COMPONENTWISE ASYMPTOTIC STABILITY INDUCED BY SYMMETRICAL POLYHEDRAL TIME-DEPENDENT CONSTRAINTS

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Abstract In this paper the concepts of componentwise asymptotic stability with respect to a differentiable vector function $\mathbf{h}(t)$ (approaching 0 as $t \to \infty$) (CWAS$_\mathrm{h}$) and componentwise exponentially asymptotic stability (CWEAS), previously introduced, have been extended to Q-CWAS$_\mathrm{h}$ and Q-CWEAS ($\mathbf{Q}$ being a $q \times n$ real matrix), respectively, in order to cover the more general situation of polyhedral time-dependent flow-invariant sets, defined by $|\mathbf{Qx}| \leq \mathbf{h}(t), \mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}_+$, symmetrical with respect to the equilibrium point of a given continuous-time linear system $\dot{\mathbf{x}} = \mathbf{Ax}, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{R}^n$. It is proved that Q-CWAS$_\mathrm{h}$ is equivalent with the existence of a $q \times q$ matrix $\mathbf{E}$ such that $\mathbf{EQ} = \mathbf{QA}, \bar{\mathbf{E}}\mathbf{h}(t) \leq \dot{\mathbf{h}}(t)$, where the bar operator $(^-)$ transforms only the extra diagonal elements of $\mathbf{E}$ into their corresponding absolute values and does not change its diagonal elements. By specializing vector function $\mathbf{h}(t)$ in an exponentially decaying form, the concept of Q-CWEAS is characterized by the above mentioned matrix equation and an algebraic inequality. For $\mathbf{Q} = \mathbf{I}_n$ these results consistently yield the earlier ones. As in this case, there exists a strong connection between Q-CWAS$_\mathrm{h}$ (Q-CWEAS) and the asymptotic stability, but now this connection is amended by the observability of the pair $(\mathbf{Q}, \mathbf{A})$.

Keywords: Stability analysis, Flow-invariant sets, Time-invariant linear systems, Continuous-time systems, Discrete-time systems

1. Introduction

Consider the nonlinear dynamical system described by the differential equation:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad t \in \mathbb{R}_+, \quad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

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with

$$\mathbf{f}(t,0)=0,\quad t\in\mathbb{R}_+, \tag{2}$$

and the initial condition:

$$\mathbf{x}(t_0)=\mathbf{x}_0,\quad t_0\in\mathbb{R}_+, \tag{3}$$

where $\mathbf{f}$ ensures the existence and the uniqueness of the Cauchy solution on the time interval $[t_0,+\infty)$.

The purpose of this paper is to extend the concepts of componentwise asymptotic stability (CWAS) and of componentwise exponential asymptotic stability (CWEAS), defined and characterized in previous works ([15], [16], [17]), to polyhedral flow-invariant sets, symmetrical with respect to the equilibrium point of system (1) (according to (2)):

$$\mathbf{x}=0. \tag{4}$$

In order to consistently specify the extensions taken into consideration, let us remind first some notations already used in the above mentioned works. Let $\mathbf{v}=:(v_i)$ and $\mathbf{w}=:(w_i)$ be two vectors of the same dimension. We denote by $|\mathbf{v}|$ the vector with the components $|v_i|$ and by $\mathbf{v}\le\mathbf{w}$ or by $\mathbf{v}>\mathbf{w}$ the componentwise inequalities $v_i\le w_i$ or $v_i>w_i$ respectively.

Given a matrix $\mathbf{Q}\in\mathbb{R}^{q\times n}$, with $\mathrm{rank}\mathbf{Q}=\min(n,q)\ge 1$, and a continuous differentiable vector function:

$$\mathbf{h}:\mathbb{R}_+\to\mathbb{R}^q, \tag{5}$$

assume that the following conditions hold:

$$\mathbf{h}(t)>0,\quad t\in\mathbb{R}_+, \tag{6}$$

$$\lim_{t\to\infty}\mathbf{h}(t)=0. \tag{7}$$

The envisaged extensions refer to the following two definitions.

