OPTIMAL CONTROL OF NON STATIONARY, THREE DIMENSIONAL MICROPOLAR FLOWS

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Abstract We study an optimal control problem associated with a nonstationary, three dimensional flow of a micropolar fluid. We consider a suitable formulation of the control problem which allows us to prove the existence of a solution of this problem and to obtain the necessary conditions of optimality.

Keywords: micropolar fluid, weak solution, strong solution, control problem.

1. Introduction

The flow of micropolar fluids is a problem of physical interest since animal blood, liquid crystals, certain polymeric fluids, etc may be represented by the mathematical model of these fluids. This model was introduced by Eringen in [1]. From the physical point of view, a micropolar fluid is characterized by the following property: fluid points contained in a small volume element, in addition to its usual rigid motion, can rotate about the centroid of the volume element in an average sense, the rotation being described by a skew-symmetric gyration tensor, ω.

In this paper we are concerned with the nonstationary, three dimensional incompressible motion of a micropolar fluid. As in the 3-D case of Navier-Stokes equations (see [2]), we define weak solutions and strong solutions of the system describing the micropolar flow, and it is known that in the class of weak solutions we cannot prove the uniqueness, while for strong solutions we obtain the uniqueness, but there is no an existence result.

The aim of this paper is to study an optimal control problem associated with the evolution system describing the flow of a micropolar fluid.

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This type of problems, for two dimensional flows, has been studied by Stavre in [3], [4], [5]. For the 3-D case, the study is more complicated, since we cannot prove the existence of a strong solution. To overcome this difficulty, we consider a suitable formulation of the control problem (as in [6], [7]), which allows us to prove the existence of a solution of the control problem and to obtain the necessary conditions of optimality.

The paper is organized as follows: in Section 2 we introduce the system of coupled equations which describes the nonstationary, three dimensional flow of an incompressible micropolar fluid and its variational formulation. We discuss about weak and strong solutions of this system and about their existence and uniqueness. By proving a general result, we obtain the desired regularity for the unknowns of the problem. In the next section we formulate the control problem such that to every optimal control we can associate a strong solution. The existence of a solution of the considered control problem is investigated. The last section deals with the first order optimality conditions.

2. Analysis of the motion system

The nonstationary, incompressible, three dimensional motion of a micropolar fluid with non-homogeneous initial data is described by the following coupled system:

$$\begin{cases} \vec{v}' + (\vec{v}\cdot\nabla)\vec{v} - (\mu+\chi)\,\triangle\,\vec{v} + \nabla p - \chi \text{curl}\,\vec{\omega} = \vec{f} \ \text{ in } \Omega_T, \\ j\vec{\omega}' + j(\vec{v}\cdot\nabla)\vec{\omega} - \gamma\triangle\vec{\omega} - (\alpha+\beta)\nabla(\text{div}\,\vec{\omega}) + 2\chi\vec{\omega} - \chi\text{curl}\,\vec{v} = \vec{g} \text{ in } \Omega_T, \\ \text{div}\,\vec{v} = 0 \ \text{ in } \Omega_T, \\ \vec{v} = \vec{0},\ \vec{\omega} = \vec{0} \ \text{ on } \partial\Omega\times(0,T), \\ \vec{v}(x,0) = \vec{v}_0(x),\ \vec{\omega}(x,0) = \vec{\omega}_0(x) \ \text{ in } \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset I\!R^3$ is an open, bounded, connected set, with $\partial\Omega$ of class $\mathcal{C}^2$, T a positive given constant and $\Omega_T = \Omega \times (0,T)$, χ, μ, j, α, β, γ are positive given constants associated with the properties of the material, $\vec{f}$, $\vec{g}$ are the given external fields, $\vec{v}_0$, $\vec{\omega}_0$ are the initial data and $\vec{v}$, $\vec{\omega}$, p are the unknown of the system: the velocity, the microrotation and the pressure of the micropolar fluid, respectively.

