Rosenblatt's Contributions to Random Walks on Compact Semigroups

By T. C. Sun

Murray Rosenblatt's interest in random walks on compact semigroups probably came from his work on representations of stationary processes as shifts of functions of independent random variables described in [11], where products of matrices were studied. His papers [12],[5],[13],[14] generalized the work of Lévy [2] on random walks on the circle and the work of Kawada and Itô [4] on random walks on compact groups. In [14], he also completely characterized the structure of the limit measures for the special case of compact semigrougs of $n \times n$ stochastic matrices.

To describe Rosenblatt's work in this area in more detail, we first introduce some definitions and basic facts about compact topological semigroups (see [15] and [3]).

1. Every compact topological semigroup S has a minimal two sided ideal K which is called the *kernel* of S.
2. The kernel K is a compact completely simple semigroup which has a Rees structure $X \times G \times Y$, where
   (a) X is a compact left-zero semigroup, Y is a compact right-zero semigroup and G is a compact group.
   (b) The topology on $X \times G \times Y$ is the product topology.
   (c) The binary operation on $X \times G \times Y$ is given by

   $$(x, g, y)(x', g', y') = (x, g\phi(x', y)g', y'),$$

   where $\phi(x', y) : X \times Y \to G$ is a continuous function which, in our case, can be chosen to be $\phi(x', y) = yx'$. The identity in each group $\{(x, g, y), g \in G\}$ is $(x, (yx)^{-1}, y)$.
3. The convolution of two regular probability measures ν and μ on S is defined by

$$\nu * \mu(A) = \int \mu(s^{-1}A)\nu(ds) = \int \nu(s^{-1}A)\mu(ds).$$

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4. Let ν be a regular probability measure on S and denote its n^{th}-fold convolution with itself by $\nu^{(n*)}$, i.e.,

$$\nu^{(2*)} = \nu * \nu, \quad \nu^{(n*)} = \nu^{((n-1)*)} * \nu.$$

5. Suppose $X_1, X_2, \cdots, X_n$ are independent and identically distributed random variables with values in S and common distribution ν on S. Then the distribution of $W_n = X_1 + X_2 + \cdots + X_n$ is $\nu^{(n*)}$, which has support contained in S and $\{W_n, n = 1, 2, \ldots\}$ is called a *random walk* on S.

In the following we shall always assume that ν is a regular probability measure on S and

$$\overline{\cup_{n=1}^{\infty} \text{supp}(\nu^{(n*)})} = S$$

or S is the compact semigroup generated by the support of ν, because the random walk can never exit the semigroup which is generated by the support of ν.

In [12], Rosenblatt first showed that the sequence $(1/n) \sum_{i=1}^{n} \nu^{(i*)}$ converges weakly to a measure μ on S satisfying,

$$\mu * \nu = \nu * \mu = \mu^{(2)} = \mu.$$

and where the support of μ is the kernel K of S. So, unlike the case of compact groups where the limit measure is the Haar measure on all of S, here in the case of compact semigroups the probability measure is absorbed into the kernel K of S in the limiting process.

In the next step, Heble and Rosenblatt in [5] determined the structure of this limit measure μ on $K = X \times G \times Y$. They showed that if a regular idempotent probability measure μ on S has a completely simple subsemigruoup $K = X \times G \times Y$ as its support, then μ is a product measure and

$$\mu = \alpha \times \chi \times \beta$$

where χ is the Haar measure of the group G and α, β are regular probability measures on X, Y respectively. So the components α and β of the product measure are the extra elements in the semigroup case. J. S. Pym also obtained this results in [10].

Later in [13], Rosenblatt found a necessary and sufficient condition for the sequence $\{\nu^{(n*)}\}$ to converge weakly. He showed that if ν is a regular probability measure on S whose support generates S, then the sequence $\{\nu^{(n*)}\}$ will not converge as $n \to \infty$ if and only if there is a proper closed subgroup G' (proper inclusion) of G such that $YX \subset G'$ and the support of ν is contained in

$$(X \times G' \times Y)^{-1}(X \times gG' \times Y),$$

where $g \notin G'$ and further, $\overline{\cup_{j=1}^{\infty} g^j G'} = G$. In the compact group case, Kawada and Itô's result says the sequence $\nu^{(n*)}$ will not converge if and only if there is a proper closed normal subgroup G' of G such that for some element $g \notin G$ the support of ν is contained in the coset gG'. This is because the support of $\nu^{(n*)}$ jumps from one coset of G' to another coset of G' as n changes. However the sequence $(1/n) \sum_{i=1}^{n} \nu^{(i*)}$ always converges.

