NEW RESULTS ABOUT THE H-MEASURE OF A SET

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Abstract The aim of this paper is to generalize our results ([1], [2]) related to the Hasudorff measure of a plane set.

Keywords: Hausdorff measure, $\delta -$ class Lipschitz function, equivalence.

1. Introduction

We denote by R^n the Euclidean n - dimensional space and by $d(E)$ - the diameter of a set $E \subset R^n$.

Definition 1.1 If $r_0 > 0$ is a fixed number, a continuous function $h(r)$, defined on $[0, r_0)$, nondecreasing and such that $\lim_{r \to 0} h(r) = 0$ is called a *measure function.* If $\delta \in R_+$, $E \subset R^n$ is a bounded set, the *Hausdorff h-measure of E* is defined by:

$$H_h(E) = \lim_{\delta \to 0} \inf \sum_i h(\rho_i)$$

inf being considered over all coverings of E with a countable number of spheres of radii $\rho_i \leq \delta$.

Definition 1.2 $f : D(\subset R^n) \to \overline{R}$ is a δ - *class Lipschitz function* if

$$|f(x+\alpha) - f(x)| \leq M\,|\alpha|^{\delta}, x \in D, \alpha \in R^n, x+\alpha \in D, M > 0. \quad (1)$$

Definition 1.3 Let $\varphi_1, \varphi_2 > 0$ be functions defined in a neighborhood of $0 \in R^n$. We say that φ_1 and φ_2 are equivalent and we denote by:

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$\varphi_1 \sim \varphi_2$, for $x \to 0$, if there exist $r > 0$, $Q > 0$, satisfying:

$$\frac{1}{Q}\varphi_1(x) \leq \varphi_2(x) \leq Q\varphi_1(x), (\forall) x \in R^n, |x| < r. \tag{2}$$

An analogous definition can be given for $x \to \infty$. In this case, $\varphi_1 \sim \varphi_2$ means that the previous inequalities have place in all the space.

Definition 1.4 The graph of the function $f : [0,1] \longrightarrow \overline{R}$ is the set:

$$\Gamma = \{(x, f(x)) | x \in [0,1]\}.$$

2. Known results

Theorem 2.1 *If h is a measure function such that*

$$h(t)\tilde{\ } t^p, p \geq 2, \tag{3}$$

and $f : [0,1] \to \overline{R}$ is a δ - class Lipschitz function, with $\delta \in [0,1]$, then: $H_h(\Gamma) < +\infty$. In the same hypothesis about h and f, the result remains true if $p \geq 1$ and $\delta > 1$.

Consider

$$g(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2} \\ -2(x-1), & \frac{1}{2} \leq x < \frac{3}{2} \\ 2(x-2), & \frac{3}{2} \leq x < 2 \end{cases}$$

and:

$$f(x) = \sum_{i=1}^{\infty} \lambda_i^{-\delta} g(\lambda_i x), (\forall) x \in [0,1], \tag{4}$$

where $\{\lambda_i\}_{i \in N^*}$ is a sequence of positive numbers.

Theorem 2.2

a. *If f is the function defined in (4), $\delta \in [0,1]$, $\varepsilon > 1$, $\{\lambda_i\}_{i \in N^*}$ is a sequence of positive numbers, such that $\lambda_{i+1} > \varepsilon \lambda_i$, $(\forall)$ $i \in N^*$ and h is a measure function, such that: $h(t)\tilde{\ } t^p$, $p \geq 2$, then: $H_h(\Gamma) < +\infty$.*

b. *In the same hypothesis about h, f and $\{\lambda_i\}_{i \in N^*}$, the result remains true if $\delta > 1$ and $\varepsilon > 1$.*

3. New results

Theorem 3.1 *If h is a measure function such that*

$$h(t)\tilde{\ }e^t t^p, p \geq 2, \tag{5}$$

and $f : [0,1] \to \overline{R}$ *is a* δ *- class Lipschitz function, with* $\delta \in [0,1]$, *then:* $H_h(\Gamma) < +\infty$. *The result remains true if* $p \geq 1$ *and* $\delta > 1$.

Proof. The first part of the proof follows that of [4].

First, we suppose that the coefficient in the Lipschitz inequality (1) can be taken 1, so that to any x corresponds an interval $(x-k, x+k)$ such that, for any $x+\alpha$ of this interval:

$$|f(x+\alpha) - f(x)| \leq |\alpha|^\delta .$$

Because $[0,1]$ is a compact set, there exists a finite set of overlapping intervals covering (0, 1):

$$(0,k_0), (x_1 - k_1, x_1 + k_1), ..., (x_{n-1} - k_{n-1}, x_{n-1} + k_{n-1}), (1 - k_n, 1).$$

