-2} dt.$$

Let

$$\hat{B}(u) := -\sqrt{2/\kappa} \int_{t=0}^{T_u} |z|^{-1} d\xi.$$

Then \hat{B} is a local martingale and

$$d\langle \hat{B} \rangle = -2|z|^{-2} dt = du.$$

Consequently, $\hat{B}(u)$ is Brownian motion (with respect to u). Set $M_u := \psi(u)^a \hat{F}(z(u))$. Itô's formula gives

$$dM_u = -2M \frac{bx}{x^2 + y^2} d\xi = \sqrt{2\kappa} M \frac{bx}{\sqrt{x^2 + y^2}} d\hat{B}.$$

Thus M is a local martingale. In fact, Lemma 2.2 then tells us that M is a martingale. Consequently, we have

$$(3.11) \quad \psi(\hat{u})^a \hat{F}(\hat{z}) = \mathbf{E}[\psi(0)^a \hat{F}(z(0))]$$

and the theorem clearly follows. □

In Section 8 we will estimate the expected number of disks needed to cover the boundary of K_t . To do this, we need the following lower bound on the expectation of the derivative.

LEMMA 3.3. *Let $\kappa > 0$ and $b < (\kappa + 4)/(4\kappa)$, and define a and λ by (3.6) with $\nu = 1$. Then there is a constant $c = c(\kappa, b) > 0$ such that*

$$\mathbf{E}\left[|\hat{f}'_1(\hat{z})|^a 1_{\text{Im}(\hat{f}_1(\hat{z})) \geq c}\right] \geq c\left(1 + (\hat{x}/\hat{y})^2\right)^b \hat{y}^{\lambda-a}$$

holds for every $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$ satisfying $|\hat{z}| \leq c$.

Proof. As before, let $u := \log y$. Set $v = v(u) := \sinh^{-1}(x/y)$; that is, $x = y \sinh(v)$. Then Itô's formula gives

$$dv = -\frac{d\xi}{|z|} + \frac{1}{2}(8 + \kappa\nu)|z|^{-3}x dt.$$

The formula for dv in terms of \hat{B} and du is

$$(3.12) \quad dv = \sqrt{\kappa/2} d\hat{B} - \frac{x}{4|z|}(8 + \kappa\nu) du.$$

Recall that $\psi(u)^a F(z(u))$ is a martingale, since $F = \hat{F}$. Define a new probability measure by

$$\tilde{\mathbf{P}}[\mathcal{A}] := \frac{\mathbf{E}\left[1_{\mathcal{A}}(1 + x(0)^2)^b \psi(0)^a\right]}{\mathbf{E}\left[(1 + x(0)^2)^b \psi(0)^a\right]},$$

for every event \mathcal{A} . This is the so-called Doob-transform (or h-transform) corresponding to the martingale $\psi(u)^a F(z(u))$. Recall that if $\alpha(w)$ is a positive martingale for a diffusion process $dw = q_1(w, t) dB(t) + q_2(w, t) dt, t \leq t_1$, where $B(t)$ is Brownian motion, then for $t < t_1, dw = q_1(w, t) d\tilde{B}(t) + q_2(w, t) dt + q_1(w, t) \partial_w \log \alpha(w) q_1(w, t) dt$, where \tilde{B} is Brownian motion with respect to the probability measure weighted by $\alpha(w(t_1))$; that is, the Doob transform of α . This follows, for example, from Girsanov's Formula [Dur96, §2.12]. We apply this with $w = (v, \psi)$ and u as the time parameter (in this case, q_1 and q_2 are vectors and $\partial_w \log \alpha(w)$ is a linear functional), and get by (3.12)

$$dv = \sqrt{\kappa/2} d\tilde{B} - \frac{x}{4|z|}(8 + \kappa) du + \frac{1}{2} \kappa \partial_v \log F du,$$

where $\tilde{B}(u)$ is Brownian motion under $\tilde{\mathbf{P}}$. This simplifies to

$$(3.13) \quad dv = \sqrt{\kappa/2} d\tilde{B} - \tilde{b} \tanh(v) du,$$

where

$$\tilde{b} := 2 - \left(b - \frac{1}{4}\right)\kappa > 1.$$

Thus, $v(u)$ is a very simple diffusion process. When $|v|$ is large, $\tanh(v)$ is close to $\text{sign}(v)$, and v has almost constant drift $-\tilde{b} \text{sign}(v)$, pushing it towards 0.

At this point, do not assume $|\hat{z}| \leq c$, but only $|\hat{z}| \leq 1$. Let $\Psi : [\hat{u}, 0] \rightarrow \mathbb{R}$ be the continuous function that is equal to 1 at 0, equal to $|\hat{u}| + 1$ at \hat{u} , has slope $-(\tilde{b} \wedge 2 + 1)/2$ in the interval $[\hat{u}, \hat{u}/2]$ and has constant slope in the interval $[\hat{u}/2, 0]$, and let \mathcal{A} be the event

$$\mathcal{A} := \left\{ \forall u \in [\hat{u}, 0], \quad |v(u)| \leq \Psi(u) \right\}.$$

Note that our assumption $|\hat{z}| \leq 1$ implies that $|\hat{v}| \leq |\hat{u}| + \log 2$. Since $\tilde{b} > 1$ it easily follows from (3.13) that there is a constant $c_1 = c_1(b, \kappa) > 0$ such that $\tilde{\mathbf{P}}[\mathcal{A}] > c_1$. In particular, c_1 does not depend on \hat{z} . This means

$$\mathbf{E}\left[1_{\mathcal{A}} (1 + x(0)^2)^b \psi(0)^a\right] \geq c_1 \mathbf{E}\left[(1 + x(0)^2)^b \psi(0)^a\right] = c_1 \left(1 + (\hat{x}/\hat{y})^2\right)^b \hat{y}^\lambda.$$

However, note that $v(0)$ and hence $x(0)$ are bounded on \mathcal{A} . Therefore, there is some constant $c_2 > 0$ such that

$$(3.14) \quad \mathbf{E}\left[1_{\mathcal{A}} |g'_{T_0}(\hat{z})|^a\right] \geq c_2 \left(1 + (\hat{x}/\hat{y})^2\right)^b \hat{y}^{\lambda-a}.$$

We now estimate T_0 on the event \mathcal{A} . Recall that $T_0 \leq 0$. From (3.4) we have $\partial_u T_u = -|z|^2/2$ and therefore on \mathcal{A}

$$\begin{aligned} T_0 &= - \int_{\hat{u}}^0 \frac{y^2}{2} du - \int_{\hat{u}}^0 \frac{x^2}{2} du = \frac{\hat{y}^2}{4} - \frac{1}{4} - \frac{1}{2} \int_{\hat{u}}^0 \sinh(v)^2 e^{2u} du \\ &\geq -\frac{1}{4} - \frac{1}{2} \int_{\hat{u}}^0 e^{2\Psi(u)+2u} du \geq -c_3, \end{aligned}$$

where $c_3 = c_3(b, \kappa) < \infty$ is some constant. That is, we have $T_0 \in [-c_3, 0]$ on \mathcal{A} . On the event $T_0 \geq -c_3$, we clearly have $\text{Im}(g_{-c_3}(\hat{z})) \geq 1$ and also $|g'_{T_0}(\hat{z})| \leq |g'_{-c_3}(\hat{z})|$, by (3.3). Consequently, Lemma 3.1 and (3.14) give

$$(3.15) \quad \mathbf{E}\left[|\hat{f}'_{c_3}(\hat{z})|^a 1_{\text{Im}(\hat{f}_{c_3}(\hat{z})) \geq 1}\right] \geq c_4 \left(1 + (\hat{x}/\hat{y})^2\right)^b \hat{y}^{\lambda-a}.$$

This is almost the result that we need. However, we want to replace c_3 by 1. For this, we apply scale invariance. In this procedure, the assumption $|\hat{z}| \leq 1$ needs to be replaced by $|\hat{z}| \leq 1/\sqrt{c_3}$. The proof of the lemma is now complete. \square

It is not too hard to see that for every constant $C > 0$ the statement of the lemma can be strengthened to allow $c < |\hat{z}| < C$. The constant c must then also depend on C .

Remark 3.4. Suppose that we take

$$b = \frac{1}{4} + \frac{2\nu}{\kappa},$$

and define a and λ using (3.6). Define $\tilde{\mathbf{P}}$ as in the proof of the lemma. Then as the proof shows, v becomes Brownian motion times $\sqrt{\kappa/2}$ under $\tilde{\mathbf{P}}$, since \tilde{b} vanishes. This allows a detailed understanding of the distribution of $|g'_{T_0}(\hat{z})|$. For example, one can determine in this way the decay of $\mathbf{P}[\psi(0) > 1/2]$ as $\hat{y} \searrow 0$. It also follows that for such a, b, λ , and every $A \subset \mathbb{R}$ one can write down an explicit expression for $\mathbf{E}\left[|g'_{T_0}(\hat{z})|^a 1_{x(0) \in A}\right]$, since for every $v_0 \in \mathbb{R}$ we have

$$\Lambda\left((1 + (x/y)^2)^b y^\lambda \frac{1}{\sqrt{\nu u}} \exp\left(\frac{(v_0 - v)^2}{\nu \kappa u}\right)\right) = 0.$$

This equation is the PDE facet of the fact that v is Brownian motion times $\sqrt{\kappa/2}$ under $\tilde{\mathbf{P}}$. However, these results will not be needed in the present paper.

3.2. *Derivative upper bounds at a fixed time t .* From Theorem 3.2 it is not hard to obtain estimates for $|\hat{f}'_t|$:

COROLLARY 3.5. *Let $b \in [0, 1 + 4\kappa^{-1}]$, and define λ and a by (3.6) with $\nu = 1$. There is a constant $C(\kappa, b)$, depending only on κ and b , such that the following estimate holds for all $t \in [0, 1]$, $y, \delta \in (0, 1]$ and $x \in \mathbb{R}$.*

$$(3.16) \quad \mathbf{P}[|\hat{f}'_t(x + iy)| \geq \delta y^{-1}] \leq C(\kappa, b) (1 + x^2/y^2)^b (y/\delta)^\lambda \vartheta(\delta, a - \lambda),$$

where

$$\vartheta(\delta, s) = \begin{cases} \delta^{-s} & s > 0, \\ 1 + |\log \delta| & s = 0, \\ 1 & s < 0. \end{cases}$$

Proof. Note that $a \geq 0$. We assume that $\delta > y$, for otherwise the right-hand side is at least $C(\kappa, b)$, and we take $C(\kappa, b) \geq 1$. Take $z = x + iy$. By Lemma 3.1, $\hat{f}'_t(z)$ has the same distribution as $g'_{-t}(z)$. Let $u_1 := \log \text{Im}(g_{-t}(x + iy))$. Recall the notation T_u from (3.2). Note that

$$(3.17) \quad |g'_{-t}(z)/g'_{T_u(z)}(z)| \leq \exp(|u - u_1|),$$

since $|\partial_u \log |g'_t(z)|| \leq 1$, by (3.5). Moreover, it is clear that there is a constant c such that $u_1 \leq c$, because $t, y \leq 1$. Consequently,

$$\mathbf{P}[|g'_{-t}(z)| \geq O(1)\delta y^{-1}] \leq O(1) \sum_{j=\lceil \log y \rceil}^0 \mathbf{P}[|g'_{T_j(z)}(z)| \geq \delta y^{-1}].$$

The Schwarz lemma implies that $y|g'(z)| \leq \text{Im}(g(z))$ if $g : \mathbb{H} \rightarrow \mathbb{H}$. Therefore, the above gives

$$(3.18) \quad \mathbf{P}[|g'_{-t}(z)| \geq O(1) \delta y^{-1}] \leq O(1) \sum_{j=\lceil \log \delta \rceil}^0 \mathbf{P}[|g'_{T_j(z)}(z)| \geq \delta y^{-1}].$$

By scale invariance, we have for all $j \in [\log y, 0]$

$$\mathbf{E}[y^a e^{-ja} |g'_{T_j(z)}(z)|^a] \leq F_b(e^{-j} z),$$

where $F = F_b$ is as in Theorem 3.2. Hence

$$\begin{aligned} \mathbf{P}[|g'_{T_j(z)}(z)| \geq \delta y^{-1}] &= \mathbf{P}[|g'_{T_j(z)}(z)|^a y^a \delta^{-a} \geq 1] \\ &\leq \mathbf{E}[|g'_{T_j(z)}(z)|^a y^a \delta^{-a}] \leq \delta^{-a} e^{ja} F_b(e^{-j} z). \end{aligned}$$

Consequently, by (3.18) and Theorem 3.2,

$$\mathbf{P}[|g'_{-t}(z)| \geq O(1) \delta y^{-1}] \leq O(1) (1 + x^2/y^2)^b \delta^{-a} y^\lambda \sum_{j=\lceil \log \delta \rceil}^0 e^{j(a-\lambda)}.$$

The corollary follows. □

3.3. *Continuity of $\hat{f}_t(0)$.* In the deterministic example of nonlocally connected hulls described in the introduction, there is a time t_0 for which $\lim_{z \rightarrow 0} \hat{f}_{t_0}(z)$ does not exist (the limit set is the outer boundary of the prescribed compact set). Even when the SLE trace is a continuous path, it is not always true that $(z, t) \mapsto f_t(z)$ extends continuously to $\overline{\mathbb{H}} \times [0, \infty)$ (this is only true for simple paths). The next theorem shows that $\hat{f}_t(0) = f_t(\xi(t))$ exists as a radial limit and is continuous. Together with the result of Section 4 below, this is enough to show that the SLE trace is a path.

THEOREM 3.6. *Define*

$$H(y, t) := \hat{f}_t(iy), \quad y > 0, t \in [0, \infty).$$

If $\kappa \neq 8$, then a.s. $H(y, t)$ extends continuously to $[0, \infty) \times [0, \infty)$.