Definition 1 The system (1) is called *Q-componentwise asymptotically stable with respect to* $\mathbf{h}(t)$ (Q-CWAS$_h$), if for each $t_0\in\mathbb{R}_+$ and for each $\mathbf{x}_0$ with

$$|\mathbf{Q}\mathbf{x}_0|\le\mathbf{h}(t_0), \tag{8}$$

the Cauchy solution of (1) satisfies:

$$|\mathbf{Q}\mathbf{x}(t)|\le\mathbf{h}(t)\quad\text{for each } t\ge t_0. \tag{9}$$

Definition 2 The system (1) is called *Q-componentwise exponential asymptotically stable* (Q-CWEAS) if there exist a positive vector $\mathbf{d}>0$, $\mathbf{d}\in\mathbb{R}^q$, and a negative scalar $r<0$ such that system (1) is Q-CWAS$_h$ for:

$$\mathbf{h}(t)=\mathbf{d}e^{rt}. \tag{10}$$

The characterization of Q-CWAS$_{\mathrm{h}}$ and Q-CWEAS will be performed by using the flow-invariance method for which the following basic result is available ([6]).

Theorem 1 *A time-dependent compact set $X(t) \subset \mathbb{R}^n$, $t \in \mathbb{R}_+$, is flow-invariant for system* (1) (*i.e. for each $t_0 \in \mathbb{R}_+$ and for each $\mathbf{x}_0 \in X(t_0)$ the solution of* (1), (2) *satisfies $\mathbf{x}(t) \in X(t)$ for each $t \geq t_0$*) *if and only if*

$$\lim_{\tau \downarrow 0} \tau^{-1} \text{dist}(\mathbf{v} + \tau\, \mathbf{f}(t, \mathbf{v}); X(t + \tau)) = 0 \tag{11}$$

for each $t \in \mathbb{R}_+$ and for each $\mathbf{v} \in X(t)$.

In relation (11) $\text{dist}(\mathbf{v}; X) = \inf_{\mathbf{w} \in X} \text{dist}(\mathbf{v}; \mathbf{w})$ denotes the distance from $\mathbf{v} \in \mathbb{R}^n$ to the set X.

The concept of flow-invariant time-dependent sets has been exploited in several works for studying particular properties of the solutions of various types of differential equations and is based on the pioneering researches in ([5], [2], [3], [4]). A remarkable monograph on this field is due to Pavel ([6]). The use of time-dependent rectangular sets $X(t) \subset \mathbb{R}^n$, $t \in \mathbb{R}_+$, has been proposed by ([15], [16]) for continuous-time linear constant systems, resulting in the definition and analysis of special types of stability, namely the CWAS$_\mathrm{h}$ and CWEAS. An overview on the application of the flow-invariance method in control theory and design is presented in ([17]), including the case of continuous-time nonlinear dynamical systems. Exploiting the inequality-form of the characterizations generated by time-dependent rectangular sets $X(t) \subset \mathbb{R}^n$, $t \in \mathbb{R}_+$, further results on linear interval matrix systems, disturbed systems , uncertain systems, and a class of nonlinear systems have been reported in ([7],[8],[9],[10],[11],[12],[13],[14]).

In order to characterize the concepts of Q-CWAS$_\mathrm{h}$ and Q-CWEAS (Definitions 1 and 2) by using Theorem 1, the following special type of time-dependent polyhedral set will be considered as flow-invariant set:

$$X(t) =: \{\mathbf{v} \in \mathbb{R}^n; |\mathbf{Q}\mathbf{v}| \leq \mathbf{h}(t)\} \subset \mathbb{R}^n, \quad t \in \mathbb{R}_+. \tag{12}$$

Remark 1 Under these circumstances it is obvious that, by taken $q = n$ in (5) and $\mathbf{Q} = \mathbf{I}_n$ (the unit matrix of order n) in (8) and (9) (but originally in (12)), the Definitions 1 and 2 consistently yield the previously introduced concepts of CWAS$_h$ and CWEAS respectively, defined and characterized in ([15],[16],[17]) and ([7],[8],[9],[10],[11],[12],[13],[14]). In the case of an arbitrary $\mathbf{Q} \in \mathbb{R}^{n \times n}$, with rank $\mathbf{Q} = n$, Q-CWAS$_h$ and CWEAS operate in $Im\mathbf{Q} = \mathbb{R}^n$ but in a new vector basis of $\mathbb{R}^n$, different from that one in which system (1) is initially expressed. As mentioned in ([17]), by using the state similarity transformation $\tilde{\mathbf{x}} = \mathbf{Q}\mathbf{x}$ for system (1), the special CWAS$_h$ and CWEAS and their characterizations for the corresponding transformed system are to be approached. ∎

In Section 2 the characterizations in view of Definitions 1 and 2 only for the linear constant continuous-time dynamical systems are performed. The concluding remarks and several comments related to some already existing particular results – see the survey paper ([1]) and the papers cited therein – are included in Section 3.