We shall need the following spaces (for their properties see, e.g. [2])

$$\begin{cases} V = \{\vec{u} \in (H_0^1(\Omega))^3 \,/\, \text{div}\,\vec{u} = 0\}, \\ H = \{\vec{u} \in (L^2(\Omega))^3 \,/\, \text{div}\,\vec{u} = 0,\ \vec{u}\cdot\vec{n}/_{\partial\Omega} = 0\}, \\ H^{2,1}(\Omega_T) = \{u \in L^2(\Omega_T) / u', \dfrac{\partial u}{\partial x_i}, \dfrac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega_T);\ i,j=1,2\}, \end{cases}$$

The following notation will be used throughout the paper:

$(\cdot,\cdot)$ the scalar product , $|\cdot|$ the norm in $L^2(\Omega)$ or $(L^2(\Omega))^3$,
$((\cdot,\cdot))_0$ the scalar product , $\|\cdot\|_0$ the norm in $H_0^1(\Omega)$ or $(H_0^1(\Omega))^3$,
$\langle\cdot,\cdot\rangle_{X',X}$ the duality pairing between a space X and its dual X',
$b(\vec{u},\vec{v})=(\vec{u}\cdot\nabla)\vec{v},\ \ \forall\,\vec{u},\vec{v}\in(H_0^1(\Omega))^3$.

For $\vec{f}\in L^2(0,T;H)$, $\vec{g}\in L^2(0,T;(L^2(\Omega))^3)$, $\vec{v}_0\in V$, $\vec{\omega}_0\in(H_0^1(\Omega))^3$, the variational formulation of the problem (2.1) is given by

$$\begin{cases} \langle\vec{v}'(t),\vec{z}\rangle_{V',V}+\langle b(\vec{v}(t),\vec{v}(t)),\vec{z}\rangle_{V',V}+(\mu+\chi)((\vec{v}(t),\vec{z}))_0 \\ \qquad -\chi(\mathrm{curl}\,\vec{\omega}(t),\vec{z})=(\vec{f}(t),\vec{z})\ \forall\vec{z}\in V, \\ j\langle\vec{\omega}'(t),\vec{\eta}\rangle_{(H^{-1}(\Omega))^3,(H_0^1(\Omega))^3}+j\langle b(\vec{v}(t),\vec{\omega}(t)),\vec{\eta}\rangle_{(H^{-1}(\Omega))^3,(H_0^1(\Omega))^3} \\ +\gamma((\vec{\omega}(t),\vec{\eta}))_0+(\alpha+\beta)(\mathrm{div}\,\vec{\omega}(t),\mathrm{div}\,\vec{\eta})+2\chi(\vec{\omega}(t),\vec{\eta}) \\ -\chi(\mathrm{curl}\,\vec{v}(t),\vec{\eta})=(\vec{g}(t),\vec{\eta})\ \forall\vec{\eta}\in(H_0^1(\Omega))^3, \\ \vec{v}(0)=\vec{v}_0,\ \vec{\omega}(0)=\vec{\omega}_0. \end{cases} \tag{2.2}$$

The next theorem gives the existence (without the uniqueness) of a weak solution and the uniqueness (without the existence) of a strong solution of the variational formulation (2.2).

Theorem 2.1. a) *There exists at least a pair* $(\vec{v},\,\vec{\omega})$ *with the regularity* $\vec{v}\in L^2(0,T;V)\cap L^\infty(0,T;H)$, $\vec{v}'\in L^{4/3}(0,T;V')$, $\vec{\omega}\in L^2(0,T;(H_0^1(\Omega))^3)\cap L^\infty(0,T;(L^2(\Omega))^3)$, $\vec{\omega}'\in L^{4/3}(0,T;(H^{-1}(\Omega))^3)$, *satisfying* (2.2) *a. e. in* $(0,T)$. *Such a solution is called a weak one.*

b) *There exists at most a pair* $(\vec{v},\,\vec{\omega})$ *which is a weak solution of* (2.2) *and satisfies* $\vec{v}\in L^8(0,T;(L^4(\Omega))^3)$. *This solution is called a strong solution of* (2.2).

Proof. The main steps in obtaining the results of point a) are similar to those for Navier-Stokes equations (see [2]). For proving the second assertion, we need further regularity of the the function $\vec{\omega}$, (i. e. $\vec{\omega}\in L^8(0,T;(L^4(\Omega))^3)$). This regularity will be obtained in Corollary 2.3, proved below and will allow us to obtain the uniqueness of the strong solution.

In the sequel, we shall prove a general result, which will give the regularity of the solutions throughout the paper.