Proof. Fix $\kappa \neq 8$. By scale invariance, it is enough to show continuity of H on $[0, \infty) \times [0, 1)$. Given $j, k \in \mathbb{N}$, with $k < 2^{2j}$, let $R(j, k)$ be the rectangle

$$R(j, k) := [2^{-j-1}, 2^{-j}] \times [k 2^{-2j}, (k + 1) 2^{-2j}],$$

and set

$$d(j, k) := \text{diam}H(R(j, k)).$$

Take $b = (8 + \kappa)/(4\kappa)$ and let a and λ be given by (3.6) with $\nu = 1$. Note that $\lambda > 2$. Set $\sigma_0 := (\lambda - 2)/\max\{a, \lambda\}$, and let $\sigma \in (0, \sigma_0)$. Our objective is to

prove

$$(3.19) \quad \sum_{j=0}^{\infty} \sum_{k=0}^{2^{2j}-1} \mathbf{P}[d(j, k) \geq 2^{-j\sigma}] < \infty.$$

Fix some such pair (j, k) . The idea of the proof is to decompose the time interval $[k 2^{-2j}, (k + 1) 2^{-2j}]$ into (random) sub-intervals using partition points $\hat{t}_N, \hat{t}_{N-1}, \dots, \hat{t}_0$ such that the change of ξ on each interval is controlled. Then it is not hard to control \hat{f}_t on each sub-interval. Now, the details. Set $t_0 = (k + 1) 2^{-2j}$. Inductively, let

$$t_{n+1} := \sup\{t < t_n : |\xi(t) - \xi(t_n)| = 2^{-j}\},$$

and let N be the least $n \in \mathbb{N}$ such that $t_N \leq t_0 - 2^{-2j}$. Also set $t_\infty := t_0 - 2^{-2j} = k 2^{-2j}$ and

$$\hat{t}_n := \max\{t_n, t_\infty\}.$$

Observe that the scaling property of Brownian motion shows that there is a constant $\rho < 1$, which does not depend on j or k , such that $\mathbf{P}[N > 1] = \rho$. Moreover, the Markov property implies that $\mathbf{P}[N \geq m + 1 \mid N \geq m] \leq \rho$ for every $m \in \mathbb{N}$. In particular, $\mathbf{P}[N > m] \leq \rho^m$.

For every $s \in [0, \infty)$, the map \hat{f}_s is measurable with respect to the σ -field generated by $\xi(t)$ for $t \in [0, s]$, while \hat{t}_n is determined by $\xi(t)$ for $t \geq t_n$ (i.e., \hat{t}_n is a stopping time for the reversed time filtration). Therefore, by the strong Markov property, for every $n \in \mathbb{N}$, $s \in [t_\infty, t_0]$ and $\delta > 0$,

$$\mathbf{P}\left[|\hat{f}'_{\hat{t}_n}(i 2^{-j})| > \delta \mid \hat{t}_n = s\right] = \mathbf{P}\left[|\hat{f}'_s(i 2^{-j})| > \delta\right],$$

and consequently,

$$\mathbf{P}\left[|\hat{f}'_{\hat{t}_n}(i 2^{-j})| > \delta \mid \hat{t}_n > t_\infty\right] \leq \sup_{s \in [0, 1]} \mathbf{P}\left[|\hat{f}'_s(i 2^{-j})| > \delta\right].$$

This allows us to estimate,

$$\begin{aligned} & \mathbf{P}\left[\exists n \in \mathbb{N} : |\hat{f}'_{\hat{t}_n}(i 2^{-j})| > \delta\right] \\ & \leq \mathbf{P}\left[|\hat{f}'_{t_\infty}(i 2^{-j})| > \delta\right] + \sum_{n=0}^{\infty} \mathbf{P}[\hat{t}_n > t_\infty] \mathbf{P}\left[|\hat{f}'_{\hat{t}_n}(i 2^{-j})| > \delta \mid \hat{t}_n > t_\infty\right] \\ & \leq \mathbf{E}[N + 1] \sup_{s \in [0, 1]} \mathbf{P}\left[|\hat{f}'_s(i 2^{-j})| > \delta\right] \leq O(1) \sup_{s \in [0, 1]} \mathbf{P}\left[|\hat{f}'_s(i 2^{-j})| > \delta\right]. \end{aligned}$$

By Corollary 3.5, and because $\sigma < (\lambda - 2)/\max\{a, \lambda\}$, this gives

$$(3.20) \quad \mathbf{P}\left[\exists n \in \mathbb{N} : |\hat{f}'_{\hat{t}_n}(i 2^{-j})| > 2^j 2^{-j\sigma} / j^2\right] \leq O(1) 2^{-2j} 2^{-\varepsilon j},$$

for some $\varepsilon = \varepsilon(\kappa) > 0$. Let S be the rectangle

$$S := \{x + iy : |x| \leq 2^{-j+3}, y \in [2^{-j-1}, 2^{-j+3}]\}.$$

We claim that

$$(3.21) \quad H(R(j, k)) \subset \bigcup_{n=0}^N \hat{f}_{\hat{t}_n}(S),$$

and

$$(3.22) \quad \forall n \in \mathbb{N} \quad \hat{f}_{\hat{t}_n}(S) \cap \hat{f}_{\hat{t}_{n+1}}(S) \neq \emptyset.$$

Indeed, to prove (3.21), let $t \in [\hat{t}_{n+1}, \hat{t}_n]$ and $y \in [2^{-j-1}, 2^{-j}]$. Then we may write

$$(3.23) \quad \hat{f}_t(iy) = \hat{f}_{\hat{t}_{n+1}}(g_{\hat{t}_{n+1}}(\hat{f}_t(iy)) - \xi(\hat{t}_{n+1})).$$

We want to prove (3.21) by showing that $g_{\hat{t}_{n+1}}(\hat{f}_t(iy)) - \xi(\hat{t}_{n+1}) \in S$. Set $\varphi(s) := g_s(\hat{f}_t(iy))$ for $s \leq t$. Then $\varphi(t) = iy + \xi(t)$ and by (2.1)

$$\partial_s \varphi(s) = 2(\varphi(s) - \xi(s))^{-1}.$$

Note that $\partial_s \text{Im}(\varphi(s)) < 0$, and hence $\text{Im}(\varphi(s)) \geq \text{Im}(\varphi(t)) \geq 2^{-j-1}$. This then implies that $|\partial_s \varphi(s)| \leq 2^{j+2}$, and therefore $|t - \hat{t}_{n+1}| \leq 2^{-2j}$ gives $|\varphi(\hat{t}_{n+1}) - \varphi(t)| \leq 2^{2-j}$. Since $|\xi(t) - \xi(\hat{t}_{n+1})| \leq 2^{1-j}$, by (3.23) this gives $\hat{f}_t(iy) \in \hat{f}_{\hat{t}_{n+1}}(S)$ and verifies (3.21). We also have (3.22), because taking $t = \hat{t}_n$ in the above gives $\hat{f}_{\hat{t}_n}(iy) \in \hat{f}_{\hat{t}_{n+1}}(S)$. Since $|f'_t(z)|/|\hat{f}'_t(i2^{-j})|$ is bounded by some constant if $z \in S$ (this follows from the Koebe distortion theorem, see [Pom92, §1.3]), we find that

$$\text{diam}(\hat{f}_t(S)) \leq O(1) 2^{-j} |\hat{f}'_t(i2^{-j})|.$$

Therefore, the relations (3.21) and (3.22) give

$$(3.24) \quad \begin{aligned} d(j, k) &\leq \sum_{n=0}^N \text{diam}(\hat{f}_{\hat{t}_n}(S)) \leq O(1) 2^{-j} \sum_{n=0}^N |\hat{f}'_{\hat{t}_n}(i2^{-j})| \\ &\leq O(1) 2^{-j} N \max\{|\hat{f}'_{\hat{t}_n}(i2^{-j})| : n = 0, 1, \dots, N\}. \end{aligned}$$

By (3.20), we get

$$\begin{aligned} \mathbf{P}[d(j, k) > 2^{-j\sigma}] &\leq \mathbf{P}[N > j^2] + O(1) 2^{-2j} 2^{-\varepsilon j} \\ &\leq \rho^{j^2} + O(1) 2^{-2j} 2^{-\varepsilon j} \leq O(1) 2^{-2j} 2^{-\varepsilon j}, \end{aligned}$$

which proves (3.19).

From (3.19) we conclude that a.s. there are at most finitely many pairs $j, k \in \mathbb{N}$ with $k \leq 2^{2j} - 1$ such that $d(j, k) > 2^{-j\sigma}$. Hence $d(j, k) \leq C(\omega) 2^{-j\sigma}$ for all j, k , where $C = C(\omega)$ is random (and the notation $C(\omega)$ is meant to suggest that). Let (y', t') and (y'', t'') be points in $(0, 1)^2$. Let j_1 be the smallest integer larger than $\min\{-\log_2 y', -\log_2 y'', -\frac{1}{2} \log_2 |t' - t''|\}$. Note that a rectangle $R(j_1, k')$ that intersects the line $t = t'$ is adjacent to a rectangle

$R(j_1, k'')$ that intersects the line $t = t''$. Consequently, $|H(y', t') - H(y'', t'')| \leq \sum_{j \geq j_1} (d(j, k'_j) + d(j, k''_j)) \leq O(1) C(\omega) 2^{-\sigma j_1}$, where $R(j, k'_j)$ is a rectangle meeting the line $t = t'$ and $R(j, k''_j)$ is a rectangle meeting the line $t = t''$. This shows that for every $t_0 \in [0, 1)$ the limit of $H(y, t)$ as $(y, t) \rightarrow (0, t_0)$ exists, and thereby extends the definition of H to $[0, \infty) \times [0, 1)$. Since H is clearly continuous in $(0, \infty) \times [0, t)$, the proof is now complete. \square

For $\kappa = 8$, we are unable to prove Theorem 3.6. However, a weaker result does follow easily, namely, a.s. for almost every $t \geq 0$ the limit $\lim_{y \searrow 0} \hat{f}_t(iy)$ exists.

Update. It follows from [LSW] and the results of the current paper that the Theorem holds also when $\kappa = 8$.

4. Reduction

The following theorem provides a criterion for hulls to be generated by a continuous path. In this section we do not assume that ξ is a (time scaled) Brownian motion.

THEOREM 4.1. *Let $\xi : [0, \infty) \rightarrow \mathbb{R}$ be continuous, and let g_t be the corresponding solution of (2.1). Assume that $\beta(t) := \lim_{y \searrow 0} g_t^{-1}(\xi(t) + iy)$ exists for all $t \in [0, \infty)$ and is continuous. Then g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ and H_t is the unbounded connected component of $\mathbb{H} \setminus \beta([0, t])$, for every $t \in [0, \infty)$.*

In the proof, the following basic properties of conformal maps will be needed. Suppose that $g : D \rightarrow \mathbb{U}$ is a conformal homeomorphism. If $\alpha : [0, 1) \rightarrow D$ is a path such that the limit $l_1 = \lim_{t \nearrow 1} \alpha(t)$ exists, then $l_2 = \lim_{t \nearrow 1} g \circ \alpha$ exists, too. (However, if $\alpha : [0, 1) \rightarrow \mathbb{U}$ is a path such that the above limit exists, it does not follow that $\lim_{t \nearrow 1} g^{-1} \circ \alpha(t)$ exists. In other words, it is essential that the image of g is a nice domain such as \mathbb{U} .) Moreover, $\lim_{t \nearrow 1} g^{-1}(tl_2)$ exists and equals l_1 . Consequently, if $\tilde{\alpha} : [0, 1) \rightarrow D$ is another path with $\lim_{t \nearrow 1} \tilde{\alpha}$ existing and with $\lim_{t \nearrow 1} g \circ \alpha(t) = \lim_{t \nearrow 1} g \circ \tilde{\alpha}(t)$, then $\lim_{t \nearrow 1} \alpha(t) = \lim_{t \nearrow 1} \tilde{\alpha}(t)$.

These statements are well known and easily established, for example with the notion of extremal length. See [Pom92, Prop. 2.14] for the first statement and [Ahl73, Th. 3.5] implies the second claim.

Proof. Let $S(t) \subset \overline{\mathbb{H}}$ be the set of limit points of $g_t^{-1}(z)$ as $z \rightarrow \xi(t)$ in \mathbb{H} . Fix $t_0 \geq 0$, and let $z_0 \in S(t_0)$. We want to show that $z_0 \in \overline{\beta([0, t_0])}$, and hence $z_0 \in \beta([0, t_0])$. Fix some $\varepsilon > 0$. Let

$$t' := \sup\{t \in [0, t_0] : K_t \cap \overline{D(z_0, \varepsilon)} = \emptyset\},$$

where $D(z_0, \varepsilon)$ is the open disk of radius ε about z_0 . We first show that

$$(4.1) \quad \beta(t') \in \overline{D(z_0, \varepsilon)}.$$

Indeed, $D(z_0, \varepsilon) \cap H_{t_0} \neq \emptyset$ since $z_0 \in S(t_0)$. Let $p \in D(z_0, \varepsilon) \cap H_{t_0}$, and let $p' \in K_{t'} \cap D(z_0, \varepsilon)$. Let p'' be the first point on the line segment from p to p' which is in $K_{t'}$. We want to show that $\beta(t') = p''$. Let L be the line segment $[p, p'')$, and note that $L \subset H_{t'}$. Hence $g_{t'}(L)$ is a curve in \mathbb{H} terminating at a point $x \in \mathbb{R}$. If $x \neq \xi(t')$, then $g_t(L)$ terminates at points $x(t) \neq \xi(t)$ for all $t < t'$ sufficiently close to t' . Because $g_\tau(p'')$ has to hit the singularity $\xi(\tau)$ at some time $\tau \leq t'$, this implies $p'' \in K_t$ for $t < t'$ close to t' . This contradicts the definition of t' and shows $x = \xi(t')$. Now $\beta(t') = p''$ follows because the conformal map $g_{t'}^{-1}$ of \mathbb{H} cannot have two different limits along two arcs with the same terminal point.