2. Linear constant continuous-time dynamical systems

System (1) is described by the following differential equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \ \ t \in \mathbb{R}_+, \ \ \mathbf{x} \in \mathbb{R}^n. \tag{13}$$

Remark 2 In terms of the definition of flow-invariant set $X(t)$ (given by (12) with (5) – (7)), which is equivalent to Definition 1, the following state transformation is considered:

$$\mathbf{y} = \mathbf{Q}\mathbf{x}, \ \ \mathbf{x} \in \mathbb{R}^n, \ \mathbf{y} \in \mathbb{R}^q, \tag{14}$$

and, corresponding to system (13), the possible existence of the transformed system has to be taken into account:

$$\dot{\mathbf{y}} = \mathbf{E}\mathbf{y}, \ \ t \in \mathbb{R}_+, \ \ \mathbf{y} \in \mathbb{R}^q. \tag{15}$$

Actually system (15) may represent system (13) in $Im\mathbf{Q}$. For this purpose there exists a system matrix $\mathbf{E} \in \mathbb{R}^{q\times q}$ if and only if the following consistency condition holds:

$$\mathbf{Q}\mathbf{A}(\mathbf{Q}^I\mathbf{Q} - \mathbf{I}_n) = 0, \tag{16}$$

and $\mathbf{E}$ can be calculated with:

$$\mathbf{E} = \mathbf{Q}\mathbf{A}\mathbf{Q}^I + \hat{\mathbf{E}}(\mathbf{Q}\mathbf{Q}^I - \mathbf{I}_q), \tag{17}$$

where $\hat{\mathbf{E}}$ is an arbitrary matrix of order q, $\mathbf{I}_q$ is the unit matrix of the same order, and $\mathbf{Q}^I$ is an inverse of $\mathbf{Q}$, namely (according to the case): it is a right inverse $\mathbf{Q}^R$, $\mathbf{Q}\mathbf{Q}^R = \mathbf{I}_q$ (for $q < n$), the regular inverse $\mathbf{Q}^{-1}$ (for $q = n$) or a left inverse $\mathbf{Q}^L$, $\mathbf{Q}^L\mathbf{Q} = \mathbf{I}_n$ (for $q > n$).

It is a simple matter to see that:

(i) in the case $q < n$ there exists a matrix $\mathbf{E}$ if and only if (16) is satisfied with $\mathbf{Q}^I = \mathbf{Q}^R$, i.e. the following consistency condition is met:

$$\mathbf{Q}\mathbf{A}(\mathbf{Q}^R\mathbf{Q} - \mathbf{I}_n) = 0; \tag{18}$$

(ii) in the other case, $q \geq n$, always exists a matrix $\mathbf{E}$ given by (17) because (16) is satisfied for any $\mathbf{Q}^I = \mathbf{Q}^L$. ■

For the concise writing of the next result, let us remind that for a given square real matrix $\mathbf{M} =: (m_{ij})$ we denote by $\bar{\mathbf{M}} =: (\bar{m}_{ij})$ the matrix with $\bar{m}_{ii} = m_{ii}$ and $\bar{m}_{ij} = |m_{ij}|$, $i \neq j$.

