Some Purely Deterministic Processes

MURRAY ROSENBLATT

1. Introduction. The linear prediction problem as it arises in the case of stationary processes has attracted much attention. The problem has been studied most intensively when the prediction error is known to be positive. It is of some interest to consider a few simple situations in which the prediction error is zero, especially as a situation of this sort arises in NEUMANN's theoretical model of storm-generated ocean waves [3]. I shall, unfortunately, not be able to discuss the problem that arises in the context of NEUMANN's model in any detail.

Let x_t , $Ex_t \equiv 0$, be a weakly stationary process, that is,

$$r_{t,\tau} = Ex_t x_\tau = r_{t-\tau}$$

depends only on the time difference $t - \tau$. Our process is assumed to be real-valued. The time parameter t may either range over the real numbers or it may range over the integers. The first case is the continuous parameter case and the second is that of a discrete parameter. Examples of both continuous parameter and discrete parameter processes will be discussed. The discrete parameter processes discussed will be of much greater interest than the continuous parameter processes.

2. Preliminary Remarks. If x_t, $Ex_t \equiv 0$, is a weakly stationary process, it has a random Fourier representation of the form

$$x_t = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ(\lambda), \qquad E\, dZ(\lambda)\, \overline{dZ(\mu)} = \delta_{\lambda\mu}\, dF(\lambda) \tag{1}$$

in the continuous parameter case $-\infty < t < \infty$, and a random Fourier representation

$$x_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dZ(\lambda), \qquad E\, dZ(\lambda)\, \overline{dZ(\mu)} = \delta_{\lambda\mu}\, dF(\lambda) \tag{2}$$

in the discrete parameter case $t = \cdots, -1, 0, 1, \cdots$. Here $\delta_{\lambda\mu}$ is the Kronecker delta

$$\delta_{\lambda\mu} = \begin{cases} 1 & \text{if} \quad \lambda = \mu \\ 0 & \text{if} \quad \lambda \neq \mu. \end{cases}$$

Journal of Mathematics and Mechanics, Vol. 6, No. 6 (1957).

R.A. Davis et al. (eds.), *Selected Works of Murray Rosenblatt*, Selected Works in Probability and Statistics, DOI 10.1007/978-1-4419-8339-8_15,

Note that $dZ(\lambda)$ is the random amplitude of the harmonic $e^{i\lambda}$. The covariances

$$r_t = \int_{-\infty}^{\infty} e^{it\lambda}\, dF(\lambda), \qquad -\infty < t < \infty, \tag{3}$$

in the continuous parameter case. In the discrete parameter case

$$r_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dF(\lambda), \qquad t = \cdots, -1, 0, 1, \cdots. \tag{4}$$

$F(\lambda)$ is a bounded nondecreasing function and is called the spectral distribution function of the process x_t. Note that $dF(\lambda)$ is the variance of the random amplitude $dZ(\lambda)$,

$$dF(\lambda) = E\,|dZ(\lambda)|^2.$$

The case of greatest interest is that in which $F(\lambda)$ is absolutely continuous, that is,

$$F(\lambda) = \int_{-\infty}^{\lambda} f(\mu)\, d\mu$$

in the continuous parameter case, and

$$F(\lambda) = \int_{-\pi}^{\lambda} f(\mu)\, d\mu$$

in the discrete parameter case. The function $f(\lambda) \geqq 0$ is called the spectral density of the process x_t.

In the linear prediction problem, the parameter t is thought of as time. The process x_t has been observed over the time $t \leqq \tau$ and one wishes to predict s time units into the future ($t = \tau$ is thought of as the present) by a predictor that is linear in the observations. Thus, one wishes to predict $x_{\tau+s}$, $s > 0$ by a predictor $px_{\tau+s}$ which is linear in the observations x_t, $t \leqq \tau$. The desired predictor is the one that minimizes the mean square error of prediction

$$E\,|x_{\tau+s} - px_{\tau+s}|^2.$$

Note that this prediction problem is somewhat unrealistic as it assumes knowledge of the entire past of the process. Nonetheless, in certain situations the solution to this problem will lead to reasonable procedures in the more realistic problem in which only a finite amount of the past of the process is known. A process x_t is called "purely deterministic" if the mean square error of prediction is zero in this idealized problem, that is, if one can predict perfectly by linear techniques when the entire past of the process is known.