Theorem 2.2. *Let* $\vec{f}\in L^2(0,T;H)$, $\vec{g}\in L^2(0,T;(L^2(\Omega))^3)$, $\vec{v}_0\in V$, $\vec{\omega}_0\in(H_0^1(\Omega))^3$, $\vec{u}\in L^8(0,T;(L^4(\Omega))^3)$, $\vec{y}\in L^\infty(0,T;V)\cap L^{3/2}(0,T;(H^2(\Omega))^3)$, *and* $\vec{\rho}\in(H^{2,1}(\Omega_T))^3\cap C([0,T];(H_0^1(\Omega))^3)$. *Then there exists an unique pair* $(\vec{v},\,\vec{\omega})\in(H^{2,1}(\Omega_T))^3\cap C([0,T];V)\times(H^{2,1}(\Omega_T))^3\cap C([0,T];(H_0^1(\Omega))^3)$ *satisfying, together with a function* $p\in L^2(0,T;H^1(\Omega))$, *unique up to the addition of a function of* t, *the following system:*

$$\left\{\begin{array}{l} \vec{v}'+b(\vec{v},\vec{y})+b(\vec{u},\vec{v})-(\mu+\chi)\triangle\vec{v}+\nabla p-\chi\text{curl}\,\vec{\omega}=\vec{f}\ \text{ in } \Omega_T, \\ j\vec{\omega}'+jb(\vec{v},\vec{\rho})+jb(\vec{u},\vec{\omega})-\gamma\triangle\vec{\omega}-(\alpha+\beta)\nabla(\text{div}\,\vec{\omega})+ \\ \qquad\qquad\qquad\qquad +2\chi\vec{\omega}-\chi\text{curl}\,\vec{v}=\vec{g}\ \text{in } \Omega_T, \\ \text{div}\,\vec{v}=0\ \text{ in } \Omega_T, \\ \vec{v}=\vec{0},\ \vec{\omega}=\vec{0}\ \text{ on } \partial\Omega\times(0,T), \\ \vec{v}(x,0)=\vec{v}_0(x),\ \vec{\omega}(x,0)=\vec{\omega}_0(x)\ \text{ in } \Omega, \end{array}\right. \tag{2.3}$$

Proof. For proving the existence and the regularity, we approximate the functions $\vec{v}$, $\vec{\omega}$ with

$$\vec{v}_m=\sum_{i=1}^{m} g_{im}(t)\vec{u}_i,\ \vec{\omega}_m=\sum_{i=1}^{m} h_{im}(t)\vec{\varphi}_i, \tag{2.4}$$

where $\{\vec{u}_i\}_i$ is a base of V and $\{\vec{\varphi}_i\}_i$ a base of $(H_0^1(\Omega))^3$. We consider the variational formulation of (2.3) corresponding to $(\vec{v}_m,\ \vec{\omega}_m)$:

$$\left\{\begin{array}{l} (\vec{v}_m'(t),\vec{u}_j)+(b(\vec{v}_m(t),\vec{y}(t)),\vec{u}_j)+(b(\vec{u}(t),\vec{v}_m(t)),\vec{u}_j) \\ +(\mu+\chi)((\vec{v}_m(t),\vec{u}_j))_0-\chi(\text{curl}\,\vec{\omega}_m(t),\vec{u}_j)=(\vec{f}(t),\vec{u}_j), \\ j(\vec{\omega}_m'(t),\vec{\varphi}_j)+j(b(\vec{v}_m(t),\vec{\rho}(t)),\vec{\varphi}_j)+j(b(\vec{u}(t),\vec{\omega}_m(t)),\vec{\varphi}_j) \\ +\gamma((\vec{\omega}_m(t),\vec{\varphi}_j))_0+(\alpha+\beta)(\text{div}\,\vec{\omega}_m(t),\text{div}\,\vec{\varphi}_j) \\ +2\chi(\vec{\omega}_m(t),\vec{\varphi}_j)-\chi(\text{curl}\vec{v}_m(t),\vec{\varphi}_j)=(\vec{g}(t),\vec{\varphi}_j)\ \forall j=1,...,m, \\ \vec{v}_m(0)=\vec{v}_{0_m},\ \vec{\omega}_m(0)=\vec{\omega}_{0_m}, \end{array}\right. \tag{2.5}$$