Theorem 2 *System* (13) *is Q-CWAS*$_h$ *if and only if there exists a matrix* $\mathbf{E} \in \mathbb{R}^{q\times q}$ *such that:*

$$\mathbf{EQ} = \mathbf{QA}, \tag{19}$$

$$\bar{\mathbf{E}}\mathbf{h}(t) \leq \dot{\mathbf{h}}(t), \ \ t \in \mathbb{R}_+. \tag{20}$$

Proof. The Q-CWAS$_\mathrm{h}$ of system (13), i.e. the flow-invariance of the set $X(t)$ (given by (12) with (5) – (7)) is equivalent to the following two conditions:

(i) on the one hand (according to Remark 2), the existence of system (15), i.e. of matrix $\mathbf{E}$ given by (19) which is expressed by (17) (either for any $q \geq n$, or for any $q < n$ if and only if (18) holds);

(ii) on the other hand (according to Theorem 1), the inequality:

$$|\mathbf{Q}[\mathbf{v} + \tau(\mathbf{Av} + \mathbf{w}(\tau)]| \leq \mathbf{h}(t+\tau), \tag{21}$$

which must be componentwise fulfilled for each $t \in \mathbb{R}_+$, for each $\mathbf{v}$ with $|\mathbf{Qv}| \leq \mathbf{h}(t)$, for $\tau > 0$, small enough, and for a certain $\mathbf{w} : \mathbb{R}_+ \to \mathbb{R}^n$, with $\mathbf{w}(\tau) \to 0$ as $\tau \downarrow 0$.

Now, combining (19) and (21) it equivalently results:

$$|\mathbf{Qv}+\tau(\mathbf{EQv}+\mathbf{Qw}(\tau)]| \leq \mathbf{h}(t+\tau), \tag{22}$$

which must be componentwise fulfilled for each $t \in \mathbb{R}_+$, for each $\mathbf{v}$ with $|\mathbf{Qv}| \leq \mathbf{h}(t)$, for $\tau > 0$, small enough, and for $\mathbf{w}(\tau) \to 0$ as $\tau \downarrow 0$.

According to the differentiability of $\mathbf{h}(t)$ there exists $\mathbf{z} : \mathbb{R}_+ \to \mathbb{R}^n$, with $\mathbf{z}(\tau) \to 0$ as $\tau \downarrow 0$, such that$\mathbf{h}(t+\tau) - \mathbf{h}(t) = \tau\, \dot{\mathbf{h}}(t) + \tau\, \mathbf{z}(\tau), \ \ t \in \mathbb{R}_+$.

Thus, (22) is equivalent to

$$|\mathbf{Qv}+\tau(\mathbf{EQv} + \mathbf{Qw}(\tau)]| \leq \mathbf{h}(t) + \tau\, \dot{\mathbf{h}}(t) + \tau\, \mathbf{z}(\tau), \tag{23}$$

which must be componentwise fulfilled for each $t \in \mathbb{R}_+$, for each $\mathbf{v}$ with $|\mathbf{Qv}| \leq \mathbf{h}(t)$, for $\tau > 0$, small enough, for $\mathbf{w}(\tau) \to 0$ as $\tau \downarrow 0$, and for $\mathbf{z}(\tau) \to 0$ as $\tau \downarrow 0$.

Using transformation (14), rewritten as $\mathbf{u} = \mathbf{Qv}$, it follows that the vectorial inequality (23) is equivalent to:

$$|\mathbf{u}+\tau(\mathbf{Eu} + \mathbf{Qw}(\tau)]| \leq \mathbf{h}(t) + \tau\, \dot{\mathbf{h}}(t) + \tau \mathbf{z}(\tau) \tag{24}$$

and this must componentwise hold for each $t \in \mathbb{R}_+$, for each $\mathbf{u} \in Im\mathbf{Q}$ with $|\mathbf{u}| \leq \mathbf{h}(t)$, for $\tau > 0$, small enough, for $\mathbf{w}(\tau) \to 0$ as $\tau \downarrow 0$, and for $\mathbf{z}(\tau) \to 0$ as $\tau \downarrow 0$.

It is obvious that (24) must also hold for the maximum value and for the minimum value of each component of $\mathbf{u}+\tau\,\mathbf{Eu}$ for $\tau > 0$, small enough, for $t \in \mathbb{R}_+$ and for each $\mathbf{u} \in Im\mathbf{Q}$ with $|\mathbf{u}| \leq \mathbf{h}(t)$. Since $\mathbf{u}+\tau\,\mathbf{Eu}$ is linear for $\mathbf{u}$ and the set $X(t)$, given by (12) and rewritten as:

$$X(t) =: \{\mathbf{u} \in Im\mathbf{Q};\ |\mathbf{u}| \leq \mathbf{h}(t)\},\ t \in \mathbb{R}_+, \tag{25}$$