Some specific discrete parameter purely deterministic processes will be discussed. A finite amount of the past of the process x_t, $\tau - m \leqq t \leqq \tau$, is assumed known. The prediction error in these cases is positive since only a finite amount of the past is known. The rate at which the prediction error decreases to zero as $m \to \infty$ is obtained. The discussion of these discrete parameter processes is based on the helpful comments of G. Szegö and some computations he made.

3. Continuous Parameter Processes. Let x_t, $-\infty < t < \infty$, be a continuous parameter process. Assume that its spectral distribution function $F(\lambda)$ satisfies the condition

$$\int_{-\infty}^{\infty} e^{t|\lambda|}\, dF(\lambda) < \infty \tag{5}$$

for some $t > 0$. It is then clear that the process x_t is purely deterministic (see J. L. Doob [1] p. 584). Let the k^{th} derivative of x_t, if it exists, be denoted by $x_t^{(k)}$. The derivative $x_t^{(1)}$ is the limit in the mean square of $(x_{t+h} - x_t)/h$ as $h \to 0$, that is, the random variable $\hat{x}$ such that

$$E\left|\hat{x} - \frac{x_{t+h} - x_t}{h}\right|^2 \to 0$$

as $h \to 0$. Condition (5) implies that

$$x = x_t^{(1)}$$

exists. The derivative $x_t^{(2)}$ is defined similarly in terms of $x_t^{(1)}$. By proceeding recursively in this manner one can define $x_t^{(k)}$ in terms of $x_t^{(k-1)}$. Condition (5) implies that all the derivatives $x_t^{(k)}$ exist and are given by

$$x_t^{(k)} = \int_{-\infty}^{\infty} (i\lambda)^k e^{it\lambda}\, dZ(\lambda).$$

Let

$$\int_{-\infty}^{\infty} e^{T|\lambda|}\, dF(\lambda) < \infty, \qquad T > 0. \tag{6}$$

This implies that

$$E\left|x_t - \sum_{k=0}^{\infty} \frac{x_\tau^{(k)}(t-\tau)^k}{k!}\right|^2 = 0$$

if $|t - \tau| < \frac{1}{2}T$, or that

$$x_t = \sum_{k=0}^{\infty} \frac{x_\tau^{(k)}(t-\tau)^k}{k!} \tag{7}$$

if $|t - \tau| < \frac{1}{2}T$. In particular, if $x_0^{(k)}$, $k = 0, 1, 2, \cdots$, is known,

$$x_t = \sum_{k=0}^{\infty} \frac{x_0^{(k)} t^k}{k!}$$

if $|t| < \frac{1}{2}T$. Given that we now know x_t, $|t| < \frac{1}{2}T$, by applying (7) to t values in the range $|t| < \frac{1}{2}T$ we see that x_t is determined in the range $|t| < T$. By applying formula (7) repeatedly x_t is determined for all t. Thus, *knowledge of* $x_0^{(k)}$, $k = 0, 1, 2, \cdots$, *when condition* (5) *is satisfied, determines the past and future of* x_t *completely*. This means that knowledge of the process x_t over any small interval $|t| < \epsilon$, $\epsilon > 0$, determines the full history of the process. This is unrealistic as in practice there is always some noise perturbing the process of interest so that condition (5) is not satisfied. Nonetheless, the study of this idealized situation implies that in practical work one might be able to use a truncated power series expansion with approximations of the derivatives inserted in the expansion. The point at which the power series is truncated and the type of approximation of the derivatives to be used (one might use difference quotients) would be dictated by some *a priori* knowledge available about the form of the spectrum of the process and the perturbing noise.