Having established (4.1), since $\varepsilon > 0$ was arbitrary, we conclude that $z_0 \in \overline{\beta([0, t_0])}$ and hence $z_0 \in \beta([0, t_0])$. This gives $S(t) \subset \beta([0, t])$ for all $t \geq 0$. We now show that H_t is the unbounded component of $\mathbb{H} \setminus \overline{\bigcup_{\tau \leq t} S(\tau)}$. First, H_t is connected and disjoint from $\overline{\bigcup_{\tau \leq t} S(\tau)}$. On the other hand, as the argument in the previous paragraph shows, $\partial H_t \cap \mathbb{H}$ is contained in $\overline{\bigcup_{\tau \leq t} S(\tau)}$. Therefore, H_t is a connected component of $\mathbb{H} \setminus \overline{\bigcup_{\tau \leq t} S(\tau)}$; that is, H_t is the unbounded connected component of $\mathbb{H} \setminus \beta([0, t])$. Since β is a continuous path, it follows from [Pom92, Th. 2.1] that g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ (which also proves that $S(t) = \{\beta(t)\}$). \square

5. Continuity

We have now established all the results needed to show that the SLE_κ trace is a continuous path a.s.

THEOREM 5.1 (Continuity). *Let $\kappa \neq 8$. The following statements hold almost surely. For every $t \geq 0$ the limit*

$$\gamma(t) := \lim_{z \rightarrow 0, z \in \mathbb{H}} \hat{f}_t(z)$$

exists, $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ is a continuous path, and H_t is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$.

We believe the theorem to be valid also for $\kappa = 8$. (This is stated as Conjecture 9.1.) Despite repeated efforts, the proof eluded us.

Update. This extension to $\kappa = 8$ is proved in [LSW].

Proof of Theorem 5.1. By Theorem 3.6, a.s. $\lim_{y \searrow 0} \hat{f}_t(iy)$ exists for all t and is continuous. Therefore we can apply Theorem 4.1, and the theorem follows. \square

It follows from Theorem 5.1 that f_t extends continuously to $\overline{\mathbb{H}}$ a.s. The next result gives more information about the regularity of f_t on $\overline{\mathbb{H}}$. It neither follows from Theorem 5.1, nor does it imply 5.1.

THEOREM 5.2 (Hölder continuity). *For every $\kappa \neq 4$ there is some $h(\kappa) > 0$ such that for every bounded set $A \subset \mathbb{H}$ and every $t > 0$, a.s. \hat{f}_t is Hölder continuous with exponent $h(\kappa)$ on A ,*

$$|\hat{f}_t(z) - \hat{f}_t(z')| \leq C|z - z'|^{h(\kappa)} \quad \text{for all } z, z' \in A,$$

where $C = C(\omega, t, A)$ is random and may depend on t and A . Moreover, $\lim_{\kappa \searrow 0} h(\kappa) = \frac{1}{2}$ and $\lim_{\kappa \nearrow \infty} h(\kappa) = 1$.

Since $f_t(z) - z \rightarrow 0$ as $z \rightarrow \infty$, it easily follows that for every t a.s.

$$(5.1) \quad |\hat{f}_t(z) - \hat{f}_t(z')| \leq C(\omega, t) \max(|z - z'|, |z - z'|^{h(\kappa)}).$$

We do not believe that the theorem holds for $\kappa = 4$, for then the trace is a simple path which “almost” touches itself.

Update. For $\kappa \leq 4$, the fact that γ is a simple path in \mathbb{H} , Theorem 6.1 below, implies $h(\kappa) \leq 1/2$. Thus the estimate $\lim_{\kappa \searrow 0} h(\kappa) = \frac{1}{2}$ is best possible. On the other hand, this nonsmoothness of \hat{f}_t is localized at $\hat{f}_t^{-1}(0)$: Joan Lind (manuscript in preparation) has shown that the Hölder exponent $h(\kappa)$ of $(\hat{f}_t(\sqrt{z}))^2$ satisfies $\lim_{\kappa \searrow 0} h(\kappa) = 1$.

Proof. Fix $\kappa \neq 4$ and $t > 0$. By scaling, we may assume $0 < t \leq 1$ and $A = [-1, 1] \times (0, 1]$. Denote

$$z_{j,n} = (j + i)2^{-n}, \quad 0 \leq n < \infty, \quad -2^n < j < 2^n.$$

We will first show that there is an $h = h(\kappa) > 0$ such that a.s.

$$(5.2) \quad |\hat{f}'_t(z_{j,n})| \leq C(\omega, t)2^{n(1-h)}, \quad \forall j, n.$$

Using Corollary 3.5 with $\delta = 2^{-nh}$ we have

$$\mathbf{P} [|\hat{f}'_t(z_{j,n})| \geq 2^{n(1-h)}] \leq C(\kappa, b) (1 + 2^{2n})^b 2^{-n(1-h)\lambda} \vartheta(2^{-nh}, a - \lambda).$$

Hence

$$\sum_{n=0}^{\infty} \sum_{j=-2^n}^{2^n} \mathbf{P} [|\hat{f}'_t(z_{j,n})| \geq 2^{n(1-h)}] < \infty$$

provided that

$$1 + 2b - (1 - h)\lambda < 0 \quad \text{and} \quad a - \lambda \leq 0,$$

or that

$$1 + 2b - \lambda + ah < 0 \quad \text{and} \quad a - \lambda \geq 0.$$

If $0 < \kappa \leq 12$, $b = 1/4 + 1/\kappa$ and $h < (\kappa - 4)^2/((\kappa + 4)(\kappa + 12))$, the first condition is satisfied. For $\kappa > 12$, $b = 4/\kappa$ and $h < 1/2 - 4/\kappa$ the second condition is satisfied, and (5.2) follows.

To see that one can actually achieve $h(\kappa) \rightarrow 1/2$ as $\kappa \searrow 0$, set $b := -1/2 + \sqrt{1/2 + 2/\kappa}$ for $0 < \kappa < 4$, and let h be smaller than but close to $(\lambda - 1 - 2b)/\lambda$. To get $\lim_{\kappa \nearrow \infty} h(\kappa) = 1$, take $b := (\sqrt{2\kappa\sqrt{\kappa^2 + 10\kappa + 16}} - 2\kappa)/(\kappa^2 + 4\kappa)$ for $\kappa \geq 2(3 + \sqrt{17})$ and let h be smaller than but close to $(\lambda - 1 - 2b)/a$.

From the Koebe distortion Theorem and (5.2) we obtain

$$|\hat{f}'_t(z)| \leq O(1)C(\omega, t)y^{h-1}$$

for all $z \in A$. It is well-known and easy to see, by integrating $|f'|$ over the hyperbolic geodesic from z to z' (similarly to the end of the proof of Theorem 3.6), that this implies Hölder continuity with exponent h on A . The theorem follows. □

The following corollary is an immediate consequence of Theorem 5.2. In Section 8 below, we will present more precise estimates.

COROLLARY 5.3. *For $\kappa \neq 4$ and every t , the Hausdorff dimension of ∂K_t a.s. satisfies*

$$\dim \partial K_t < 2.$$

In particular, a.s. $\text{area } \partial K_t = 0$.

Proof. By [JM95] (see also [KR97] for an easier proof), the boundary of the image of a disk under a Hölder continuous conformal map has Hausdorff dimension bounded away from 2. Consider the conformal map $T(z) = (z - i)/(z + i)$ from \mathbb{H} onto \mathbb{U} . By (5.1), $T \circ f_t \circ T^{-1}$ is a.s. Hölder continuous in \mathbb{U} . Since T preserves Hausdorff dimension, the corollary follows. □

6. Phases

In this section, we will investigate the topological behavior of SLE, and will identify three very different phases for the parameter κ , namely, $[0, 4]$, $(4, 8)$, and $[8, \infty)$.

The following result was conjectured in [Sch00]. There, it was proved that for $\kappa > 4$, a.s. K_t is not a simple path. The proof was based on the calculation of the harmonic measure $F(x)$ below, which we will repeat here for the convenience of the reader.

THEOREM 6.1. *In the range $\kappa \in [0, 4]$, the SLE_κ trace γ is a.s. a simple path and $\gamma[0, \infty) \subset \mathbb{H} \cup \{0\}$.*

LEMMA 6.2. *Let $\kappa \in [0, 4]$, and let γ be the SLE_κ trace. Then a.s.*

$$\gamma[0, \infty) \subset \mathbb{H} \cup \{0\}.$$

Set $Y_x(t) := g_t(x) - \xi(t)$, $x \in \mathbb{R}$, $t \geq 0$. Then $Y_x(t)/\sqrt{\kappa}$ is a Bessel process of dimension $1 + 4/\kappa$. The basic theory of Bessel processes then tells us that a.s. $Y_x(t) \neq 0$ for $t \geq 0$, $x \neq 0$ and $\kappa \in [0, 4]$, which amounts to $x \neq K_t$. However, we will prove this for the convenience of the reader unfamiliar with Bessel processes.

Proof. Let $b > a > 0$. For $x \in [a, b]$, let

$$T = T_x := \inf\{t \geq 0 : Y_x(t) \notin (a, b)\}.$$

Set

$$\hat{F}(x) := \begin{cases} x^{(\kappa-4)/\kappa} & \kappa \neq 4, \\ \log(x) & \kappa = 4, \end{cases}$$

and

$$F(x) := \frac{\hat{F}(x) - \hat{F}(a)}{\hat{F}(b) - \hat{F}(a)}.$$

Itô's formula shows that $F(Y_x(t \wedge T))$ is a local martingale, and since F is bounded in $[a, b]$, this is a martingale. Consequently, the optional sampling theorem gives $F(x) = \mathbf{E}[F(Y_x(T))] = \mathbf{P}[Y_x(T) = b]$.

Note that $F(x) \rightarrow 1$ when $a \searrow 0$. (That's where the assumption $\kappa \leq 4$ is crucial.) Hence, given $x > 0$, a.s. for all $b > x$, there is some $s > 0$ such that $g_t(x)$ is well defined for $t \in [0, s]$, $\inf\{Y_x(t) : t \in [0, s]\} > 0$ and $Y_x(s) = b$. Note that the Itô derivative of $Y_x(t)$ (with respect to t) is

$$dY_x = (2/Y_x) dt + d\xi.$$

It follows easily that a.s. $Y_x(t)$ does not escape to ∞ in finite time. Observe that if $x' > x$, then $Y_{x'}(t) \geq Y_x(t)$. Therefore, a.s. for every $x > 0$ we have $Y_x(t)$ well defined and in $(0, \infty)$ for all $t \geq 0$. This implies that a.s. for every $x > 0$ and every $s > 0$ there is some neighborhood N of x in \mathbb{C} such that the differential equation (2.1) has a solution in the range $z \in N$, $t \in [0, s]$. This proves that a.s. $\gamma[0, \infty)$ does not intersect $(0, \infty)$. The proof that it a.s. does not intersect $(-\infty, 0)$ is the same. \square

Proof of Theorem 6.1. Let $t_2 > t_1 > 0$. The theorem will be established by proving that $\gamma[0, t_1] \cap \gamma[t_2, \infty) = \emptyset$. Let $s \in (t_1, t_2)$ be rational, and set

$$\hat{g}_t(z) := g_{t+s} \circ g_s^{-1}(z + \xi(s)) - \xi(s).$$

By Proposition 2.1 ($\hat{g}_t : t \geq 0$) has the same distribution as $(g_t : t \geq 0)$. Let $\hat{\gamma}_s(t)$ be the trace for the collection $(\hat{g}_t : t \geq 0)$; that is,

$$\hat{\gamma}_s(t) = g_s \circ g_{t+s}^{-1}(\xi(t+s)) - \xi(s) = g_s \circ \gamma(t+s) - \xi(s).$$

We know from Lemma 6.2 that a.s. for all rational $s \in [t_1, t_2]$,

$$\hat{\gamma}_s[0, \infty) \subset \mathbb{H} \cup \{0\}.$$

Consequently, by applying g_s^{-1} , we conclude that a.s. for every rational $s > 0$,

$$\gamma[s, \infty) \cap (\mathbb{R} \cup K_s) = \{\gamma(s)\}.$$

Since $g_{t_2} \circ g_{t_1}^{-1}$ is not the identity, it follows from Theorem 5.1 that there is some $s \in (t_1, t_2)$ such that $\gamma(s) \notin \mathbb{R} \cup K_{t_1}$, and this holds for an open set of $s \in (t_1, t_2)$. In particular, there is some rational $s \in (t_1, t_2)$ with $\gamma(s) \notin \mathbb{R} \cup K_{t_1}$. Since $\gamma[0, t_1] \subset K_{t_1} \cup \mathbb{R}$, it follows that $\gamma[0, t_1] \cap \gamma[t_2, \infty) = \emptyset$, proving the theorem. \square

Recall the definition of the hypergeometric function in $|z| < 1$,

$$(6.1) \quad {}_2F_1(\eta_0, \eta_1, \eta_2, z) := \sum_{n=0}^{\infty} \frac{(\eta_0)_n (\eta_1)_n}{(\eta_2)_n n!} z^n,$$

where $(\eta)_n := \prod_{j=0}^{n-1} (\eta + j)$. Let

$$\hat{G}(x + iy) = \hat{G}_{a,\kappa}(x + iy) := {}_2F_1(\eta_0, \eta_1, 1/2, x^2/(x^2 + y^2)),$$

where

$$\eta_j = \eta_j(a, \kappa) := \frac{1}{2} - \frac{2}{\kappa} - (-1)^j \frac{\sqrt{32a\kappa + (2\kappa - 8)^2}}{4\kappa}.$$