is symmetrical with respect to $\mathbf{x} = 0$, the extrema of the $i-$th component of $\mathbf{u} + \tau\,\mathbf{E}\mathbf{u}$ for $\tau > 0$, small enough, can be reached, respectively, for

$$\mathbf{u}_{ex}^i = \pm\ \mathrm{diag}\{\mathrm{sgn}e_{i1}, ..., \mathrm{sgn}e_{ii-1}, 1, \mathrm{sgn}e_{ii+1}, ..., \mathrm{sgn}e_{iq}\}\mathbf{h}(t) \in X(t), \quad i=1,2,...,q, \tag{26}$$

where e_{ij}, $i,j = 1,2,...,q$, are the elements of matrix $\mathbf{E}$. Thus, for $\mathbf{u} = \mathbf{u}_{ex}^i$ the $i-$th inequality from (24), after simplification by $\tau > 0$, is equivalent to:

$$e_{ii}h_i(t) + \sum_{j=1,j\neq i}^{q} |e_{ij}|h_j(t) \leq \dot{h}_i(t) + z_i(\tau) \mp w_i(\tau),\ i=1,2,...,q, \tag{27}$$

for each $t \in \mathbb{R}_+$, for $\tau > 0$, small enough, for $\mathbf{w}(\tau) \to 0$ and for $\mathbf{z}(\tau) \to 0$ as $\tau \downarrow 0$, where $h_i(t)$, $z_i(\tau)$ and $w_i(\tau)$ are the components of $\mathbf{h}(t)$, $\mathbf{z}(\tau)$ and $\mathbf{w}(\tau)$, respectively.

Now, taking into account that $\mathbf{z}(\tau) \to 0$ and $\mathbf{w}(\tau) \to 0$ as $\tau \downarrow 0$, the equivalence between (24) and (15) is proved. ■

To this extent it is obvious that, according to Theorems 2 and 3 in ([16]), the following results can be stated.

Theorem 3 *System* (1) *is Q-CWAS*$_h$ *if and only if there exists a matrix* $\mathbf{E} \in \mathbb{R}^{q\times q}$ *such that* (19) *and the following inequality are met:*

$$e^{\bar{\mathbf{E}}(t-\vartheta)}\mathbf{h}(\vartheta) \leq \mathbf{h}(t),\ \ t \geq \vartheta \geq 0. \tag{28}$$

Theorem 4 *A necessary and sufficient condition for the existence of* $\mathbf{h}(t)$ *such that system* (1) *be Q-CWAS*$_h$ *is the existence of matrix* $\mathbf{E} \in \mathbb{R}^{q\times q}$ *satisfying* (19) *and* $\bar{\mathbf{E}}$ *be Hurwitzian.*

Remark 3 Let $\mathcal{H}$ be the Abelian semigroup of the solutions of (20) in the conditions of Theorem 4. Obviously, system (13) is Q-CWAS$_h$ for each $\mathbf{h} \in \mathcal{H}$. Moreover, for each pair $\mathbf{h}_1$ and $\mathbf{h}_2$ the Q-CWAS$_{h1}$ is equivalent to CWAS$_{h2}$. This allows us to specialize $\mathbf{h}(t)$ and to characterize in a more explicit manner the free response of system (13), namely according to Definition 2. ■

Theorem 5 *System* (1) *is Q-CWEAS if and only if there exists a matrix* $\mathbf{E} \in \mathbb{R}^{q\times q}$ *such that* (19) *and the following inequality are met:*

$$\bar{\mathbf{E}}\mathbf{d} \leq r\mathbf{d}. \tag{29}$$

Proof. It is immediate by replacing (10) into (20). ■

In view of Remark 3 the following statement is obvious.

Theorem 6 *System* (13) *is Q-CWAS*$_h$ *if and only if it is Q-CWEAS.*

In the light of these results and according to Theorem 4 in ([7]) it is quite natural to state next the conditions that ensure the compatibility of inequality (29) regardless of its meaning in connection with CWEAS of system (13). For this purpose let us denote by $\lambda_i(\bar{\mathbf{E}}),\ i = 1, ..., q$, the eigenvalues of $\bar{\mathbf{E}}$.