4. Prediction Error for Discrete Parameter Processes when the Spectral Density is Zero on an Interval. Let x_t , $t = \cdots, -1, 0, 1, \cdots$, be a discrete parameter weakly stationary process with mean value $Ex_t \equiv 0$. The process is assumed to have an absolutely continuous spectral distribution function with spectral density $f(\lambda)$. It will be convenient for us to let λ vary from zero to 2π instead of $-\pi$ to π. Further let $f(\lambda)$ be continuous and positive on the closed interval $[\frac{1}{2}\pi - \alpha, \frac{1}{2}\pi + \alpha]$ and zero elsewhere. Thus $f(\lambda)$ is zero on a set of length $2\pi - 2\alpha$. This may disturb the reader a little as such a spectral density corresponds in general to a complex-valued process since it is not symmetric about π, and real-valued processes are of interest. However, this can easily be remedied by adding $\frac{1}{2}\pi$ to λ. This convention is held to because of the convenience in the later discussion. The prediction error for such a spectral density is the same as in the corresponding problem with λ shifted by $\frac{1}{2}\pi$ units.

We would like to consider $f(\lambda)$ as a function on the unit circle $|z| = 1$ in the complex plane. This can be accomplished by setting

$$f(\lambda) = f\left(\frac{1}{i} \log (e^{i\lambda})\right) = W(e^{i\lambda}).$$

$W(e^{i\lambda})$ is now a positive continuous function on the arc $z = e^{i\lambda}$, $\frac{1}{2}\pi - \alpha \leqq \lambda \leqq \frac{1}{2}\pi + \alpha$, of the unit circle. The error in predicting x_0 , given that $x_{-1}, x_{-2}, \cdots, x_{-n}$ are known, is

$$\begin{aligned}\sigma_n^2 &= \min_{c_1, \cdots, c_n} E\left|x_0 - \sum_{j=1}^{n} c_j x_{-j}\right|^2 \\ &= \min_{c_1, \cdots, c_n} \int_{-\pi}^{\pi} \left|e^{in\lambda} - \sum_{j=1}^{n} c_j e^{i(n-j)\lambda}\right|^2 f(\lambda)\, d\lambda \\ &= \min_{c_1, \cdots, c_n} \int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi+\alpha} \left|e^{in\lambda} - \sum_{j=1}^{n} c_j e^{i(n-j)\lambda}\right|^2 W(e^{i\lambda})\, d\lambda.\end{aligned}$$

This means that the prediction error σ_n^2 is equal to

$$\min_{p_n} \int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi+\alpha} |p_n(e^{i\lambda})|^2\, W(e^{i\lambda})\, d\lambda$$

where $p_n(z)$ is a polynomial of degree n in z with the coefficient of z^n one. In our problem the process x_t is purely deterministic since

$$\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda = -\infty .$$

But we want to study the rate at which the prediction error σ_n^2 approaches zero as $n \to \infty$. In evaluating the rate at which $\sigma_n^2 \to 0$ as $n \to \infty$ we will lean heavily on some results of G. SZEGÖ that are given in Chapter 16 of his monograph *Orthogonal Polynomials*. The crucial results will be given in our discussion.

SZEGÖ considers the following problem. $W(z)$ is a positive continuous function defined on a Jordan curve C in the complex plane. C is the boundary of a simply connected region T in the complex z-plane containing $z = \infty$ as an interior point. Let L be the length of the curve C. Now let $p_n(z)$ be a polynomial of degree n in z with the coefficient of z^n one. One wishes to obtain

$$\min \frac{1}{L} \int_C |p_n(z)|^2\, W(z)\, |dz|$$

over the class of such polynomials. In our problem the curve C is the arc of the unit circle $e^{i\lambda}$, $\frac{1}{2}\pi - \alpha \leqq \lambda \leqq \frac{1}{2}\pi + \alpha$, and $W(e^{i\lambda}) = f((1/i) \log (e^{i\lambda}))$ on this curve. T consists of all the points of the plane off the arc.