LEMMA 6.3. *Let $\kappa > 0$ and $z = x + iy \in \mathbb{H}$. Then the limit*

$$Z(z) := \lim_{t \nearrow \tau(z)} y |g'_t(z)| / \text{Im}(g_t(z))$$

exists a.s. We have $Z(z) = \infty$ a.s. if $\kappa \geq 8$ and $Z(z) < \infty$ a.s. if $\kappa < 8$. Moreover, for all $a \in \mathbb{R}$ we have

$$\mathbf{E}[Z(z)^a] = \begin{cases} \hat{G}_{a,\kappa}(z) / \hat{G}_{a,\kappa}(1) & a < 1 - \kappa/8, \kappa < 8, \\ \infty & a \geq 1 - \kappa/8, a > 0, \\ 0 & a < 0, \kappa \geq 8. \end{cases}$$

Before proving the lemma, let us discuss its geometric meaning. For $z_0 \in \mathbb{H}$ and $t < \tau(z_0)$ we have $z_0 \in H_t$. Set $r_t := \text{dist}(z_0, \partial H_t)$, and let $\phi : \mathbb{H} \rightarrow \mathbb{U}$ be a conformal homeomorphism satisfying $\phi(g_t(z_0)) = 0$. Note that $|\phi'(g_t(z_0))| = (2 \text{Im } g_t(z_0))^{-1}$. Hence, the Schwarz lemma applied to the map $z \mapsto \phi \circ g_t(z_0 + r_t z)$ proves $r_t |g'_t(z_0)| \leq 2 \text{Im } g_t(z_0)$. On the other hand, the Koebe 1/4 Theorem applied to the function $g_t^{-1} \circ \phi^{-1}$ implies that $\text{Im } g_t(z_0) \leq 2 r_t |g'_t(z_0)|$. Therefore, since $\lim_{t \nearrow \tau(z_0)} r_t = \text{dist}(z_0, \gamma[0, \infty) \cup \mathbb{R})$,

$$(6.2) \quad Z(z_0)^{-1} \text{Im } z_0 / 2 \leq \text{dist}(z_0, \gamma[0, \infty) \cup \mathbb{R}) \leq 2 Z(z_0)^{-1} \text{Im } z_0.$$

In particular, the lemma tells us that $\gamma[0, \infty)$ is dense in \mathbb{H} if and only if $\kappa \geq 8$. (For $\kappa = 8$ we have not proven that γ is a path. In that case, $\gamma[0, \infty)$ is defined as the union over t of all limit points of sequences $\overline{f_t(z_j)}$ where $z_j \rightarrow 0$ in \mathbb{H} . By an argument in the proof of Theorem 4.1, $\bigcup_{t \geq 0} \partial K_t = \overline{\gamma[0, \infty)}$.)

The way one finds the function \hat{G} is explained within the proof below, and is similar to the situation in Theorem 3.2.

Proof. Let $G(z) = G_{a,\kappa}(z) := \mathbf{E}[Z(z)^a]$. Fix some $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$, and abbreviate $Z := Z(\hat{z})$. Let $z_t = x_t + iy_t := g_t(\hat{z}) - \xi(t)$. From (3.9) we find that

$$\partial_t \log |g'_t(z)/y_t| = 4y_t^2 |z_t|^{-4},$$

which implies that

$$(6.3) \quad Z = \exp\left(\int_0^{\tau(\hat{z})} 4y_t^2 |z_t|^{-4} dt\right) = \exp\left(\int_0^{\tau(\hat{z})} \frac{-2y_t^2}{x_t^2 + y_t^2} d \log y_t\right).$$

In particular, the limit in the definition of Z exists a.s. and $G(\hat{z}) = \mathbf{E}[Z^a]$ (possibly $+\infty$).

Let \mathcal{F}_t denote the σ -field generated by $(\xi(s) : s \leq t)$. A direct application of Itô's formula shows that

$$M_t := (\hat{y} |g'_t(\hat{z})|/y_t)^a \hat{G}(z_t)$$

is a local martingale¹.

As before, let $u_t = \log y_t$. Note that $w(u) = w_t := x_t/y_t$ is a time-homogeneous diffusion process as a function of u . It is easy to see that $\tau(\hat{z})$ is a.s. the time at which u hits $-\infty$. (For example, see the argument following (3.2).)

If $t_1 < \tau(\hat{z})$ is a stopping time, then by (6.3) we may write

$$(6.4) \quad \log Z(\hat{z}) = \int_0^{t_1} + \int_{t_1}^{\tau(\hat{z})} 4y_t^2 |z_t|^{-4} dt.$$

Note that if $s > 0$ then the distribution of $Z(\hat{z})$ is the same as that of $Z(s\hat{z})$, by scale invariance. It is clear that for every $z' \in \mathbb{H}$ there is positive probability that there will be some first time $t_1 < \tau(\hat{z})$ such that $z_{t_1}/|z_{t_1}| = z'/|z'|$. Conditioned on $t_1 < \tau(\hat{z})$ and on \mathcal{F}_{t_1} , the second integral in (6.4) has the same distribution as $\log Z(z')$, by the strong Markov property and scale invariance. Consequently, we find that $\mathbf{P}[Z(\hat{z}) > s] \geq c \mathbf{P}[Z(z') > s]$ holds for every s , where $c = c(\hat{z}, z', \kappa) > 0$. In fact, later we will need the following stronger

¹The Markov property can be used to show that $(\hat{y} |g'_t(\hat{z})|/y_t)^a \hat{G}(z_t)$ is a local martingale. One then assumes that G is smooth and applies Itô's formula, to obtain a PDE satisfied by G . Scale invariance can then be used to transform the PDE to the ODE (6.9) appearing below. Then one obtains \hat{G} as a solution. That's how the function \hat{G} was arrived at.

statement (which is not needed for the present lemma, but we find it convenient to give the argument here). Let $\bar{w} \in (0, \infty)$. Then assuming $|w_0| \leq \bar{w}$, there is a $c = c(\bar{w}, \kappa) > 0$ such that

$$(6.5) \quad \forall s > 0, \quad \mathbf{P}[Z(i) > s] \geq \mathbf{P}[Z(\hat{z}) > s] \geq c \mathbf{P}[Z(i) > 2s].$$

The right-hand inequality is proved by taking t_1 to be the first time smaller than $\tau(z)$ such that $\operatorname{Re} z_{t_1} = 0$ and the first integral in (6.4) is greater than $\log 2$, on the positive probability event that there is such a time. The left-hand inequality is clear, for if $|\operatorname{Re} g_t(i)/\operatorname{Im} g_t(i)|$ never hits $|w_0|$, then we have $Z(i) = \infty$, by (6.3).

Note that $\hat{G}(i) = 1$. Assume, for now, that $G(i) < \infty$. We claim that in this case $\hat{G}(x+i) > 0$ for each $x \in \mathbb{R}$. Let $s_0 > 0$, let $T := \inf\{t \geq 0 : |w_t| = s_0\}$. Since $w(u)$ is time-homogeneous, clearly, $T < \tau(\hat{z})$. If $\hat{G}(x+i) \leq 0$ for some $x \in \mathbb{R}$, we may choose $s_0 := \inf\{x > 0 : \hat{G}(x+i) = 0\}$. Then $M_T = 0$ if $|w_0| \leq s_0$. Note that $1 \leq \hat{y} |g'_T(\hat{z})|/y_T < Z$. Since M_t is a local martingale on $t < \tau(\hat{z})$, there is an increasing sequence of stopping times $t_n < \tau(\hat{z})$ with $\lim_n t_n = \tau(\hat{z})$ a.s. such that $M_{t \wedge t_n}$ is a martingale. Set $\bar{t}_n := T \wedge t_n$. The optional sampling theorem then gives for $\hat{z} = i$

$$1 = \hat{G}(\hat{z}) = M_0 = \mathbf{E}[M_{\bar{t}_n}] = \mathbf{E}[(\hat{y} |g'_{\bar{t}_n}(\hat{z})|/y_{\bar{t}_n})^a \hat{G}(z_{\bar{t}_n})] \\ \leq \mathbf{E}[\max\{1, Z^a\} \hat{G}(z_{\bar{t}_n})].$$

Since $\hat{G}(z_{t \wedge T})$ is bounded, $\lim_{n \rightarrow \infty} \hat{G}(z_{\bar{t}_n}) = 0$ a.s., and $\mathbf{E}[Z^a] < \infty$, we have $\lim_{n \rightarrow \infty} \mathbf{E}[\max\{1, Z^a\} \hat{G}(z_{\bar{t}_n})] = 0$, contradicting the above inequality. Consequently, $\hat{G}(x+i) > 0$ for every $x \in \mathbb{R}$.

We maintain the assumption $G(i) < \infty$. By (6.5), this implies $G(z) < \infty$ for all $z \in \mathbb{H}$. Let s_1, s_2, \dots be a sequence in $(0, \infty)$ with $s_n \rightarrow \infty$ such that $L := \lim_n \hat{G}(s_n + i)$ exists (allowing $L = \infty$). Now set $T_m := \inf\{t \geq 0 : |w_t| = s_m\}$. The argument of the previous paragraph shows that $M_0 = \mathbf{E}[M_{T_m \wedge t_n}] \rightarrow \mathbf{E}[M_{T_m}]$, as $n \rightarrow \infty$. Therefore,

$$\hat{G}(\hat{z}) = M_0 = \mathbf{E}[M_{T_m}] \rightarrow L G(\hat{z}), \quad m \rightarrow \infty,$$

unless $L = \infty$, which happens if and only if $G(\hat{z}) = 0$. Moreover, L does not depend on the particular sequence s_n , and we may write $\hat{G}(1)$ instead of L . Thus, when $G(i) < \infty$,

$$(6.6) \quad G(\hat{z}) = \hat{G}(\hat{z})/\hat{G}(1).$$

When $a = 1 - \kappa/8$, \hat{G} simplifies to $(y/|z|)^{(8-\kappa)/\kappa}$, with $\hat{G}(1) = \infty$ if $\kappa > 8$ and $\hat{G}(1) = 0$ if $\kappa < 8$. It follows that $G(z) = \infty$ if $a = 1 - \kappa/8$ and $\kappa < 8$. Clearly, $G(z) = \infty$ also if $a > 1 - \kappa/8$, and $\kappa < 8$. Similarly, we have $G(z) = 0$ if $\kappa > 8$ and $a = 1 - \kappa/8$. This implies that $Z = \infty$ a.s. when $\kappa > 8$, which implies $G = 0$ when $\kappa > 8$ and $a < 0$. There are various ways to argue that $Z = \infty$ a.s. when $\kappa = 8$. In particular, it is easy to see that $\hat{G}_{\frac{1}{4}, 8}(1) = \infty$

(if a_n denotes the n -th coefficient of ${}_2F_1(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, z)$, then $(n + 1)a_{n+1} = na_n b_n$ with $b_n = (n + 1/4)^2 / (n(n + 1/2))$ and it follows from $\prod_n b_n > 0$ that na_n is bounded away from zero) so that the above argument applies. It remains to prove that $G(i) < \infty$ when $a < 1 - \kappa/8$ and $\kappa < 8$.

Assume now that $\kappa < 8$ and $a < 1 - \kappa/8$ and a is sufficiently close to $1 - \kappa/8$ so that $\eta_1 > 0$. We prove below that in this range

$$(6.7) \quad \inf_{z \in \mathbb{H}} \hat{G}(z) > 0.$$

Since $\hat{G}(\hat{z}) = M_0 = \mathbf{E}[M_{t_n}]$ when $\hat{z} = i$, we immediately get $G(i) < \infty$ from (6.7). Since $G(i) < \infty$ for some a implies $G(i) < \infty$ for any smaller a , it only remains to establish (6.7) in the above specified range.

To prove (6.7), first note that $\eta_0 \leq \eta_1$, $0 < \eta_1 < 1/2$ and $\eta_0 + \eta_1 = 1 - 4/\kappa < 1/2$. By [EMOT53, 2.1.3.(14)], when $\eta_2 > \eta_1 + \eta_0$ and $\eta_2 > \eta_1 > 0$, the right-hand side of (6.1) converges at $z = 1$ and is equal to

$$(6.8) \quad \frac{\Gamma(\eta_2) \Gamma(\eta_2 - \eta_1 - \eta_0)}{\Gamma(\eta_2 - \eta_1) \Gamma(\eta_2 - \eta_0)}.$$

This gives $\hat{G}(1) > 0$. Now write

$$g(x) := \hat{G}_{a,\kappa}(x + i) / \hat{G}_{1-\kappa/8,\kappa}(x + i) = (1 + x^2)^{(8-\kappa)/(2\kappa)} \hat{G}_{a,\kappa}(x + i).$$

Since \hat{G} satisfies

$$(6.9) \quad \frac{4ay^2}{|z|^4} \hat{G} + \frac{\kappa}{2} \partial_x^2 \hat{G} + \frac{4x}{|z|^2} \partial_x \hat{G} = 0$$

for $y = 1$, it follows that

$$(6.10) \quad (\kappa - 8 + 8a)g + 2(\kappa - 4)x(1 + x^2)g' + \kappa(1 + x^2)^2 g'' = 0.$$

It is immediate that $g(0) = 1$, $g'(0) = 0$ and $g''(0) > 0$, e.g., by differentiation or by considering (6.9). Consequently, $g > 1$ for small $|x| > 0$. Suppose that there is some $x \neq 0$ such that $g(x) = 1$. Then there must be some $x_0 \neq 0$ such that $g(x_0) > 1$ and g has a local maximum at x_0 . Then $g'(x_0) = 0 \geq g''(x_0)$. But this contradicts (6.10), giving $g > 1$ for $x \neq 0$. Together with $\hat{G}(1) > 0$, this implies (6.7), and completes the proof. \square

We say that $z \in \overline{\mathbb{H}}$ is *swallowed* by the SLE if $z \notin \gamma[0, \infty)$ but $z \in K_t$ for some $t \in (0, \infty)$. The smallest such t will be called the *swallowing time* of z .