If c_i are arbitrary points, satisfying:

$$\begin{aligned} c_1 &\in (0,x_1), c_i \in (x_{i-1}, x_i), i = 1,2,...,n-1, c_n \in (x_{n-1}, 1) \\ c_i &\in (x_{i-1} - k_{i-1}, x_{i-1} + k_{i-1}) \frown (x_i - k_i, x_i + k_i), i = 1,...,n-1. \end{aligned}$$

we have: $0 < c_1 < x_1 < c_2 < x_2 < ... < x_{n-1} < c_n < 1\cdot$

The oscillation of $f(x)$ in the interval (c_{i-1}, c_i) is less than $2\,(c_i - c_{i-1})^\delta$ and thus the part of the curve corresponding to the interval (c_{i-1}, c_i) can be enclosed in a rectangle of height $2\ (c_i - c_{i-1})^\delta$ and of base $c_i - c_{i-1}$, and consequently in $\left[2\,(c_i - c_{i-1})^{\delta-1}\right] + 1$ squares of side $c_i - c_{i-1}$ or in the number of circles of radius $\frac{c_i - c_{i-1}}{\sqrt{2}}$ circumscribed about each of these squares.

We denoted by $[x]$ the integer part of x.

Given an arbitrary $r \in \left(0, \frac{1}{2}\right)$we can always assume: $c_i - c_{i-1} < r$, $i = 2,3,...,n$.

Denote by C_r the set of all the above circles and consider

$$\sum_{C_r} h(2r) = \sum_{C_r} \left\{ \frac{h(2r)}{e^{2r}(2r)^p} \cdot e^{2r}(2r)^p \right\}. \tag{6}$$

$$r \in \left(0, \frac{1}{2}\right), e^{2r} \in (1,e) \tag{7}$$

We have to estimate $\sum_{C_r}(2r)^p$. The sum of the terms corresponding to the interval (c_{i-1}, c_i) is:

$$S = \left\{\left[2\left(c_i - c_{i-1}\right)^{\delta-1}\right] + 1\right\}\left\{\left(c_i - c_{i-1}\right)\sqrt{2}\right\}^p \Leftrightarrow$$

$$S = 2^{\frac{p}{2}}\left(c_i - c_{i-1}\right)^p\left\{\left[2\left(c_i - c_{i-1}\right)^{\delta-1}\right] + 1\right\}.$$

$$S \leq 2^{\frac{p}{2}}\left(c_i - c_{i-1}\right)^p\left\{2\left(c_i - c_{i-1}\right)^{\delta-1} + 1\right\} \Rightarrow$$

$$S \leq 2^{\frac{p}{2}+1}\left(c_i - c_{i-1}\right)^{p+\delta-1} + 2^{\frac{p}{2}}\left(c_i - c_{i-1}\right)^p \tag{8}$$

$$c_i - c_{i-1} < 1, p \geq 2, \delta \in [0,1] \Rightarrow$$

$$\begin{cases} \left(c_i - c_{i-1}\right)^p < c_i - c_{i-1} \\ p + \delta - 1 \geq 1 \Rightarrow \left(c_i - c_{i-1}\right)^{p+\delta-1} < c_i - c_{i-1} \end{cases} \tag{9}$$

From (8) and (9) it results:

$$S \leq 2^{\frac{p}{2}+1}\left(c_i - c_{i-1}\right) + 2^{\frac{p}{2}}\left(c_i - c_{i-1}\right) = 3 \cdot 2^{\frac{p}{2}}\left(c_i - c_{i-1}\right) \Rightarrow$$

$$\sum_{C_r}(2r)^p \leq \sum_{i=2}^{n} 3 \cdot 2^{\frac{p}{2}}\left(c_i - c_{i-1}\right) = 3 \cdot 2^{\frac{p}{2}}\sum_{i=2}^{n}\left(c_i - c_{i-1}\right) \leq 3 \cdot 2^{\frac{p}{2}} \Leftrightarrow$$

$$\sum_{C_r}(2r)^p \leq 3 \cdot 2^{\frac{p}{2}} \tag{10}$$

Using the definition 2 and the relations (7) and (10), (6) gives:

$$\sum_{C_r} h(2r) = \sum_{C_r}\left\{\frac{h(2r)}{(2r)^p}(2r)^p\right\} < Qe\sum_{C_r}(2r)^p \leq 3 \cdot 2^{\frac{p}{2}} \cdot Q,$$

where $Q > 0$ and $r \in \left(0, \frac{1}{2}\right)$, small enough.

Then $H_h(\Gamma) < +\infty$.

If $M \neq 1$, then $\sum_{C_r} h(2r) \leq 3 \cdot 2^{\frac{p}{2}} \cdot QM \Rightarrow H_h(\Gamma) < +\infty$.