Define the inner product of two functions $h(z)$, $g(z)$ where z is on C by

$$(h, g) = \frac{1}{L} \int_C h(z)\overline{g(z)} W(z)\, |dz|.$$

If the system $1, z, z^2, \cdots$ is orthogonalized by the Gramm-Schmidt orthogonalization procedure, one obtains an orthonormal set of polynomials $\phi_n(z)$,

$$\frac{1}{L} \int_C \phi_n(z)\overline{\phi_m(z)} W(z)\, |dz| = \delta_{nm}, \qquad n, m = 0, 1, 2, \cdots ,$$

where $\phi_n(z)$ is a polynomial of degree n with the coefficient of z^n real and positive. Let k_n be the coefficient of z^n in $\phi_n(z)$. Then

$$\frac{1}{\sigma_n^2} = \frac{L}{k_n^2} = \frac{2\alpha}{k_n^2}$$

in our case since $L = 2\alpha$.

Let

$$z = \eta(\mu) = a\mu + a_0 + a_1\mu^{-1} + a_2\mu^{-2} + \cdots$$

be the analytic function, regular and one-valued for $|\mu| > 1$, which maps $|\mu| > 1$ conformally onto the region T, preserving the point at infinity and the direction there. The function $\eta(\mu)$ is uniquely determined and a is called the capacity of the curve C. Let $\mu = N(z)$ be the inverse function of $z = \eta(\mu)$. Now let $W(z)$ be the positive continuous function on C. Then $W\{\eta(e^{-i\theta})\}$ is positive and continuous on the unit circle $\mu = e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$. Let

$$D(W\eta; \mu) = \exp\left\{\frac{1}{4\pi} \int_0^{2\pi} \log\,[W\{\eta(e^{-it})\}]\, \frac{1 + \mu e^{-it}}{1 - \mu e^{-it}}\, dt\right\}.$$

On substituting into the analytic function $D(W\eta;\mu)$ the function $\mu = \{N(z)\}^{-1}$ we obtain the analytic function $\Delta(z)$. $\Delta(z)$ is regular in T including $z = \infty$. Moreover $\Delta(z) \neq 0$, $\Delta(\infty)$ is real and positive, and

$$\lim_{z \to z_0} |\Delta(z)|^2 = W(z_0)$$

where z_0 is a point on C and z approaches z_0 from within the region T. SZEGÖ reduces the minimum problem on the curve C to the problem on the full unit circle by the mapping given above (see pages 355–356, 270 of SZEGÖ [4]).

We now cite two interesting results of SZEGÖ (see pages 365–367 [4]). If $\Delta(z)$ is regular in the closed exterior of C (the closure of T)

$$\phi_n(z) \cong \left(\frac{L}{2\pi}\right)^{\frac{1}{2}} \frac{\{N'(z)\}^{\frac{1}{2}}\{N(z)\}^n}{\Delta(z)} \tag{8}$$

when z is in the closure of T. Moreover

$$\frac{1}{k_n^2} \cong \frac{2\pi}{L} \{\Delta(\infty)\}^2 a^{2n+1}.$$

Now let us see what we can derive from these general results in our problem. In our problem

$$z = \eta(\mu) = \mu \sin\frac{\alpha}{2} + i\cos^2\frac{\alpha}{2} + \frac{\cos^2\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\mu + i\sin\frac{\alpha}{2}}$$

maps the region $|\mu| > 1$ onto the complex z-plane cut by the circular arc $z = e^{i\lambda}$, $\frac{1}{2}\pi - \alpha \leqq \lambda \leqq \frac{1}{2}\pi + \alpha$, $0 < \alpha < \pi$. The function

$$\mu = N(z) = \frac{-(i - z) - \sqrt{(i-z)^2 + 4i\sin^2\frac{\alpha}{2}}}{2\sin\frac{\alpha}{2}}.$$