THEOREM 6.4. *If $\kappa \in (4, 8)$ and $z \in \overline{\mathbb{H}} \setminus \{0\}$, then a.s. z is swallowed. If $\kappa \notin (4, 8)$, then a.s. no $z \in \overline{\mathbb{H}} \setminus \{0\}$ is swallowed.*

LEMMA 6.5. *Let $z \in \overline{\mathbb{H}} \setminus \{0\}$. If $\kappa > 4$, then $\mathbf{P}[\tau(z) < \infty] = 1$.*

In fact, in this range $\kappa > 4$, the stronger statement that a.s. for all $z \in \overline{\mathbb{H}}$ we have $\tau(z) < \infty$ holds as well. This follows from Theorem 7.1 (and its extension to $\kappa = 8$ based on [LSW]).

Proof. Let $b := 1 - 4/\kappa$, and $\theta := \exp(i\pi(1 - b)/2)$. Set

$$h(z) := \operatorname{Im}(\theta z^b), \quad z \in \overline{\mathbb{H}}.$$

A direct calculation shows that the drift term in Itô's formula for $(g_t(z) - \xi(t))^b$ is zero. Consequently, $h(g_t(z) - \xi(t))$ is a local martingale. Note that $\kappa > 4$ implies that $h(z) \rightarrow \infty$ as $z \rightarrow \infty$ in $\overline{\mathbb{H}}$, that $h(z) > 0$ in $\overline{\mathbb{H}} \setminus \{0\}$ and $h(0) = 0$.

Fix some $z_0 \in \mathbb{H}$ and some $\varepsilon > 0$. For $t \geq 0$ set $z_t := g_t(z_0) - \xi(t)$. We shall prove that there is some $T = T(z_0, \varepsilon)$ such that $\mathbf{P}[z_0 \in K_T] > 1 - \varepsilon$. Indeed, let $R > 1$ be sufficiently large so that $\varepsilon \inf\{h(z) : |z| = R\} > 2h(z_0)$. Let

$$\tau_R := \inf\left(\{t \in [0, \tau(z_0)) : |z_t| \geq R\} \cup \{\tau(z_0)\}\right).$$

We claim that if $T = T(z_0, \varepsilon)$ is sufficiently large, then $\mathbf{P}[\tau_R > T] < \varepsilon/2$. Let $t \geq 0$ and let $\lceil t \rceil := \inf\{k \in \mathbb{Z} : k \geq t\}$. Say that t is a *sprint time* if $|\xi(t) - \xi(\lceil t \rceil)| = R$ and there is no $s \in [t, \lceil t \rceil]$ such that $|\xi(s) - \xi(\lceil t \rceil)| = R$. Note that for every integer $k \geq 0$ there is positive probability that there is a sprint time in $[k, k + 1]$ and this event is independent from $(\xi(t) : t \in [0, k])$. Therefore, there are infinitely many sprint times a.s. Let $t_0 < t_1 < t_2 < \dots$ be the sequence of sprint times. Let \mathcal{A}_j be the event that $\xi(\lceil t_j \rceil) > \xi(t_j)$. Observe that for all $j \in \mathbb{N}$,

$$(6.11) \quad \mathbf{P}[\mathcal{A}_j \mid t_j, \mathcal{F}_{t_j}] = 1/2,$$

where \mathcal{F}_t denotes the σ -field generated by $(\xi(s) : s \in [0, t])$, because given t_j and $(\xi(t) : t \in [0, t_j])$, the distribution of $(\xi(t) : t > t_j)$ is the same as that of $(2\xi(t_j) - \xi(t) : t > t_j)$, by reflection symmetry. Let $x_t := \operatorname{Re}(z_t)$, and note that

$$dx_t = 2x_t |z_t|^{-2} dt - d\xi(t).$$

Consequently, for $s \geq t_j$,

$$(6.12) \quad x_s = x_{t_j} + 2 \int_{t_j}^s x_t |z_t|^{-2} dt - \xi(s) + \xi(t_j).$$

On \mathcal{A}_j we have $\xi(s) - \xi(t_j) > 0$ for all $s \in [t_j, \lceil t_j \rceil]$. Hence, it follows from (6.12) that on \mathcal{A}_j and assuming $x_{t_j} \leq 0$, we have $x_s \leq 0$ for all $s \in [t_j, \lceil t_j \rceil]$ and moreover $x_{\lceil t_j \rceil} < -R$. Similarly, if $x_{t_j} \geq 0$ and $\neg \mathcal{A}_j$ holds, then $x_{\lceil t_j \rceil} > R$. Hence, by (6.11) we have $\mathbf{P}[\tau_R > \lceil t_j \rceil] \leq 2^{-j}$, which implies that we may choose T such that $\mathbf{P}[\tau_R > T] < \varepsilon/2$. Set $\tau^* := \tau_R \wedge T$, where T is as above.

Since $h(z_t)$ is a local martingale, and $h(z_{t \wedge \tau^*})$ is bounded, $h(z_{t \wedge \tau^*})$ is a martingale. Therefore,

$$h(z_0) = \mathbf{E}[h(z_{\tau^*})] \geq \inf\{h(z) : |z| = R\} \mathbf{P}[|z_{\tau^*}| = R].$$

By our choice of R , this gives $\mathbf{P}[|z_{\tau^*}| = R] < \varepsilon/2$. Since $\mathbf{P}[\tau_R > T] < \varepsilon/2$, it follows that $\mathbf{P}[\tau(z_0) \leq T] \geq 1 - \varepsilon$, and the lemma is thereby established. \square

We now prove a variant of Cardy’s formula. (See [Car] for a survey of Cardy’s formula.) A proof of (a generalized) Cardy’s formula for SLE_κ appears in [LSW01a]. However the following variant appears to be different, except in the special case $\kappa = 6$, in which they are equivalent due to the locality property of SLE_6 . (See [LSW01a] for an explanation of the locality property.)

LEMMA 6.6. *Fix $\kappa \in (4, 8)$ and let*

$$X := \inf([1, \infty) \cap \gamma[0, \infty)).$$

Then for all $s \geq 1$,

$$(6.13) \quad \mathbf{P}[X \geq s] = \frac{4^{(\kappa-4)/\kappa} \sqrt{\pi} {}_2F_1(1 - 4/\kappa, 2 - 8/\kappa, 2 - 4/\kappa, 1/s) s^{(4-\kappa)/\kappa}}{\Gamma(2 - 4/\kappa) \Gamma(4/\kappa - 1/2)}.$$

The proof will be brief, since it is similar to the detailed proof of the more elaborate result in [LSW01a].

Proof. Let $s > 1$, and set

$$Y_t := \frac{g_t(1) - \xi(t)}{g_t(s) - \xi(t)},$$

for $t < \tau(1)$. It is not hard to see that $\lim_{t \nearrow \tau(1)} Y_t = 1$ if $X > s$ and $\lim_{t \nearrow \tau(1)} Y_t = 0$ if $X < s$: Just notice that for $X > s$ and $t \nearrow \tau(1)$, the extremal distance within H_t from $[1, s]$ to $[\text{Re } \gamma(t), \gamma(t)]$ tends to ∞ , whereas for $X < s$ we have $\tau(1) < \tau(s)$. Direct inspection shows that the right-hand side of (6.13) is zero when $s = \infty$ and is 1 when $s = 1$. In particular, it is bounded for $s \in [1, \infty)$. Moreover, Itô’s formula shows that when we plug $1/Y_t$ in place of s in the right-hand side of (6.13), we get a local martingale. Since $Y_0 = 1/s$ and $Y_t \in (0, 1)$ for $t < \tau(1)$, it therefore suffices to prove that $\lim_{t \nearrow \tau(1)} Y_t \in \{0, 1\}$ a.s. The only way this could fail is if $X = s$ occurs with positive probability. However, scale invariance shows that $\mathbf{P}[X = s'] \geq \mathbf{P}[X = s]$ when $s' \in (1, s)$. Hence $\mathbf{P}[X = s] = 0$ for all $s \neq 1$. \square

Proof of Theorem 6.4. The case $\kappa \in [0, 4]$ is covered by Theorem 6.1 and Theorem 5.1.

Suppose that $\kappa \in (4, 8)$ and $z \in \overline{\mathbb{H}}$. Lemma 6.5 shows that $z \in \bigcup_{t>0} K_t$ a.s. If $z \in \mathbb{H}$, then Lemma 6.3 and (6.2) show a.s. $z \notin \gamma[0, \infty)$. Lemma 6.6 shows that $1 \notin \gamma[0, \infty)$ a.s., and the same follows for every $z \in \mathbb{R} \setminus \{0\}$ by scale and reflection invariance. This covers the range $\kappa \in (4, 8)$.

Now suppose that z_0 is swallowed, and let t_0 be the swallowing time. If $\delta > 0$ is smaller than the distance from z_0 to $\gamma[0, t_0]$, then all the points within distance δ from z_0 are swallowed with z_0 . Therefore, if the probability that some $z_0 \in \overline{\mathbb{H}} \setminus \{0\}$ is swallowed is positive, then by (6.2) there is some z_0 such that $\mathbf{P}[Z(z_0) < \infty] > 0$. By Lemma 6.3, this implies that for $\kappa \geq 8$ a.s. there is no $z \in \overline{\mathbb{H}} \setminus \{0\}$ which is swallowed. \square

Exercise 6.7. Prove that $\mathbf{P}[\tau(z) < \infty] = 0$ when $\kappa \in (0, 4]$ and $z \in \mathbb{H}$.

7. Transience

THEOREM 7.1 (Transience). *For all $\kappa \neq 8$ the SLE_κ trace $\gamma(t)$ is transient a.s. That is,*

$$\mathbf{P}\left[\liminf_{t \rightarrow \infty} |\gamma(t)| = \infty\right] = 1.$$

LEMMA 7.2. *Suppose that $\kappa \leq 4$, and let $x \in \mathbb{R} \setminus \{0\}$. Then a.s. $x \notin \overline{\gamma[0, \infty)}$.*

Proof. By symmetry, we may assume that $x > 0$. Moreover, scaling shows that it is enough to prove the lemma in the case $x = 1$. So assume that $x = 1$. Let $s \in (0, 1/4)$ and suppose that there is some first time $t_0 > 0$ such that $|\gamma(t_0) - 1| = s$. This implies that $|g_{t_0}(1/2) - \xi(t_0)| = O(s)$. Indeed, $|g_{t_0}(1/2) - \xi(t_0)|$ is the limit as $y \rightarrow \infty$ of y times the harmonic measure in $\mathbb{H} \setminus \gamma[0, t_0]$ from iy of the union of $[0, 1/2]$ and the “right-hand side” of $\gamma[0, t_0]$. By the maximum principle, this harmonic measure is bounded above by the harmonic measure in \mathbb{H} of the interval $[1, \gamma(t_0)]$, which is $O(s/y)$.

In the proof of Lemma 6.2, we have determined the probability that $Y_x(t) = g_t(x) - \xi(t)$ hits b before a , where $x \in [a, b]$ and $0 < a < b$. For $\kappa < 4$, that probability goes to 1 as $a \searrow 0$, uniformly in b . Consequently, a.s. there is a random $\varepsilon > 0$ such that for all $b > x$, $Y_x(t)$ hits b before ε . Since $\sup_{t \in [0, T]} Y_x(t) < \infty$ a.s. for all finite T , it follows that $\inf_{t \geq 0} Y_x(t) > 0$ a.s. Using this with $x = 1/2$, we conclude from the previous paragraph that a.s. $1 \notin \overline{\gamma[0, \infty)}$, at least when $\kappa < 4$.

Assume now that $\kappa = 4$. Although the above proof does not apply (since $Y_x(t)$ is recurrent), there are various different ways to handle this case. Let $0 < y < x < \infty$. Let $\bar{\gamma}(t)$ denote the complex conjugate of $\gamma(t)$, and set $D_t := \mathbb{C} \setminus (\gamma[0, t] \cup \bar{\gamma}[0, t] \cup (-\infty, y])$. Using Schwarz reflection, g_t may be conformally extended to D_t . Note that $g_t(D_t) = \mathbb{C} \setminus (-\infty, g_t(y)]$. By the Koebe 1/4 Theorem applied to g_t^{-1} , the distance from x to ∂D_t is at least $\frac{1}{4}(g_t(x) - g_t(y))/g_t'(x)$. Consequently, it suffices to show that a.s.

$$\sup_{t \geq 0} g_t'(x)/(g_t(x) - g_t(y)) < \infty.$$

Set $Q(t) = Q_{x,y}(t) := \log g_t'(x) - \log(g_t(x) - g_t(y))$. Then the above reduces to verifying that $\sup_t Q(t) < \infty$. The time derivative of Q is

$$\partial_t Q(t) = -2 Y_x(t)^{-2} + 2 Y_x(t)^{-1} Y_y(t)^{-1}.$$

In particular, $Q(t)$ is increasing in t . Define

$$G(s) := \log s \log(1 + s) - \frac{1}{2} \log^2(1 + s) + \text{Li}_2(-s),$$

where $\text{Li}_2(z) = \int_z^0 s^{-1} \log(1 - s) ds$. Note that G satisfies $s(1 + s)^2 G''(s) + s(1 + s) G'(s) = 1$. A straightforward application of Itô's formula shows that $Q(t) - G_t$ is a local martingale, where

$$G_t := G\left(\frac{g_t(x) - g_t(y)}{g_t(y) - \xi(t)}\right).$$

Consequently, there is an increasing sequence of stopping times t_n tending to ∞ a.s. such that $\mathbf{E}[Q(t_n)] = \mathbf{E}[G_{t_n}] + Q(0) - G_0$. It is immediate to verify that $\sup\{G(x) : x > 0\} < \infty$. Hence, $\limsup_{n \rightarrow \infty} \mathbf{E}[Q(t_n)] < \infty$. Because Q is increasing in t , it follows that $\mathbf{E}[\sup_t Q(t)] < \infty$ and hence $\sup_t Q(t) < \infty$ a.s., which completes the proof. \square

LEMMA 7.3. *Suppose that $\kappa > 4$, $\kappa \neq 8$, and let $t > 0$. Then a.s. there is some $\varepsilon > 0$ such that $K_t \supset \{z \in \mathbb{H} : |z| < \varepsilon\}$.*

Proof. Consider first the case $\kappa \in (4, 8)$. We know that $\gamma(t)$ is a continuous path, by Theorem 5.1. Note that by scale and reflection invariance, Lemma 6.6 shows that for $a < b < 0$ there is a positive probability that the interval $[a, b]$ is swallowed all at once; i.e., $\mathbf{P}[[a, b] \cap \gamma[0, \infty) = \emptyset] > 0$, but a.s. there is some t such that $[a, b] \subset K_t$. When this happens, we have $[a, b] \cap \overline{H}_t = \emptyset$.