with $\vec{v}_{0_m}\to\vec{v}_0$ in V and $\vec{\omega}_{0_m}\to\vec{\omega}_0$ in $(H_0^1(\Omega))^3$. If we introduce (2.4) into (2.5) we obtain a linear system of ordinary differential equations with the unique solution g_{im}, h_{im}, $i=1,...,m$. For obtaining the existence of $(\vec{v},\ \vec{\omega})$ we establish some *a priori* estimates. The estimates in $L^\infty(0,T;H)\times L^\infty(0,T;(L^2(\Omega))^3)$ and in $L^2(0,T;V)\times L^2(0,T;(H_0^1(\Omega))^3)$ are obtained in the classical way (see e.g. [2]). We establish next the estimates in $L^\infty(0,T;V)\times L^\infty(0,T;(H_0^1(\Omega))^3)$ and in $(L^2(0,T;(H^2(\Omega))^3))^2$. For this purpose, we define the linear operator $L{:}(H_0^1(\Omega))^3\mapsto(H^{-1}(\Omega))^3$,

$$L\vec{\omega}=-\gamma\triangle\vec{\omega}-(\alpha+\beta)\nabla(\text{div}\vec{\omega}),\ \forall\vec{\omega}\in(H_0^1(\Omega))^3. \tag{2.6}$$

The linear operators $-\triangle$ and L being compact and self adjoint, they have an orthonormal sequence of eigenfunctions, which can be taken as a base in V and in $(H_0^1(\Omega))^3$, respectively. We take in $(2.4)_1$ the sequence of eigenfunctions of $-\triangle$ and in $(2.4)_2$ the sequence of eigenfunctions of L. Hence $\vec{u}_i$ and $\vec{\varphi}_i$ satisfy:

$$\left\{\begin{array}{l} -\triangle\vec{u}_i+\nabla p_i=\lambda_i\vec{u}_i,\ \vec{u}_i\in V \\ L\vec{\varphi}_i=\nu_i\vec{\varphi}_i,\ \vec{\varphi}_i\in(H_0^1(\Omega))^3\ \forall i\geq 1. \end{array}\right. \tag{2.7}$$

Since Ω is of class $\mathcal{C}^2$, we can apply the regularity results for elliptic equations and it follows that $\vec{u}_i \in V \cap (H^2(\Omega))^3$, $\vec{\varphi}_i \in (H_0^1(\Omega))^3 \cap (H^2(\Omega))^3$ and

$$\begin{cases} \|\vec{v}_m(t)\|_{(H^2(\Omega))^3} \leq c(\Omega)|-\Delta\vec{v}_m(t)|, \\ \|\vec{\omega}_m(t)\|_{(H^2(\Omega))^3} \leq c(\Omega)|L\vec{\omega}_m(t)|. \end{cases} \tag{2.8}$$

We multiply now $(2.5)_1$ with $\lambda_j g_{jm}(t)$ and we add the equalities. It follows:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\vec{v}_m(t)\|_0^2 + (\mu+\chi)|-\Delta\vec{v}_m(t)|^2 = (\vec{f}(t), -\Delta\,\vec{v}_m(t)) \\ &+\chi(\text{curl}\;\vec{\omega}_m(t) - \Delta\vec{v}_m(t)) - (b(\vec{v}_m(t),\vec{y}(t)), -\Delta\,\vec{v}_m(t)) \\ &-(b(\vec{u}(t),\vec{v}_m(t)), -\Delta\,\vec{v}_m(t)). \end{aligned} \tag{2.9}$$

For the right-hand side of (2.9) we use the following inequalities (taking into account the regularity of the functions $\vec{y}$, $\vec{u}$, $(2.8)_1$ and the properties of b):

$$\begin{aligned} -(b(\vec{v}_m(t),\vec{y}(t)), -\Delta\vec{v}_m(t)) &\leq \|\vec{v}_m(t)\|_{(L^4(\Omega))^3}\|\nabla\vec{y}(t)\|_{(L^4(\Omega))^9}|-\Delta\vec{v}_m(t)| \\ &\leq c(\Omega)\|\vec{v}_m(t)\|_{(L^4(\Omega))^3}\|\vec{y}(t)\|_0^{1/4}\|\vec{y}(t)\|_{(H^2(\Omega))^3}^{3/4}|-\Delta\vec{v}_m(t)| \\ &\leq \frac{\mu+\chi}{8}|-\Delta\vec{v}_m(t)|^2 + c\|\vec{y}\|_{L^\infty(0,T;V)}^{1/2}\|\vec{v}_m(t)\|_0^2\;\|\vec{y}(t)\|_{(H^2(\Omega))^3}^{3/2}, \\ -(b(\vec{u}(t),\vec{v}_m(t)), -\Delta\vec{v}_m(t)) &\leq \|\vec{u}(t)\|_{(L^4(\Omega))^3}\|\nabla\vec{v}_m(t)\|_{(L^4(\Omega))^9}|-\Delta\vec{v}_m(t)| \\ &\leq \frac{\mu+\chi}{8}|-\Delta\vec{v}_m(t)|^2 + c\|\vec{u}(t)\|_{(L^4(\Omega))^3}^2\|\vec{v}_m(t)\|_0^{1/2}\|\vec{v}_m(t)\|_{(H^2(\Omega))^3}^{3/2} \\ &\leq \frac{\mu+\chi}{4}|-\Delta\vec{v}_m(t)|^2 + c\|\vec{u}(t)\|_{(L^4(\Omega))^3}^8\|\vec{v}_m(t)\|_0^2 \end{aligned}$$