Here

$$\begin{aligned}\Delta(\infty) = D(W\eta; \{N(\infty)\}^{-1}) &= D(W\eta; 0)\\ &= \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log\,[W\{\eta(e^{-it})\}]\,dt\right\}\\ &= \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \pi\log W\left\{\frac{e^{-2it}}{i}\,\frac{\sin\frac{\alpha}{2} + ie^{it}}{\sin\frac{\alpha}{2} - ie^{-it}}\right\}dt\right\}\end{aligned}$$

where

$$W(e^{i\lambda}) = f(\lambda), \qquad \tfrac{1}{2}\pi - \alpha \leqq \lambda \leqq \tfrac{1}{2}\pi + \alpha.$$

Note that the capacity of the arc in our problem is $a = \sin \frac{1}{2}\alpha$. Thus

$$\sigma_n^2 \cong 2\pi\{\Delta(\infty)\}^2 \left(\sin \frac{\alpha}{2}\right)^{2n+1}.$$

By using (8) we can also obtain information about the orthogonal polynomials $\phi_n(z)$ as $n \to \infty$ (or equivalently the predictors). However, we shall not pursue this topic any further. We have already evaluated the asymptotic behavior of the prediction error and we see that it approaches zero as fast as the $(2n + 1)^{\text{th}}$ power of a positive number less than one.

Note that the result obtained can be used to obtain upper and lower bounds for the rate at which the prediction error approaches zero when the spectral density is a continuous function that is zero on a set of positive measure and positive on a set of positive measure. It is clear that in such a case the prediction error approaches zero as fast as the $(2n + 1)^{\text{th}}$ power of a positive number less than one.

5. Prediction Error for Some Discrete Parameter Processes Having Spectra with a High Order Contact with Zero. The spectral density that NEUMANN (3) suggests as the theoretical spectrum of a storm-generated ocean surface is positive except for one point. However, the order of contact of the spectral density with zero at this one point is so high that the corresponding stationary process is purely deterministic. Thus far, no one has been able to discuss the prediction problem with such a spectral density in any detail.

G. SZEGÖ has suggested looking at a discrete parameter weakly stationary process x_t, $Ex_t \equiv 0$, $t = \cdots, -1, 0, 1, \cdots$, that has a spectral density positive everywhere except for one point. At this one point it has a very high contact with zero that is not exactly of the same character as that shown by NEUMANN's spectral density. Nonetheless, it is close enough to give some small insight into the asymptotic behavior of the prediction error.

The computations of this section were carried out by G. SZEGÖ. Throughout this discussion we shall refer to SZEGÖ's book *Orthogonal Polynomials* and a paper of his on POLLACZEK's polynomials.

Let the spectral density of the process x_t be

$$f(\lambda; a) = W(\cos\lambda; a)\,|\sin\lambda| = \frac{e^{(2\lambda-\pi)\varphi(\lambda)}}{\cos\lambda(\pi\varphi(\lambda))}$$

$$f(-\lambda; a) = f(\lambda; a), \qquad 0 \leqq \lambda \leqq \pi,$$

where

$$\varphi(\lambda) = \frac{\operatorname{actn}\lambda}{2}$$

Here a is a fixed parameter. Note that

$$f(\lambda; a) \sim 2\exp\left\{-\frac{\pi}{|\lambda|}\right\}|\sin\lambda|$$

as $\lambda \to 0$ so that $f(\lambda)$ has a very high order contact with zero at $\lambda = 0$. Let $\phi_n(z) = k_n z^n + \cdots + l_n$ be the orthogonal polynomial of degree n with respect to $f(\lambda)$, $z = e^{i\lambda}$, that is

$$\int_{-\pi}^{\pi} \phi_n(e^{i\lambda})\overline{\phi_m(e^{i\lambda})} f(\lambda)\, d\lambda = \delta_{nm} .$$

We know that the prediction error one step ahead, given observations at n successive time points, is

$$\sigma_n^2 = \frac{1}{k_n^2}.$$

The evaluation of the asymptotic behavior of k_n^{-1} as $n \to \infty$ is required.