If $p \geq 1$ and $\delta > 1$, then:

$$\begin{aligned} & c_i - c_{i-1} < 1,\ p \geq 1,\ \delta \geq 1 \Rightarrow \\ & \Rightarrow \begin{cases} \left(c_i - c_{i-1}\right)^p < c_i - c_{i-1} \\ p + \delta - 1 \geq 1 \Rightarrow \left(c_i - c_{i-1}\right)^{p+\delta-1} < c_i - c_{i-1} \end{cases} \end{aligned} \tag{11}$$

and the proof is the same as above if we replace the relation (9) with (11). ■

Theorem 3.2 *If h is a measure function such that*

$$h(t)\tilde{}P(t)e^{T(t)}, \tag{12}$$

where P and T are polynomials with positive coefficients:

$$\begin{aligned} P(t) &= a_1 t + a_2 t^2 + ... + a_p t^p,\ p \geq 1 \\ T(t) &= b_0 + b_1 t + ... + a_m t^m, \end{aligned}$$

$f : [0,1] \to \overline{R}$ is a δ - class Lipschitz function, with $\delta \geq 1$, then: $H_h(\Gamma) < +\infty$. The result is also true if $p \geq 2, a_1 = 0$ and $\delta \in [0,1]$.

Proof. The first part follows that of the previous theorem. We have to estimate the sum $\sum_{C_r} h(2r)$, for $r \in \left(0, \frac{1}{2}\right)$.

$$\sum_{C_r} h(2r) = \sum_{C_r} \frac{h(2r)}{P(2r)e^{T(2r)}} \cdot P(2r)e^{T(2r)} < Qe^{\sum_{k=0}^{m} \frac{b_k}{2^k}} \sum_{C_r} P(2r), \tag{13}$$

because we have used the fact that $r \in \left(0, \frac{1}{2}\right)$ and (12).

Now, we estimate $\sum_{C_r} P(2r)$. The sum of the terms corresponding to the interval (c_{i-1}, c_i) is:

$$S = \left\{\left[2\left(c_i - c_{i-1}\right)^{\delta-1}\right] + 1\right\} \sum_{k=1}^{p} a_k \left(\left(c_i - c_{i-1}\right)\sqrt{2}\right)^k,$$

where $[x]$ is the integer part of x.

$$S \leq \left\{2\left(c_i - c_{i-1}\right)^{\delta-1} + 1\right\} \sum_{k=1}^{p} a_k \left(c_i - c_{i-1}\right)^k 2^{k/2} \Leftrightarrow$$

$$S \leq 2^{\frac{p}{2}} \max_{k=\overline{1,p}} a_k \left\{2\left(c_i - c_{i-1}\right)^{\delta-1} + 1\right\} \sum_{k=1}^{p} \left(c_i - c_{i-1}\right)^k \Leftrightarrow$$

$$S \leq 2^{\frac{p}{2}} \max_{k=\overline{1,p}} a_k \sum_{k=1}^{p} \left\{2\left(c_i - c_{i-1}\right)^{k+\delta-1} + \left(c_i - c_{i-1}\right)^k\right\}. \tag{14}$$

If $p, \delta \geq 1$, then $k + \delta - 1 \geq 1$ and $\left(c_i - c_{i-1}\right)^{k+\delta-1} \leq c_i - c_{i-1}$; thus:

$$S \leq 3 \cdot 2^{\frac{p}{2}} \max_{k=\overline{1,p}} a_k \sum_{k=1}^{p} \left(c_i - c_{i-1}\right)$$

and it results, from (13):

$$\sum_{C_r} h(2r) < Qe^{\sum_{k=0}^{m} \frac{b_k}{2^k}} \cdot 3 \cdot 2^{\frac{p}{2}},$$

where $Q > 0$ and $r \in \left(0, \frac{1}{2}\right)$, small enough.

Then $H_h(\Gamma) < +\infty$.

If $p \geq 1, \delta \in [0, 1)$, then

$$1 \leq k - 1 \leq k + \delta - 1 < k$$

and

$$(c_i - c_{i-1})^k < (c_i - c_{i-1})^{k+\delta-1} \leq c_i - c_{i-1}.$$

Thus

$$S \leq 3 \cdot 2^{\frac{p}{2}} \max_{k=\overline{2,p}} a_k \sum_{k=2}^{p} (c_i - c_{i-1})$$

and, analogous, it results that $H_h(\Gamma) < +\infty$. ■

Theorem 3.3

a. *If f is the function defined in* (4), $\delta \in [0, 1], \varepsilon > 1$, $\{\lambda_i\}_{i \in N^*}$ *is a sequence of positive numbers, such that* $\lambda_{i+1} > \varepsilon \lambda_i$, $(\forall)\, i \in N^*$ *and h is a measure function, satisfying the relation* (5), *then:* $H_h(\Gamma) < +\infty$.

b. *In the same hypothesis about f and* $\{\lambda_i\}_{i \in N^*}$, *if* $\delta > 1$ *and* $\varepsilon > 1$, *then* $H_h(\Gamma) < +\infty$.

Proof. The proof is analogous with that of the Theorem 2.2. (See [3].) ■

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