Theorem 7 a. $\bar{\mathbf{E}}$ *has a real eigenvalue (simple or multiple), denoted by* $\lambda_{\max}(\bar{\mathbf{E}})$, *which fulfils the dominance condition:*

$$Re[\lambda_i(\bar{\mathbf{E}})] \le \lambda_{\max}(\bar{\mathbf{E}}),\ i = 1, ..., q. \tag{30}$$

b. *Inequality* (29) *is compatible if and only if*

$$\lambda_{\max}(\bar{\mathbf{E}}) \le r. \tag{31}$$

Now, according to Theorem 8 in ([17]), the Q-CWEAS of system (13) can be further characterized as follows.

Theorem 8 *System* (13) *is Q-CWEAS if and only if there exists a matrix* $\mathbf{E} \in \mathbb{R}^{q\times q}$ *such that* (19) *and one of the following equivalent conditions are met:*

(i) $$(-1)^k \bar{E}_k > 0,\ k = 1, ..., q, \tag{32}$$

where $\bar{E}_k,\ k = 1, ..., q$, *are the leading principal minors of* $\bar{\mathbf{E}}$*;*

(ii) $$\det \bar{\mathbf{E}} \ne 0,\ \ (-\bar{\mathbf{E}})^{-1} \ge 0, \tag{33}$$

where the inequality is to be taken elementwise;

(iii) $$\bigcup_{i=1}^{q} G_i(\mathbf{E}_d) \subset \{s \in \mathbb{C};\ Re s < 0\}, \tag{34}$$

where $G_i(\mathbf{E}_d) = \{s \in \mathbb{C};\ |s - e_{ii}| \le d_i^{-1} \sum_{j=1, j\ne i}^{q} |e_{ij}|\, d_j\},\ i = 1, ..., q$, *are the Gershgorin's discs associated to matrix* $\mathbf{E}_d = \mathrm{diag}\{d_1^{-1}, ..., d_q^{-1}\} \times \mathbf{E}\,\mathrm{diag}\{d_1, ..., d_q\}$, *i.e. to* $\mathbf{E}$ *and vector* $\mathbf{d}$ *(having the components* $d_1, ..., d_n$*);*

(iv) $$\lambda_{\max}(\bar{\mathbf{E}}) \le r < 0. \tag{35}$$

Unlike the special forms of $\mathrm{CWAS_h}$ and CWEAS, i.e.Q-$\mathrm{CWAS_h}$ and Q-CWEAS for $\mathbf{Q} = \mathbf{I}_n$, which are sufficient conditions for the asymptotic stability of systems (13) because $\mathbf{E} = \mathbf{A}$, in the case of an arbitrary $\mathbf{Q}$ according to (19), the relation between Q-$\mathrm{CWAS_h}$ (Q-CWEAS) and the asymptotic stability depends on the pair ($\mathbf{Q}$, $\mathbf{A}$). Obviously, in the context of Q-$\mathrm{CWAS_h}$ (Q-CWEAS), the dynamics of system (13) is actually observed by means of the transformation (14). As a matter of fact the state observability of system (13), (14), i.e of the pair ($\mathbf{Q}$, $\mathbf{A}$), plays here an adequate part and the following results will clarify its place in the mentioned relation.

Theorem 9 *The observability properties of the pair* $(\mathbf{Q}, \mathbf{A})$ *are as follows:*
– For $q < n$*, system* (13),(14) *(pair* $(\mathbf{Q}, \mathbf{A})$*) is partially state observable; the dimension of the completely observable part of system* (13),(14) *is* q*.*
– For $q \geq n$*, system* (13), (14) *(pair* $(\mathbf{Q}, \mathbf{A})$*) is completely state observable.*

Proof. It relies on the rank evaluation of $(\mathbf{Q}, \mathbf{A})$ - observability matrix performed as follows:

$$\text{rank}\mathbf{O} = \text{rank}\begin{bmatrix} \mathbf{Q} \\ \mathbf{QA} \\ \vdots \\ \mathbf{QA}^{n-1} \end{bmatrix} = \text{rank}\begin{bmatrix} \mathbf{Q} \\ \mathbf{EQ} \\ \vdots \\ \mathbf{E}^{q-1}\mathbf{Q} \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} \mathbf{I}_q \\ \mathbf{E} \\ \vdots \\ \mathbf{E}^{q-1} \end{bmatrix}\mathbf{Q} = \min(n, q). \tag{36}$$

The second equality in (3) becomes obvious by using (19) repeatedly. Further we take into consideration that:
for $q < n$, rank$\mathbf{Q} = \min(n, q) = q$ and rank$\mathbf{O} = q$;
for $q \geq n$, rank$\mathbf{Q} = \min(n, q) = n$, and rank$\mathbf{O} = n$. ∎
In view of this result we can naturally state the next one.

