Discussion of Bickel and Rosenblatt's work on Global Measures of Deviations for Density Estimates

By Peter Bickel

The paper [1] of Murray Rosenblatt's, which I coauthored, seems to have had a reasonably long shelf life. I believe Murray enjoyed working on this paper—I certainly did! The question posed by Murray was that of obtaining asymptotic approximations to the distribution of the maximum deviation of a kernel density estimate $\hat{f}_n$ of an unknown density f based on a sample $X_1, \dots, X_n$. If a limit distribution existed for

$$\sigma_n \left[\max_{0 \le x \le 1} \left\{ \frac{|\hat{f}_n(x) - f(x)|}{\sqrt{\hat{f}_n(x)}} \right\} - \mu_n \right],$$

where μ_n, $\sigma_n > 0$ are normalizing constants not depending on f, then it is clear how to

(a) construct approximate uniform confidence bands for f of the form $\hat{f}_n \pm c_n \sqrt{\hat{f}_n}$.

(b) construct goodness of fit tests of point hypotheses $H : f = f_0$ using

$$\max_x \frac{|\hat{f}_n(x) - f_0(x)|}{\sqrt{f_0(x)}}$$

as a test statistic.

The heuristic that Murray advanced was the following. If $U_n(x)$ denotes the stochastic process defined by

$$U_n(x) \equiv \sqrt{nh_n} \frac{(\hat{f}_n(x) - f(x)}{\sqrt{\hat{f}(x)}},$$

then $U_n(x_1), \dots, U_n(x_k)$ are asymptotically iid Gaussian and $U_n(x)$ behaves locally like a stationary Gaussian process. Hence one would expect that the statistic based on $\max_{0 \le x \le 1} U_n(x)$ would behave like the corresponding statistic for a stationary Gaussian process over an increasing stretch of the real line; namely the limit would have a Gumbel distribution. Murray and I were able to show this using strong embedding of the empirical process techniques.

We also studied another potential test statistic, the L_2 deviation, defined by

$$\int_0^1 \frac{(\hat{f}(x) - f_0(x))^2}{f_0(x)} dx.$$

We showed that under strong conditions, this statistic had a limiting Gaussian distribution consistent with its χ^2 like nature. Following this up in [5], Rosenblatt showed that this result held under very weak conditions in view of the representation of the statistic as a U statistic with a kernel dependent on the sample size. This was indeed a nice and useful observation.

Subsequently, we used our methods to consider the testing problem and examined the rate at which alternatives tend to the hypothesis with non-negligible power. Not surprisingly, the rate is less than $n^{-\frac{1}{2}}$ which is what one obtains for Kolmogorov-Smirnov and similar tests, although the quadratic deviation is consistently better. However, as results of Ingster and Suslina [4] show, minimax results for different smoothness classes of alternatives are obtained for tests which resemble quadratic density deviation measures. These results clearly depend on what class of alternatives is

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considered, $\{f : \|F - F_0\|_\infty \geq \delta\}$ where F is the cdf and f is the density is suitable for KS tests while $\{f : \ |f(x) - f(y)| \leq M|x - y|^\alpha$ all $x, y, \ \|f - f_0\|_2 \geq \delta\}$ is suitable for the test of [5].

Extensions of this work to the dependent and multivariate cases, and processes other than kernel density ones were developed quickly by Rosenblatt and others [6]. Murray and I only had one further paper in this area, [2], although we had certainly corresponded on [5] and others. The second paper was more probabilistic and derived results on the maxima of Gaussian random fields, which were later obtained more generally by Hogan and Siegmund [4] using a different method. We envisaged these results to be applied to the extension of the results of [1] to the multivariate case.

It is clear from all of the follow-up work that Rosenblatt's initial question was important and his intuition perfect.

References

[1] Bickel, P.J. and Rosenblatt M. (1973). On some global measures of the deviations of density function estimates, *Ann. Statist. 1*, 1071–1095.
[Correction: *Ann. Statist. 3* (1975), no. 6, 1370].

[2] Bickel P.J. and Rosenblatt M. (1973). Two-dimensional random fields, *Multivariate Analysis III*, 3–15, Academic Press NY.

[3] Hogan M.L. and Siegmund D. (1986). Large deviations for the maxima of some random fields, *Advances in Applied Probability 7*, 2–22.

[4] Ingster Y. I. and Suslina, I. A. (2000). Minimax nonparametric hypothesis testing for ellipsoids and Besov bodies. *ESAIM Probab. Statist. 4*, 53–135.

[5] Rosenblatt M. (1975). A quadratic measure of deviation of two-dimensional density estimates and a test of independence, *Ann. Statist. 3*, 1–14.

[6] Rosenblatt, M. (1976). On the maximal deviation of k-dimensional density estimates, *Ann. Probab. 4*, 1009–1015.