Let S be the set of limit points of $g_{\tau(1)}(z)$ as $z \rightarrow 0$ in $H_{\tau(1)}$. (If $0 \notin \overline{H_{\tau(1)}}$, set $S = \emptyset$.) Then $S \subset (-\infty, \xi(\tau(1)))$. Note that $(g_{\tau(1)}(\gamma(t + \tau(1))) - \xi(\tau(1)) : t \geq 0)$ has the same distribution as $(\gamma(t) : t \geq 0)$. Consequently, by the above paragraph, there is positive probability that $g_{\tau(1)}(\gamma[\tau(1), \infty))$ is disjoint from S , but there is some $t > \tau(1)$ such that $g_{\tau(1)}(K_t \setminus K_{\tau(1)}) \supset S$. This implies that with positive probability there is some $t > 0$ such that

$$\delta := \mathbf{P}[0 \notin \overline{H}_t] > 0.$$

By scale invariance, δ does not depend on t . Since these events are monotone in t , it follows that

$$\mathbf{P}[\forall t > 0, 0 \notin \overline{H}_t] = \delta.$$

By Blumenthal's 0-1 law it therefore follows that $\delta = 1$. This completes the proof in the case $\kappa \in (4, 8)$.

Now suppose that $\kappa > 8$. The above argument implies that it is sufficient to show that there is some $t > 0$ such that with positive probability $0 \notin \overline{H}_t$. Striving for a contradiction, we assume that for all $t > 0$ a.s. $0 \in \overline{H}_t$.

Let $z_0 \in \mathbb{H}$ be arbitrary. We know from Theorem 6.4 and Lemma 6.5 that a.s. $z_0 \in \gamma[0, \infty)$. Set $t_0 := \tau(z_0)$. Then $z_0 = \gamma(t_0)$. Let $t > 0$. Conditioned on \mathcal{F}_{t_0} , the set $g_{t_0}(K_{t+t_0} \setminus K_{t_0})$ has the same distribution as $K_t + \xi(t_0)$. It therefore follows by our above assumption that a.s. $\xi(t_0) \in \overline{g_{t_0}(H_{t+t_0})}$. Mapping by $g_{t_0}^{-1}$, this implies that a.s. $z_0 \in \overline{H_{t+t_0}}$; that is, $z_0 \in \partial K_{t+t_0}$. We conclude that for

every $z \in \mathbb{H}$ and every $t > 0$ a.s. $z \in \partial K_{t+\tau(z)}$. Therefore, a.s.

$$\text{area}(\partial K_t) = \int_{z \in \mathbb{H}} 1_{z \in \partial K_t} dx dy \geq \int_{z \in \mathbb{H}} 1_{t > \tau(z)} dx dy,$$

and Fubini implies that ∂K_t has positive measure with positive probability for large t . However, this contradicts Corollary 5.3, completing the proof of the lemma. \square

Proof of Theorem 7.1. First suppose that $\kappa > 4$. Then we know from Lemma 7.3 that there is a random but positive $r > 0$ such that $K_1 \supset \{z \in \mathbb{H} : |z| < r\}$. Let $\varepsilon > 0$, and let $r_0 = r_0(\kappa, \varepsilon) > 0$ be a constant such that $\mathbf{P}[r < r_0] < \varepsilon$. Then the scaling property of SLE shows that for all $t > 0$

$$\mathbf{P}\left[K_t \supset \{z \in \mathbb{H} : |z| < r_0\sqrt{t}\}\right] \geq 1 - \varepsilon.$$

This implies that

$$\mathbf{P}[\exists t' > t : |\gamma(t')| < r_0\sqrt{t}] < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this clearly implies transience.

Consider now the case $\kappa \in [0, 4]$. In this range, γ is a simple path. Then a.s. there are two limit points x_0, x_1 for $g_1(z)$ as $z \rightarrow 0$ in H_1 . Note that $g_1(\gamma[1, \infty))$ has the same distribution as $\gamma[0, \infty)$ translated by $\xi(1)$. Consequently, Lemma 7.2 shows that a.s. x_0 and x_1 are not in the closure of $g_1(\gamma[1, \infty))$. This means that $0 \notin \overline{\gamma[1, \infty)}$ a.s. The proof is then completed as in the case $\kappa > 4$ above. \square

COROLLARY 7.4. *Suppose that $\kappa > 8$, then $\gamma[0, \infty) = \overline{\mathbb{H}}$ a.s.*

Proof. From Lemma 6.3 and (6.2) we know that $\gamma[0, \infty)$ is a.s. dense in $\overline{\mathbb{H}}$. Since γ is a.s. transient, $\gamma[0, \infty)$ is closed in \mathbb{C} , and hence is equal to $\overline{\mathbb{H}}$. \square

Update. Corollary 7.4 and Theorem 7.1 are true also for $\kappa = 8$. The proofs are based on the extension [LSW] to $\kappa = 8$ of Theorem 5.1, and are otherwise the same.

8. Dimensions

In this section we study the size of the SLE trace and the hull boundary. When $\kappa \leq 4$, these two sets are the same, and it is interesting to compare the methods and results. For sets $A \subset \mathbb{C}$, denote by $N(\varepsilon) = N(\varepsilon, A)$ the minimal number of disks of radius ε needed to cover A .

8.1. *The size of the trace.*

THEOREM 8.1. *For every nonempty bounded open set A with $\bar{A} \subset \mathbb{H}$,*

$$\limsup_{\varepsilon \rightarrow 0} \frac{\log \mathbf{E} \left[N(\varepsilon, \gamma[0, \infty) \cap A) \right]}{|\log \varepsilon|} = \begin{cases} 1 + \kappa/8 & \kappa < 8, \\ 2 & \kappa \geq 8. \end{cases}$$

Proof. We may assume $\kappa < 8$ because $\gamma[0, \infty) = \bar{\mathbb{H}}$ if $\kappa > 8$, by Corollary 7.4, and when $\kappa = 8$ we know that for every $z \in \mathbb{H}$ we have $z \in \gamma[0, \infty)$ a.s., by Theorem 6.4 and Lemma 6.5. Fix $\varepsilon > 0$ smaller than the distance from A to \mathbb{R} , and denote those points $z \in \varepsilon\mathbb{Z}^2$ with $\text{dist}(z, A) < \varepsilon$ by $z_j, 1 \leq j \leq n$. Then n is comparable to $1/\varepsilon^2$. Set $N := N(\varepsilon, \gamma[0, \infty) \cap A)$. Obviously $N \leq \#\{z_j : \text{dist}(z_j, \gamma[0, \infty)) < \varepsilon\} \leq 100N$. From (6.2) and (6.5) we get that $\mathbf{E}[N]$ is comparable to

$$\sum_{j=1}^n \mathbf{P}[Z(z_j) > \varepsilon^{-1}].$$

Write $F(t) = \mathbf{P}[Z(i) > t]$. Using (6.5), the proof of the theorem reduces to showing that

$$\limsup_{t \rightarrow \infty} \frac{\log F(t)}{\log t} = \kappa/8 - 1.$$

By (6.8),

$$\lim_{a \nearrow 1 - \kappa/8} \frac{\hat{G}_{a, \kappa}(1)}{1 - \kappa/8 - a} = \frac{\sqrt{\pi} \Gamma(4/\kappa - 1/2)}{2 \Gamma(4/\kappa)}$$

and since $\hat{G}_{a, \kappa}(i) = 1$ from Lemma 6.3 we obtain

$$C^{-1} \frac{1}{1 - \kappa/8 - a} \leq \mathbf{E}[Z(i)^a] \leq C \frac{1}{1 - \kappa/8 - a}$$

for all $a < 1 - \kappa/8$ sufficiently close to $1 - \kappa/8$. Hence, for small $\Delta = \Delta(\kappa) > 0$,

$$C\Delta \geq \int_0^\Delta \mathbf{E}[Z(i)^{1 - \kappa/8 - \delta}] \delta d\delta = \int_0^\Delta \int_0^\infty (1 - \kappa/8 - \delta) t^{-\kappa/8 - \delta} F(t) dt \delta d\delta.$$

From $\int_0^\Delta t^{-\kappa/8 - \delta} \delta d\delta \geq Ct^{-\kappa/8}/(\log t)^2$ for $t \geq e$ (where Δ is treated as a constant), we obtain

$$\int_e^\infty \frac{t^{-\kappa/8} F(t)}{(\log t)^2} dt = O(1).$$

Using $F(t)/F(2t) = O(1)$ from (6.5) this easily implies for large t

$$F(t) \leq Ct^{\kappa/8 - 1} (\log t)^2.$$

There is a sequence $t_n \rightarrow \infty$ with

$$F(t_n) \geq \frac{t_n^{\kappa/8 - 1}}{(\log t_n)^2},$$

for otherwise we would have $\mathbf{E}[Z(i)^{1-\kappa/8}] = O(1) \int_e^\infty \frac{dt}{t(\log t)^2} < \infty$, contradicting Lemma 6.3. \square

An immediate consequence is

COROLLARY 8.2. *For $\kappa < 8$, the Hausdorff dimension of $\gamma[0, \infty)$ is a.s. bounded above by $1 + \kappa/8$.*

8.2. *The size of the hull boundary.* When studying the size of the hull boundary ∂K_t , it suffices to restrict to $t = 1$, by scaling. We will first estimate the size of $\partial H_1 \cap \{y > h\}$ for small $h > 0$ in terms of the convergence exponent of the Whitney decomposition of H_1 . A Whitney decomposition of H_1 is a covering by essentially disjoint closed squares $Q \subset H_1$ with sides parallel to the coordinate axes such that the side length $d(Q)$ is comparable to the distance of Q from the boundary of H_1 . One can always arrange the side lengths to be integer powers of 2 and that

$$(8.1) \quad d(Q) \leq \text{dist}(Q, \partial H_1) \leq 8 d(Q).$$

For example, one can take all squares of the form $Q = 2^n([k, k + 1] \times [j, j + 1])$ such that $Q \subset H_1$, $d(Q) \leq \text{dist}(Q, \partial H_1)$, and Q is not strictly contained in any other Q' satisfying these conditions. One important feature of the Whitney decomposition is that the hyperbolic diameters of the tiles in the decomposition are bounded away from 0 and ∞ . (The hyperbolic metric on H_1 is the metric for which the conformal map $g_1 : H_1 \rightarrow \mathbb{H}$ is an isometry, where \mathbb{H} is taken with the hyperbolic metric with length element $y^{-1} |dz|$.) This follows from the well-known consequence of the Koebe 1/4 theorem saying that the hyperbolic length element in H_1 is comparable to $\text{dist}(z, \partial H_1)^{-1} |dz|$. This description of the hyperbolic metric will be useful below.

Fix a Whitney decomposition of H_1 , and denote by W the collection of those squares Q for which $\text{dist}(Q, K_1) \leq 1$ and $\sup\{\text{Im } z : z \in Q\} \geq h$. For $a > 0$ let

$$S(a) = S_h(a) := \sum_{Q \in W} d(Q)^a.$$

THEOREM 8.3. *Set*

$$\delta(\kappa) := \begin{cases} 1 + \kappa/8 & \kappa \leq 4, \\ 1 + 2/\kappa & \kappa > 4. \end{cases}$$

For $\kappa > 0$, $\kappa \neq 4$, and all sufficiently small $h \in (0, 1)$ we have

$$\mathbf{E}[S(a)] \begin{cases} < \infty & a > \delta(\kappa), \\ = \infty & a \leq \delta(\kappa). \end{cases}$$

For $j, n \in \mathbb{Z}$, let $x_{j,n} := j 2^{-n}$, $y_n := 2^{-n}$ and $z_{j,n} = x_{j,n} + i y_n$. Let I be the interval $g_1(\partial K_1) - \xi(1)$; that is, $f^{-1}(\partial K_1)$.

LEMMA 8.4. *There are constants $C = C(h), c = c(h) > 0$ such that the following holds. Every $Q \in W$ is within distance C in the hyperbolic metric of H_1 from some $\hat{f}_1(z_{j,n})$ with $x_{j,n} \in I$ and $n \geq 0$, and every $\hat{f}_1(z_{j,n})$ such that $x_{j,n} \in I, n \geq 0$ and $\text{Im } \hat{f}_1(z_{j,n}) \geq h$ is within hyperbolic distance C from some $Q \in W$. Moreover, in each of these cases,*

$$c^{-1} d(Q) \leq y_n |\hat{f}'_1(z_{j,n})| \leq c d(Q).$$

Proof. Observe that the diameter of K_1 is bounded from below, because the uniformizing map $g : \mathbb{H} \setminus K \rightarrow \mathbb{H}$ with the hydrodynamic normalization (2.2) tends to the identity as K tends to a point, and the uniformizing map g_1 for K_1 satisfies $g_1(z) = z + 2/z + \dots$ near ∞ , by (2.3).