In a more concrete setting, when the semigroup S is the set of all $m \times m$ stochastic matrices, one wonders what a completely simple kernel semigroup K would look like? In [14], Rosenblatt gave an answer about the structure of such a K as the following.

1. There is a partition of integers $\{1, 2, \ldots, m\}$ into disjoint classes of integers $T, C_1, C_2, \ldots, C_r$ of numbers $n_0, n_1, n_2, \ldots, n_r$ respectively with $\sum_{i=0}^{r} n_i = m$ and $T = \{1, 2, \ldots, n_0\}$, $C_1 = \{n_0+1, n_0+2, \ldots, n_0+n_1\}, \ldots, C_r = \{n_0+n_1+\cdots+n_{r-1}+1, \ldots, m\}$.

2. There are column vectors

$$\rho^{(\alpha)} = (\rho_j^{(\alpha)}; j \in T), \quad \rho_j^{(\alpha)} \geq 0, \quad \sum_{\alpha=1}^{r} \rho_j^{(\alpha)} = 1,$$

$$u^{(\alpha)} = (u_j^{(\alpha)}; j \in C_\alpha), \quad u_j^{(\alpha)} > 0, \quad \sum_{\alpha=1}^{r} u_j^{(\alpha)} = 1,$$

$$1^{(\alpha)} = (1; j \in C_\alpha)$$

3. H is the permutation group on $\{1, 2, \ldots, r\}$.

Then a completely simple kernel semigroup K in S contains all matrices of the form (for all $h \in H$),

$$\begin{pmatrix} 0 & \rho^{(h1)} u^{(1)'} & \cdots & \rho^{(hr)} u^{(r)'} \\ 0 & \delta_{1-h1} 1^{(1)} u^{(1)'} & \cdots & \delta_{1-hr} 1^{(1)} u^{(1)'} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \delta_{r-h1} 1^{(r)} u^{(1)'} & \cdots & \delta_{r-hr} 1^{(r)} u^{(r)'} \end{pmatrix},$$

where

1. the vectors $\{u^{(\alpha)}; \alpha = 1, 2, \cdots, r\}$ corresponds to Y,

2. the vectors $\{\rho^{(\alpha)}; \alpha = 1, 2, \cdots, r\}$ corresponds to X,

3. H corresponds to the group G,

4. the map $\phi(x, y)$ maps everything into the identity in G.

One may also wonder what the limit measures α and β would look like. In [15], in the Notes of Chapter 5, Rosenblatt gave a simple example of a measure ν on the semigroup S of 2×2 stochastic matrices whose support contains only two points. Each 2×2 stochastic matrix

$$\begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}$$

can be identified with a point (a, b). In this setting both the X and G components of the kernel $K = X \times G \times Y$ of S degenerate into one point and $Y = \{(a, a); a \in [0, 1]\}$. So $K = Y$ and $\mu = \beta$. With proper choice of the two points in the support of ν, the limit measure β on Y can be discrete, singularly continuous or absolutely continuous. In [17], T. Sun generalized this result and, in particular, showed geometrically how this sequence $\nu^{(n*)}$ of measures is attracted to the kernel $K = Y$. More examples in this direction can be found in [18] and [3].

To summarize, Rosenblatt's work completely generalizes Kawada and Ito's result from compact groups to compact semigroups. Of course the result on compact semigroups is much more complicated because of the appearance of the two components X and Y in addition to the group component G. However his work on the structure of the kernel K for the semigroup of $m \times m$ stochastic matrices at least reveals some mysteries about the 2 components X and Y.

Rosenblatt's seminal research in this area inspired a number of related papers. Here we mention just a few.

1. Mukherjea and Tserpes showed in [8] that any idempotent probability measure on a locally compact semigroup must have support which is a completely simple subsemigroup and the measure is a product measure. They generalized Pym, Heble and Rosenblatt's result from compact semigroups to locally compact semigroups.

2. After Rosenblatt's results, it was natural to study recurrent random walks on compact and locally compact semigroups [19], [9]. It is not surprising that any recurrent state must belong to some subgroup and, in fact, in a compact semigroup the set of recurrent states is exactly the kernel of the semigroup. Since a completely simple semigroup is really a union of isomorphic groups, should there be any type of harmonic analysis on a completely simple semigroup?

3. A study of products of random matrices on the semigroup of all $m \times m$ matrices with positive entries [6] and on the semigroup of all $m \times m$ matrices with real entries is contained in [1], [7]. In [16] Rosenblatt posed the question of determining the structure of a completely simple semigroup in these types of matrix semigroups. It seems that this question remains unanswered.

For more recent results on the random walks on semigroups we refer to the excellent reference book [3] by Hognas and Mukherjea.

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