ON SPECTRAL ANALYSIS OF STATIONARY TIME SERIES*

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1. The present statistical theory of analysis of stationary time series (e.g., extrapolation) has assumed complete knowledge of the covariance sequence or equivalently of the spectrum of the process. It is, therefore, of great importance to be able to estimate one of these. However, knowledge of the spectrum seems to yield greater immediate insight into the structure of the process. This seems to have first been noted in a fundamental paper by Bartlett.[1] An unpublished paper by Tukey[2] deals with some aspects of the problem of estimating the spectrum.

2. Consider a discrete parameter stochastic process x_t ($t = \ldots, -1, 0, 1, \ldots$) of the following form:

$$x_t = \sum_{\nu=-\infty}^{\infty} a_{t-\nu}\xi_\nu$$

where $\{a_n\}$ is a real sequence and

$$\sum_{-\infty}^{\infty} a_n^2 < \infty.$$

The ξ_ν's are assumed to be independent and identically distributed real-valued random variables with

$$E\xi_\nu = 0,\ E\xi_\nu^2 = \sigma^2,\ E\xi_\nu^4 = \mu_4.$$

The conditions cited above imply that the stochastic process x_t is strictly

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stationary. Such a process is known to have an absolutely continuous spectral distribution function $F(\lambda)$ with a density given by

$$F'(\lambda) = f(\lambda) = \frac{1}{2\pi}\left|\sum_{-\infty}^{\infty} a_\nu e^{i\nu\lambda}\right|^2 \sigma^2.$$

Proofs of the results cited below will require $f(\lambda)$ to satisfy a smoothness condition and the ξ_ν's to have moments up to the eighth order.

As the process is real-valued, the spectral density is symmetric about zero and hence one need only consider a spectral distribution function $F(\lambda)$ as the integral of the spectral density from zero to λ, $0 \leq \lambda \leq \pi$. A reasonable estimate of the spectral distribution function $F(\lambda)$ is

$$F_N^*(\lambda) = \int_0^\lambda I_N(l)dl$$

where

$$I_N(l) = \frac{1}{2\pi N}\left|\sum_1^N x_n e^{inl}\right|^2$$

is the so-called periodogram. Professor Doob suggested that we investigate the problem of giving confidence bands for the estimation of $F(\lambda)$ in terms of $F_N^*(\lambda)$. In the case of a normally distributed process, Theorem 3 yields a solution which is non-parametric with respect to the spectrum. In the general case, Theorem 1 and an appropriate estimate of $e = \frac{\mu_4}{\sigma^4} - 3$ can be used to obtain conservative bands non-parametric with respect to the spectrum. Theorem 4 can be used to test whether two independently drawn time series come from processes having the same spectrum.

3. The theorems are as follows:

THEOREM 1.

$$\lim_{N\to\infty} P\left\{\max_{0\leq\lambda\leq\pi}\left|\sqrt{N}[F_N^*(\lambda) - F(\lambda)]\right| \leq \alpha\right\} = P\left\{\max_{0\leq\lambda\leq\pi}\left|Z(\lambda)\right| \leq \alpha\right\}$$

where $Z(\lambda)$ is a normal process whose distribution is completely determined by the mean value $EZ(\lambda) = 0$ and the covariance

$$EZ(\lambda)Z(\mu) = eF(\lambda)F(\mu) + 2\pi G\,(\min.\,(\lambda, \mu)),$$

where

$$G(\lambda) = \int_0^\lambda f^2(l)dl.$$

COROLLARY. *If $e = 0$*

$$\lim_{N\to\infty} P\left\{\max_{0\leq\lambda\leq\pi}\left|\sqrt{N}[F_N^*(\lambda) - F(\lambda)]\right| \leq \alpha\right\} = P\left\{\max_{0\leq\lambda\leq 2\pi G(\pi)}\left|W(\lambda)\right| \leq \alpha\right\}$$

where $W(\lambda)$ is the Wiener process, i.e.,

$$EW(\lambda) = 0 \qquad EW(\lambda)W(\mu) = \min.\ (\lambda, \mu)$$

and $W(\lambda)$ *is normally distributed.*

Note that $e = 0$ when the ξ_ν are normally distributed.

THEOREM 2. *The statistic*

$$R_N = \frac{1}{2}\sum_{\nu=-kN^\alpha}^{kN^\alpha} \frac{c_\nu^2}{N^2}$$

converges to $2\pi G(\pi)$ *in probability as* $N \to \infty$ *if* $k > 0, 0 < \alpha < 1$. *The* c_ν*'s are the product sums*

$$c_\nu = \sum_{1}^{N-|\nu|} x_n x_{n+|\nu|}.$$

THEOREM 3. *If* $e = 0$

$$\lim_{N\to\infty} P\left\{\max_{0\le\lambda\le\pi} \frac{\left|\sqrt{N}[F_N^*(\lambda) - F(\lambda)]\right|}{R_N^{1/2}} \le \alpha\right\} = P\left\{\max_{0\le\lambda\le 1}|W(\lambda)| \le \alpha\right\}.$$

It is well known that the probability on the right can be explicitly evaluated in a convenient form.

THEOREM 4. *If* $e = 0$ *and if* $F_{N1}^*(\lambda)$, $F_{N2}^*(\lambda)$, R_{N1}, R_{N2} *are the statistics defined above corresponding to two independently drawn samples from the same stochastic process, then*

$$\lim_{N\to\infty} P\left\{\max_{0\le\lambda\le\pi} \frac{\left|\sqrt{N}[F_{N1}^*(\lambda) - F_{N2}^*(\lambda)]\right|}{(R_{N1} + R_{N2})^{1/2}} \le \alpha\right\} = P\left\{\max_{0<\lambda\le\pi}|W(\lambda)| \le \alpha\right\}.$$

4. Although the plausibility of the basic Theorem 1 is apparent by heuristic arguments, the proof requires a long and formidable sequence of lemmas. The practical value of the theory cited above can only be judged after having completed an extensive computational program involving analyses of artificially generated time series. Some preliminary analyses have already been made and tables of the related distributions are being compiled. Considerable investigation must be made of the practical application of the theory. The proofs of the above theorems and considerable elaboration of the theory (e.g., solution of the corresponding problem for the spectral density) as well as aspects of the practical application of the theory will be contained in a forthcoming publication.

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[1] Bartlett, M. S., "Periodogram Analysis and Continuous Spectra," *Biometrika*, **37**, (1950).

[2] Tukey, J. W., "Measuring Noise Color," unpublished manuscript.