The asymptotic behavior of k_n is evaluated by studying a closely related family of orthogonal polynomials on the interval $(-1, 1)$. Let

$$\begin{aligned} (9) \qquad g(x, z) = g(\cos\lambda, z) &= \sum_{n=0}^{\infty} P_n(x; a) z^n \\ &= (1 - ze^{i\lambda})^{-\frac{1}{2}+i\varphi(\lambda)}(1 - ze^{-i\lambda})^{-\frac{1}{2}-i\varphi(\lambda)}, \qquad x = \cos\lambda, \end{aligned}$$

be the generating function of the polynomials $P_n(x)$ of degree n. Let

$$p_n(x) = p_n x^n + \cdots = \left(n + \frac{a+1}{2}\right)^{\frac{1}{2}} P_n(x).$$

The polynomials $p_n(x)$ are orthonormal with respect to the weight function

$$W(x; a) = W(\cos\lambda; a) = \frac{e^{(2\lambda-\pi)\varphi(\lambda)}}{\cosh(\pi\varphi(\lambda))}$$

on the interval $-1 \leqq x \leqq 1$. Replace z by z/x in formula (9) and let $x \to \infty$ (see [5] p. 731–732). One then obtains

$$\sum_{n=0}^{\infty} \frac{p_n}{\left(n + \frac{a+1}{2}\right)^{\frac{1}{2}}} z^n = (1 - 2z)^{-\frac{1}{2}} \exp\left\{a \sum_{m=1}^{\infty} \frac{z^m}{m} 2^{m-1}\right\} = (1 - 2z)^{-(a+1)/2}.$$

But then

$$\frac{p_n}{\left(n + \frac{a+1}{2}\right)^{\frac{1}{2}}} = (-1)^n \begin{bmatrix} -\frac{a+1}{2} \\ n \end{bmatrix} 2^n = \frac{2^n}{\Gamma\left(\frac{a+1}{2}\right)} \frac{\Gamma\left(n + \frac{a+1}{2}\right)}{\Gamma(n+1)}$$

$$\sim \frac{2^n}{\Gamma\left(\frac{a+1}{2}\right)} n^{(a-1)/2}$$

so that

$$p_n \sim \frac{2^n}{\left(\frac{a+1}{2}\right)} n^{\frac{1}{2}a}.$$

Also $p_n(1) \sim n^{\frac{1}{4}} \exp\{2\sqrt{an}\}$ (see [5] p. 733). Now let

$$q_n = q_n x^n + \cdots$$

be the orthonormal family of polynomials with weight function $(1 - x^2)W(x; a)$ on the interval $-1 \leqq x \leqq 1$. Then it is known that

$$q_n(x) = cQ_n(x)$$

where

$$Q_n(x) = \frac{1}{1 - x^2}\begin{vmatrix} p_n(x) & p_{n+2}(x) \\ p_n(1) & p_{n+2}(1) \end{vmatrix} = p_{n+2}p_n(1)x^n + \cdots = \frac{p_{n+2}}{p_n} p_n(x) + \cdots$$

(see [4] pp. 28–30). Now

$$\int_{-1}^{1} (1 - x^2)W(x)(Q_n(x))^2\, dx = \int_{-1}^{1} \begin{vmatrix} p_n(x) & p_{n+2}(x) \\ p_n(1) & p_{n+2}(1) \end{vmatrix} W(x) \frac{p_{n+2}}{p_n} p_n(1)p_n(x)\, dx$$

$$= \frac{p_{n+2}}{p_n} p_n(1)p_{n+2}(1)$$

so that

$$c = \sqrt{\frac{p_n}{p_{n+2}}}\, [p_n(1)p_{n+2}(1)]^{-\frac{1}{2}}$$

and

$$q_n = \sqrt{p_n p_{n+2}} \left[\frac{p_n(1)}{p_{n+2}(1)}\right]^{\frac{1}{2}} \sim 2p_n\,.$$