Let $\hat{z} \in Q \in W$. It follows from the definition of W and the fact that $\text{diam}(K_1)$ is bounded from zero that the harmonic measure of K_1 from \hat{z} in H_1 is bounded from zero. Therefore, the harmonic measure of I from $\hat{f}_1^{-1}(\hat{z})$ in \mathbb{H} is bounded from zero. This implies that there is some $z_{j,n}$ within bounded distance from $\hat{f}_1^{-1}(\hat{z})$ in the hyperbolic metric of \mathbb{H} such that $x_{j,n} \in I$. Then $\hat{f}_1(z_{j,n})$ is within bounded hyperbolic distance from Q . Because $\sup\{\text{Im } z : z \in K_1\} = O(1)$, we also have $\text{Im } \hat{z} = O(1)$, and therefore $\text{Im } \hat{f}_1(z_{j,n}) = O(1)$. Since $\text{Im } \hat{f}_1(z_{j,n}) \geq \text{Im } z_{j,n}$, it follows that $\max\{0, -n\} = O(1)$. Then $z' := x_{j,n} + i 2^{n \wedge 0}$ is equal to $z_{j',n'}$ for some $j' \in \mathbb{Z}, n' \in \mathbb{N}$, and satisfies the requirements in the first statement of the lemma.

From the fact that $\text{diam}(K_1)$ is bounded from zero it also follows that $|I|$, the length of I , is bounded from zero. Suppose that we have $x_{j,n} \in I, j \in \mathbb{Z}, n \in \mathbb{N}$. Then the harmonic measure of I from $z_{j,n}$ is bounded from zero. Consequently, the harmonic measure of K_1 from $\hat{f}_1(z_{j,n})$ in H_1 is bounded from zero. Now suppose also that $\text{Im } \hat{f}_1(z_{j,n}) \geq h$. Since $\sup\{\text{Im } z : z \in K_1\} = O(1)$ and $\text{Im } \hat{f}_1(z_{j,n}) = O(1)$, it follows that the Euclidean distance from $\hat{f}_1(z_{j,n})$ to K_1 is bounded from above. If $\text{dist}(\hat{f}_1(z_{j,n}), K_1) \leq 1$, then the Whitney tile Q containing $\hat{f}_1(z_{j,n})$ is in W . Otherwise, let p be a point closest to $\hat{f}_1(z_{j,n})$ among the points z satisfying $\text{Im } z \geq h$ and $\text{dist}(z, K_1) \leq 1$. Since $h < 1$ and $\text{dist}(\hat{f}_1(z_{j,n}), K_1) = O(1)$, it follows that $|p - \hat{f}_1(z_{j,n})| = O(1)$. Since the points in $[p, \hat{f}_1(z_{j,n})]$ have distance at least h from ∂H_1 , it follows that the hyperbolic distance from p to $\hat{f}_1(z_{j,n})$ is bounded above. For the second statement of the lemma we may then take Q to be the Whitney tile containing p .

The last statement holds because \hat{f}_1 is an isometry from the hyperbolic metric of \mathbb{H} to the hyperbolic metric of H_1 . □

Proof of Theorem 8.3. Set

$$\tilde{S}_h(a) := \sum_{j=-\infty}^{\infty} \sum_{n=0}^{\infty} 1_{x_{j,n} \in I} 1_{\text{Im } \hat{f}_1(z_{j,n}) > h} |\hat{f}'_1(z_{j,n}) y_n|^a.$$

We now show that this quantity is comparable to $S(a)$. Let C be the constant in the lemma. Since each square $Q \in W$ has a bounded diameter in the hyperbolic metric of H_1 , and the hyperbolic distance in \mathbb{H} between any two distinct $z_{j,n}$ is bounded from below, it follows that there are at most a bounded number of $\hat{f}_1(z_{j,n})$ within distance C , in the hyperbolic metric of H_1 , from any single $Q \in W$. Similarly, there are at most a bounded number of $Q \in W$ within hyperbolic distance C from any point in H_1 . Moreover, if z is within hyperbolic distance C from some $Q \in W$, then $\text{Im } z$ is bounded away from zero, because the hyperbolic distance in H_1 between any two points is an upper bound on the distance between them in the hyperbolic metric of \mathbb{H} (by the Schwarz lemma, for instance). Consequently, Lemma 8.4 implies

$$(8.2) \quad c_1^{-1} \tilde{S}_h(a) \leq S_h(a) \leq c_1 \tilde{S}_{h'}(a),$$

for some constant c_1, h' , which depends only on h and a . Now,

$$(8.3) \quad \mathbf{E}[\tilde{S}_h(a)] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=-\infty}^{\infty} \mathbf{E}\left[|\hat{f}'_1(z_{j,n}) y_n|^a 1_{\mathcal{A}(j,n,m)}\right]$$

where $\mathcal{A}(j, n, m)$ denotes the event $\{x_{j,n} \in I, \text{Im } \hat{f}_1(z_{j,n}) \geq h, m \leq |I| < m+1\}$. We split this sum into two pieces, depending on whether $m \leq n$ or $m > n$. Clearly, $0 \in I$. It is easy to see that

$$|I| \leq O\left(1 + \max_{0 \leq t \leq 1} |\xi(t)|\right),$$

for example, by considering the differential equation (2.1) directly. This implies $\mathbf{P}[|I| \geq m] \leq O(1) e^{-C(\kappa)m^2}$, for some $C(\kappa) > 0$. Since

$$|y \hat{f}'_1(x + iy)| \leq O(1) \text{Im } \hat{f}_1(x + iy) \leq O(1)$$

for $0 < y \leq 1$, we have the trivial estimate

$$\begin{aligned} \mathbf{E}\left[\sum_{j=-\infty}^{\infty} |\hat{f}'_1(z_{j,n}) y_n|^a 1_{\mathcal{A}(j,n,m)}\right] &\leq O(1) m 2^n \mathbf{P}[|I| \geq m] \\ &\leq O(1) m 2^n e^{-C(\kappa)m^2} \end{aligned}$$

for all $m \geq 0$. This shows that in (8.3) the terms with $m \geq n$ have finite sum. For $m \leq n$, we use Theorem 3.2 together with (3.17) and obtain

$$\mathbf{E}\left[\sum_{j=-\infty}^{\infty} |\hat{f}'_1(z_{j,n}) y_n|^a 1_{\mathcal{A}(j,n,m)}\right] \leq C(h, \kappa, b) m^{2b+1} 2^{-n(\lambda-2b-1)},$$

where a, λ, b and κ are related by (3.6) with $\nu = 1$ and $b > 0$. For b between $1/2$ and $2/\kappa$ we have $\lambda - 2b - 1 > 0$ and conclude that $\mathbf{E}[\tilde{S}(a)] < \infty$ in this range. By (8.2), $\mathbf{E}[S(a)] < \infty$ for $\kappa < 4, a > 1 + \kappa/8$ and for $\kappa > 4, a > 1 + 2/\kappa$.

Now consider $a = \delta(\kappa)$. Suppose that $2^{-n} < h/4$ and $\hat{f}_1(z_{j,n}) \geq h$. We claim that the harmonic measure in H_1 of K_1 from $\hat{f}_1(z_{j,n})$ is bounded from

zero. Indeed, fix a constant $h_1 > 1$ such that $h_1/2 \geq \sup\{\text{Im } z : z \in K_1\}$. (It is easy to see that $h_1 = 4$ works.) Consider the harmonic function $Y(z) := \text{Im } g_1(z)$ on H_1 . Then $Y(x + iy) - y \rightarrow 0$ uniformly in x , as $y \rightarrow \infty$, by the hydrodynamic normalization (2.2) of g_1 . Since $\mathbb{H} + \frac{i}{2}h_1 \subset H_1$, the maximum principle applied to $Y(z) - \text{Im } z$ implies that $Y(x + ih_1) \geq h_1/2$ for all $x \in \mathbb{R}$. Consequently, for every $z \in H_1$ with $\text{Im } z < h_1$, the harmonic measure in $H_1 \cap \{y < h_1\}$ from z of the line $y = h_1$ is at most $2Y(z)/h_1$. However, the harmonic measure from z of $y = h_1$ in the domain $\mathbb{H} \cap \{y < h_1\}$ is $\text{Im } z/h_1$. Therefore, the harmonic measure from z of K_1 is at least the difference between these two quantities, which is at least $\text{Im } z/h_1 - 2Y(z)/h_1$. Applying this to $z := \hat{f}_1(z_{j,n})$ proves our claim that the harmonic measure of K_1 from $\hat{f}_1(z_{j,n})$ in H_1 is bounded from zero. This implies that the harmonic measure of I from $z_{j,n}$ in \mathbb{H} is bounded from zero, which implies that at bounded hyperbolic distance from $z_{j,n}$ we can find a point $z_{j',n'}$ with $x_{j',n'} \in I$. Consequently, to prove that $\mathbf{E}[\hat{S}_h(\delta(\kappa))] = \infty$ for all sufficiently small $h > 0$, it suffices to prove that for all $n_0 \in \mathbb{N}$ and all sufficiently small $h > 0$

$$\mathbf{E}\left[\sum_{j=-\infty}^{\infty} \sum_{n=n_0}^{\infty} 1_{\text{Im}\hat{f}_1(z_{j,n})>h} |\hat{f}'_1(z_{j,n}) y_n|^{\delta(\kappa)}\right] = \infty.$$

This follows from Lemma 3.3 with $b = 1/2$ if $\kappa < 4$ or $b = 2/\kappa$ if $\kappa > 4$. An appeal to (8.2) gives $\mathbf{E}[S(\delta(\kappa))] = \infty$. Since $\mathbf{E}[S(a)] < \infty$ for $a > \delta(\kappa)$, it also follows that $\mathbf{E}[S(a)] = \infty$ for $a < \delta(\kappa)$, and the proof is complete. \square

COROLLARY 8.5. *For all $\kappa > 0$, $\kappa \neq 4$, and all sufficiently small $h > 0$*

$$\limsup_{\varepsilon \rightarrow 0} |\log \varepsilon|^{1+h} \varepsilon^{\delta(\kappa)} \mathbf{E}\left[N(\varepsilon, \partial K_1 \cap \{y > h\})\right] = \infty.$$

Proof. Denote by $W(n)$ the number of $Q \in W$ with $2^{-n-1} < d(Q) \leq 2^{-n}$. Then (8.1) immediately gives

$$(8.4) \quad W(n) \leq C N(2^{-n}, \partial K_1 \cap \{y > h\}).$$

The corollary now follows from $S(a) = \sum_{n=0}^{\infty} W(n)2^{-an}$ and Theorem 8.3. \square

Finally we state an upper bound for the covering dimension.

THEOREM 8.6. *For all $\kappa, h > 0$ and $a > \delta(\kappa)$, a.s.*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^a N(\varepsilon, \partial K_1 \cap \{y > h\}) = 0.$$

Proof. Suppose first that $\kappa \neq 4$. By Theorem 5.2 we know that \hat{f}_1 is a.s. Hölder continuous. There is no lower bound version of (8.4) for general domains. However, by results of Pommerenke [Pom92] and Makarov [Mak98] it is known that for Hölder domains (i.e., the conformal map from \mathbb{U} into the

domain is Hölder) the asymptotics of N can be recovered from the convergence exponent of the Whitney decomposition. Our proof follows this philosophy. We claim that there is a (random) constant $C = C(\omega) > 0$, depending on h and the Hölder norm of \hat{f}_1 , such that for every small $\varepsilon > 0$ and every point $z_0 \in \partial K_1 \cap \{y \geq 2h\}$

$$(8.5) \quad \exists Q \in W, \quad Q \subset D(z_0, \varepsilon), \quad \varepsilon \geq d(Q) \geq C\varepsilon/|\log \varepsilon|.$$

In fact, this follows from [KR97], but we include a short proof here, for the convenience of the reader.

Let $x_0 \in \mathbb{R}$ be a point such that $\hat{f}_1(x_0) = z_0$. Clearly, there is such an x_0 , because \hat{f}_1 extends continuously to $\overline{\mathbb{H}}$. Let $\beta(s) := \hat{f}_1(x_0 + is)$, $s > 0$, let $\hat{Q}(s)$ be the set of Whitney cells $Q \in W$ which intersect $\beta((0, s))$, and set

$$q(s) := \sum_{Q \in \hat{Q}(s)} d(Q).$$

Since \hat{f}_1 is an isometry between the hyperbolic metric of \mathbb{H} and that of H_1 , it follows that for every $s > 0$ the arc $\beta[s, 2s]$ meets a bounded number of Whitney cells, and each of these has Euclidean diameter comparable to the Euclidean diameter of $\beta[s, 2s]$. Suppose that \hat{f}_1 is Hölder with exponent $\alpha > 0$. Then, $d(Q) \leq O(1) s^\alpha$ for $Q \in \hat{Q}(s)$, and consequently also $q(s) \leq O(1) s^\alpha$. Observe that there is a constant c_0 so that $q(s)/\sup\{q(s') : s' < s\} < c_0$, because there are constant upper bounds on the number of Whitney cells containing a point, and on the ratio $d(Q)/d(Q')$ for neighboring Whitney cells Q and Q' . Let $s_0 := \sup\{s : q(s) < \varepsilon\}$ and $s_1 := \sup\{s : q(s) < c_0^{-1} \varepsilon/2\}$. The choice of c_0 ensures that $q(s_0) - q(s_1) \geq \varepsilon/2$. Clearly $\hat{Q}(s_0) \setminus \hat{Q}(s_1)$ has at most $O(1 + \log(s_0/s_1))$ cells. Therefore, there is some $Q \in \hat{Q}(s_0) \setminus \hat{Q}(s_1)$ with $d(Q) \geq \varepsilon/O(1 + \log(s_0/s_1))$. Since $q(s) \leq O(1) s^\alpha$, it follows that $O(1) s_1 \geq \varepsilon^{1/\alpha}$, and we get $O(1) d(Q) \geq \varepsilon/|\log \varepsilon|$, proving (8.5).