Theorem 10 *System* (13) *is asymptotically stable if it is Q-CWAS*$_h$ *(Q-CWEAS) and one of the following condition holds:*
(i) $q < n$ *and the unobservable part of system* (13), (14), *which evolves in the subspace* $Ker\,\mathbf{O} \subset \mathbb{R}^n$ *of dimension* $n-q$*, is asymptotically stable;*
(ii) $q \geq n$.

It is eminently clear that condition (i) reveals that, in the case $q < n$, Q-CWAS$_h$ (Q-CWEAS) evaluates only that part of dimension q of the dynamics of system (13),(14), which is completely state observable. In this respect the following sufficient condition for the *partial asymptotic stability*, ([18]), of system (13) in the subspace $\mathbb{R}^n \backslash Ker\,\mathbf{O}$ may be stated too.

Theorem 11 *If system* (13) *is Q-CWAS*$_h$ *(Q-CWEAS) and* $q < n$*, then the completely state observable part of system* (13), (14), *which evolves in the subspace* $\mathbb{R}^n \backslash Ker\,\mathbf{O}$ *of dimension* q*, is asymptotically stable.*

3. Concluding remarks

In this paper, the concepts of $\mathrm{CWAS_h}$ and CWEAS, previously introduced by the first author, have been extended to Q-$\mathrm{CWAS_h}$ (Definition 1) and Q-CWEAS (Definition 2), respectively, in order to cover the more general situation of polyhedral time-dependent flow-invariant sets, symmetrical with respect to the equilibrium point of a given continuous-time linear system. The main results are formulated by Theorems 2–4, where the asymptotic behavior to the infinity of such a polyhedral set is expressed by a priori defined vector function $\mathbf{h}(t)$. These novel results are consistent with those mentioned above, in the sense that the characterization of Q-$\mathrm{CWAS_h}$ relies on matrix operator $\bar{\mathbf{E}}$ involved in differential inequality (20), that is accompanied by matrix equation (19) generated by state-space transform (14). Note that the bar operator ($^{-}$) is now applied to the transformed matrix $\mathbf{E}$, resulting from the original system matrix $\mathbf{A}$. Obviously, for the particularization of matrix $\mathbf{Q}$ to the identity matrix in equation (19), Q-$\mathrm{CWAS_h}$ becomes $\mathrm{CWAS_h}$.

By specializing vector function $\mathbf{h}(t)$ in an exponentially decaying form, the concept of Q-CWEAS is characterized in Theorem 5, which, for the studied linear case, is shown to be equivalent with Q-$\mathrm{CWAS_h}$ (Theorem 6). The characterization of Q-CWEAS stated in Theorem 5 is given by matrix equation (19) and algebraic inequality (29), and for the compatibility of the latter (Theorem 7) an earlier result of the authors is used. Other algebraic (and, to some extent, parametric) characterizations of Q-CWEAS are given in Theorem 8.

As in the case of $\mathrm{CWAS_h}$ (CWEAS), there exists a strong connection between Q-$\mathrm{CWAS_h}$ (Q-CWEAS) and the asymptotic stability, which is explored by Theorems 10 – 11 where, for $q < n$, the observability of the pair $(\mathbf{Q}, \mathbf{A})$ (Theorem 9) plays an adequate part.

Finally, it is worth mentioning that the flow invariance of the polyhedral sets with respect to linear dynamic systems has been investigated by many other works as shown in the survey paper ([1]). Unlike our approach where the time-dependence of the polyhedral sets is a priori defined, these works cover only a very special case of time-dependent polyhedral sets, when they are a posteriori proved contractive via an adequate Lyapunov function. This contractiveness is only of exponential type and, therefore, relations to Q-CWEAS are straightforward. However the framework created by the mentioned works did not address concepts related to Q-$\mathrm{CWAS_h}$, just because the time-dependence of the polyhedral sets is a posteriori investigated.

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