The orthonormal polynomials $p_n(x)$, $q_n(x)$ can be expressed in terms of the orthonormal polynomials $\phi_n(z)$, $x = \frac{1}{2}(z + z^{-1})$, $z = e^{i\lambda}$,

$$p_n(x) = (2\pi)^{-\frac{1}{2}}\left\{1 + \frac{\phi_{2n}(0)}{k_{2n}}\right\}^{-\frac{1}{2}} \{z^{-n}\phi_{2n}(z) + z^n\phi_{2n}(z^{-1})\}$$

$$q_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{1 - \frac{\phi_{2n+2}(0)}{k_{2n+2}}\right\}^{-\frac{1}{2}} \frac{z^{-n-1}\phi_{2n+2}(z) - z^{n+1}\phi_{2n+2}(z^{-1})}{z - z^{-1}}$$

(see [4] p. 287). Thus

$$p_n = (2\pi)^{-\frac{1}{2}}\left(1 + \frac{l_{2n}}{k_{2n}}\right)^{-\frac{1}{2}} 2^n(k_{2n} + l_{2n}) = (2\pi)^{-\frac{1}{2}}2^n k_{2n}^{\frac{1}{2}}(k_{2n} - l_{2n})^{\frac{1}{2}}$$

and

$$q_n = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(1 - \frac{l_{2n+2}}{k_{2n+2}}\right)^{-\frac{1}{2}} 2^{n+1}(k_{2n+2} - l_{2n+2})$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} 2^{n+1}k_{2n+2}^{\frac{1}{2}}(k_{2n+2} - l_{2n+2})^{\frac{1}{2}}.$$

By (11.3.6) on p. 283 [4] we know that $| l_n | < k_n$. But

$$k_{2n}^{\frac{1}{2}}(k_{2n} + l_{2n})^{\frac{1}{2}} \sim \frac{(2\pi)^{\frac{1}{2}}}{\Gamma\left(\frac{a+1}{2}\right)} n^{\frac{1}{2}a} = \gamma_n$$

$$k_{2n}^{\frac{1}{2}}(k_{2n} - l_{2n})^{\frac{1}{2}} \sim \gamma_n .$$

Thus $k_{2n} \sim \gamma_n$. Now $k_{2n-1} \sim \gamma_n$. Therefore

$$k_n \sim \frac{\pi^{\frac{1}{2}} 2^{1-a/2}}{\Gamma\left(\frac{a+1}{2}\right)} n^{\frac{1}{2}a}$$

and

$$\sigma_n^2 \sim n^{-a}.$$

6. **Conclusion.** It is worthwhile contrasting the asymptotic behavior of the prediction error σ_n^2 for the processes dealt with in sections 4 and 5. In section 4 the spectral density is positive and continuous except for an interval of length $2\pi - 2\alpha$, $\pi > \alpha > 0$, where the spectral density is zero. The prediction error σ_n^2 approaches zero geometrically

$$\sigma_n^2 \sim \left(\sin \frac{\alpha}{2}\right)^{2n+1}. \tag{10}$$

In section 5 the spectral density $f(\lambda; a)$ is positive away from $\lambda = 0$ and has a very high order of contact with zero at $\lambda = 0$

$$f(\lambda; a) \sim 2 \exp\left\{-\frac{\pi}{|\lambda|}\right\} |\sin \lambda|.$$

The prediction error σ_n^2 approaches zero as $n \to \infty$ but at a slower rate than (10)

$$\sigma_n^2 \sim n^{-a}.$$

This paper was prepared for the Office of Naval Research under contract Nonr 285(17) at New York University. It may be reproduced in whole or in part for any purpose of the United States Government.

References

[1] J. L. Doob, *Stochastic Processes*, John Wiley, New York, 1953.
[2] U. Grenander, On Toeplitz forms and stationary processes, *Arkiv for Matematik* 1 (1951).
[3] W. J. Pierson, Jr., *Wind Generated Gravity Waves*, Advances in Geophysics, Vol. 2, Academic Press, New York, 1955.
[4] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Pub. Vol. 23, 1939.
[5] G. Szegö, On certain special sets of orthogonal polynomials, *Proc. Amer. Math. Soc.* 1, pp. 731–737 (1950).