We know that for every set $F \subset \mathbb{C}$ the covering number $N(\varepsilon, F)$ is bounded by a constant times the maximum number of disjoint disks of radius ε centered in F . Therefore, by (8.5), $N(\varepsilon, \partial K_1 \cap \{y > 2h\})$ is bounded by a constant times the number of Whitney cells in W with size between $O(1)\varepsilon$ and $\varepsilon/O(|\log \varepsilon|)$; that is,

$$N(2^{-n}, \partial K_1 \cap \{y > 2h\}) \leq C(\omega) \sum_{j=n}^{n+O(\log n)} W(j),$$

which implies for all $a' > 0$,

$$N(2^{-n}, \partial K_1 \cap \{y > 2h\}) \leq C(\omega) 2^{(n+O(\log n)) a'} S(a').$$

By Theorem 8.3, $S(a') < \infty$ a.s. for $a' > \delta(\kappa)$. Therefore, the case $\kappa \neq 4$ in Theorem 8.6 now follows by taking $a' \in (\delta(\kappa), a)$.

The following argument covers not only the case $\kappa = 4$, but the whole range $\kappa \in (0, 4]$. For such κ , Theorem 8.1 can be used, since $\partial K_1 = \gamma[0, 1]$ for $\kappa \in (0, 4]$. Although Theorem 8.1 discusses the expected number of balls needed to cover $\gamma[0, \infty) \cap A$, where A is an arbitrary bounded set satisfying $\bar{A} \subset \mathbb{H}$, one can refine the result to bound $N(\varepsilon, \partial K_1 \cap \{y > H\})$, as follows. Given a point $z = x + iy$, $y > h$, we need to estimate the probability that z is close to K_1 . We may certainly restrict attention to the case where y is bounded by some constant. There is a constant $C \geq 1$ such that the probability that z is close to K_1 is bounded by the probability that there is some time $t_1 < 1$ such that $\xi(t_1)$ is within distance C from x times the probability that $Z(z)$ is large, conditioned on the existence of such a t_1 . The proof is then completed by applying the estimates in the proof of Theorem 8.1 and the Markov property. The details are left to the reader. \square

The upper bound for the Hausdorff dimension follows immediately:

COROLLARY 8.7. *For $\kappa > 4$, the Hausdorff dimension of ∂K_1 is a.s. at most $1 + 2/\kappa$.*

9. Further discussion and some open problems

We believe Theorem 5.1 to hold also in the case $\kappa = 8$:

CONJECTURE 9.1. *The SLE_8 trace is a continuous path a.s.*

Update. This is proved in [LSW].

CONJECTURE 9.2. *The Hausdorff dimension of the SLE_κ trace is a.s. $1 + \kappa/8$ when $\kappa < 8$.*

Update. A proof of this result is announced in [Bef] for $\kappa \neq 4$.

Corollary 8.2 shows that the Hausdorff dimension cannot be larger than $1 + \kappa/8$.

CONJECTURE 9.3 (Richard Kenyon). *The Hausdorff dimension of ∂K_1 is a.s. $1 + 2/\kappa$ when $\kappa \geq 4$.*

Corollary 8.7 shows that the Hausdorff dimension cannot be larger than $1 + 2/\kappa$.

Problem 9.4. Find an estimate for the modulus of continuity for the trace of SLE_4 . Improve the estimate for the modulus of continuity for SLE_κ for other κ . What better modulus of continuity can be obtained with a different parametrization of the trace?

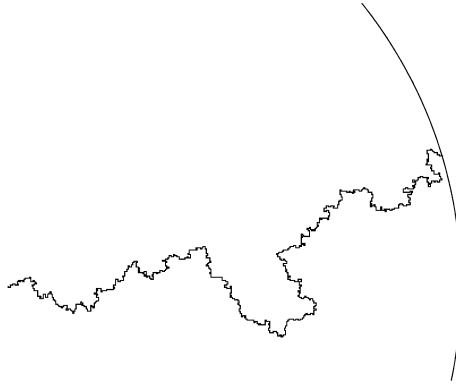


Figure 9.1: A sample of the loop-erased random walk; believed to converge to radial SLE_2 .

Based on the conjectures for the dimensions [Dup00], Duplantier suggested that $\text{SLE}_{16/\kappa}$ describes the boundary of the hull of SLE_κ when $\kappa > 4$. This has further support, some of which is discussed below. However, at this point there isn't even a precise version for this duality statement.

Problem 9.5. Understand the relation between the boundary of the SLE_κ hull and $\text{SLE}_{16/\kappa}$ when $\kappa > 4$.

Update. See [Dub] for recent progress on this problem.

We now list some conjectures about discrete processes tending to SLE_κ for various κ . Let D be a bounded simply connected domain in \mathbb{C} with smooth boundary, and let a and b be two distinct points in ∂D . Let ϕ be a conformal map from \mathbb{H} to D such that $\phi(0) = a$ and $\phi(\infty) = b$. The *SLE trace in D from a to b* is defined as the image under ϕ of the trace of chordal SLE in \mathbb{H} . Similarly, one defines radial SLE in arbitrary simply connected domains strictly contained in \mathbb{C} .

In [Sch00] it was shown that if the loop-erased random walk in simply connected domains in \mathbb{C} has a conformally invariant scaling limit, then this must be radial SLE_2 . See Figure 9.1 for a sample of the loop-erased random walk in \mathbb{U} .

In [Sch00] it was also conjectured that the UST Peano curve, with appropriate boundary conditions (see Figure 9.2), tends to SLE_8 as the mesh goes to zero. This offers additional support to Duplantier's conjectured duality 9.5, Since the outer boundary of the hull of the UST Peano curve consists of two loop-erased random walks, one in the tree and one in the dual tree. See [Sch00] for additional details.

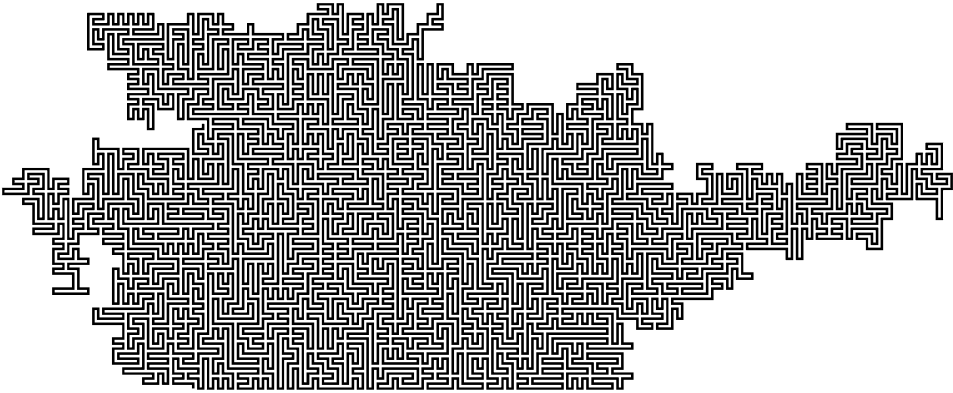


Figure 9.2: A piece of the UST Peano curve; believed to converge to SLE_8 .

Update. The convergence of LERW to SLE_2 and of the UST Peano path to SLE_8 is proved in [LSW].

CONJECTURE 9.6. *Consider a square grid of mesh ε in the unit disk \mathbb{U} . Then the uniform measure on simple grid paths from 0 to $\partial\mathbb{U}$ on this grid converges weakly to the trace of radial SLE_8 as $\varepsilon \searrow 0$, up to reparametrization. The uniform measure on simple paths joining two distinct fixed boundary points converges weakly to the trace of chordal SLE_8 , up to reparametrization.*

We have not bothered to make this conjecture entirely precise, since there is more than one natural choice for the topology on path space. However, this does not seem to be very important at this point.

Note that we do not require that the grid path be space filling (i.e., visit every vertex). Indeed, it is not hard to see that a uniformly selected path will be essentially filling, in the sense that the probability that a nonempty open set inside \mathbb{U} is unvisited goes to zero as $\varepsilon \searrow 0$.

Consider an $n \times n$ grid in the unit square, and identify the vertices of the right and bottom boundary arcs to a single vertex w_0 . Let G_n denote this graph, and note that the dual graph G_n^\dagger is obtained by rotating G_n 180° and shifting appropriately. See Figure 9.3. Let ω be a sample from the critical random-cluster measure on G_n with parameters q and $p = p(q) = \sqrt{q}/(1 + \sqrt{q})$ (which is the conjectured value of $p_c = p_c(q)$) and let ω^\dagger be the set of edges in G_n^\dagger that do not cross edges of ω . Then the law of ω^\dagger is the random-cluster measure on G_n^\dagger with the same parameters as for ω . Let β be the set of points in the unit square that are at equal distance from the component of ω containing w_0 and the component of ω^\dagger containing the wired vertex of G_n^\dagger .

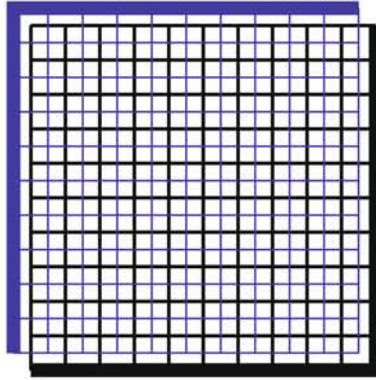


Figure 9.3: A partially wired grid and its dual.

CONJECTURE 9.7. *If $q \in (0, 4)$, then as $n \rightarrow \infty$, the law of β converges weakly to the trace of SLE_κ , up to reparametrization, where $\kappa = \frac{4\pi}{\cos^{-1}(-\sqrt{q}/2)}$.*

The above value of κ is derived by applying the predictions [SD87] of the dimensions of the perimeters of clusters in the random cluster model.

Consider a random-uniform domino tiling of the unit square, such as in the top right of Figure 9.4. Taking the union with another independent domino tiling with the two lower corner squares removed, one gets a path joining the two removed squares; see Figure 9.4. This model was considered by [RHA97]. Richard Kenyon proposed the following

Problem 9.8. Does this double-domino model converge weakly (up to reparametrization) to SLE_4 as the mesh of the grid tends to zero?

Kenyon also produced some calculations for this model that agree with the corresponding calculations for SLE_4 .

Let $a \in \mathbb{R}$, $\varepsilon > 0$, and consider the following random walk S on the grid $\varepsilon \mathbb{Z}^2$. Set $S_0 = 0$, and inductively, we construct S_{n+1} given $\{S_0, \dots, S_n\}$. If $S_n \notin \mathbb{U}$, then stop. Otherwise, let $h_n : \varepsilon \mathbb{Z}^2 \rightarrow [0, 1]$ be the function which is 1 outside of \mathbb{U} , 0 on $\{S_0, \dots, S_n\}$, and discrete-harmonic elsewhere. Now choose S_{n+1} among the neighbors v of S_n such that $h(v) > 0$, with probability proportional to $h(v)^a$. When $a = 1$, this is called *Laplacian random walk*, and is known to be equivalent to loop-erased random walk. When $a = 0$ this walk is believed to be closely related to the boundary of percolation clusters.

The following problem came up in discussions with Richard Kenyon.

Problem 9.9. For what values of a does this process converge weakly to radial SLE? What is the correspondence $\kappa = \kappa(a)$?

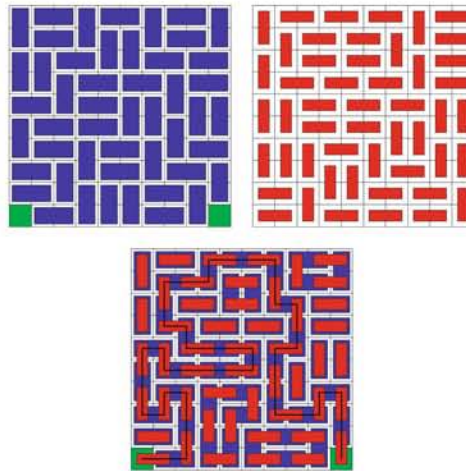


Figure 9.4: The double domino path.

Motivated in part by Duplantier’s duality conjecture 9.5, in [LSW03] it is conjectured that chordal $SLE_{8/3}$ is the scaling limit of the self-avoiding walk in a half-plane. (That is, the limit as $\delta \searrow 0$ of the limit as $n \nearrow \infty$ of the law of a random-uniform n -step simple path from 0 in $\delta\mathbb{Z}^2 \cap \overline{\mathbb{H}}$.) Some strong support to this conjecture is provided there, which in turn can be viewed as further support for the duality in 9.5.

Many of the processes believed to converge to chordal SLE have the following reversibility property. Up to reparametrization, the path from a to b in D ($a, b \in \partial D$) has the same distribution as the path from b to a . It is then reasonable to conjecture:

CONJECTURE 9.10. *The chordal SLE trace is reversible for $\kappa \in [0, 8]$.*

It is not too hard to see that this fails for $\kappa > 8$. In particular, when $\kappa > 8$ the chordal SLE path is more likely to visit 1 before visiting i . Since inversion in the unit circle preserves 1 and i , the probability for visiting i before 1 would have to be $1/2$ if one assumes reversibility. To apply this argument one has to note that a.s. the path visits 1 and i exactly once.

Update. The above conjecture is known to hold when $\kappa = 0, 2, 8/3, 6, 8$. The cases $\kappa = 2, 8$ follow from [LSW] (though some additional work is needed in the case where $\kappa = 2$, since the convergence of the LERW is to radial SLE), the case $\kappa = 6$ follows from Smirnov’s Theorem [Smi01], and the case $\kappa = 8/3$ follows from the characterization in [LSW03] of $SLE_{8/3}$ as the restriction measure with exponent $5/8$.

Some conjectures and simulations concerning the winding numbers of SLE's are described in [WW].

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