With these inequalities, (2.9) becomes

$$\begin{aligned} \frac{d}{dt}\|\vec{v}_m(t)\|_0^2 + (\mu+\chi)|-\Delta\vec{v}_m(t)|^2 &\leq c(|\vec{f}(t)|^2 + \|\vec{\omega}_m(t)\|_0^2) \\ &+A(t)\|\vec{v}_m(t)\|_0^2, \end{aligned} \tag{2.10}$$

where $A(t) = c(\|\vec{y}\|_{L^\infty(0,T;V)}^{1/2}\|\vec{y}(t)\|_{(H^2(\Omega))^3}^{3/2} + \|\vec{u}(t)\|_{(L^4(\Omega))^3}^8)$. Using the regularity of the given functions $\vec{u}$, $\vec{y}$ it follows the integrabily of A on $(0,T)$. We can then integrate (2.10) with respect to t and the boundedness of $\{\vec{\omega}_m\}_m$ in $L^2(0,T;(H_0^1(\Omega))^3)$ and of $\{\vec{v}_{0m}\}_m$ in V lead us to the estimates of $\vec{v}_m$ in $L^\infty(0,T;V)$ and in $L^2(0,T;(H^2(\Omega))^3)$. With similar computations we get the corresponding estimates for $\vec{\omega}_m$.

The last step of the proof is to obtain the estimates of $(\vec{v}_m', \vec{\omega}_m')$ in $L^2(0,T;(L^2(\Omega))^3)^2$. The inclusion $(H^{2,1}(\Omega_T))^3 \subset L^2(0,T;(L^2(\Omega))^3)$ being compact, the existence of $(\vec{v}, \vec{\omega})$ follows passing to the limit, on a subsequence, in (2.5). The uniqueness of the pair $(\vec{v}, \vec{\omega})$ is proved as usual.

Corollary 2.3. *Let $(\vec{v}, \vec{\omega})$ be a strong solution of (2.2). Then* $\vec{v} \in (H^{2,1}(\Omega_T))^3 \cap C([0,T];V)$, $\vec{\omega} \in (H^{2,1}(\Omega_T))^3 \cap C([0,T];(H_0^1(\Omega))^3)$. *Moreover, we have:*

$$\begin{cases} \|\vec{v}\|_{(H^{2,1}(\Omega_T))^3} \leq c, \\ \|\vec{\omega}\|_{(H^{2,1}(\Omega_T))^3} \leq c, \end{cases} \tag{2.11}$$

where the constant c depends on Ω, the constants of the problem, the given fields, $\vec{f}$, $\vec{g}$, $\vec{v}_0$, $\vec{\omega}_0$ and on $\|\vec{v}\|_{L^8(0,T;(L^4(\Omega))^3)}$.

Proof. We take in (2.3) $\vec{y} = \vec{0}$, $\vec{u} = \vec{v}$, $\vec{\rho} = \vec{0}$ and we apply Theorem 2.2. The regularity $\vec{\omega} \in L^8(0,T;(L^4(\Omega))^3)$ follows from the inclusions $C([0,T];(H_0^1(\Omega))^3) \subset L^q(0,T;(H_0^1(\Omega))^3) \subset L^q(0,T;(L^4(\Omega))^3)$, $\forall q$.

3. Study of the control problem

In the theory of micropolar fluids a special case appears when the microrotation is constrained by:

$$\vec{\omega} = \text{curl } \vec{v}. \tag{3.1}$$

Indeed, if we introduce (3.1) in $(2.1)_1$, the micropolar fluid becomes a Navier-Stokes one. The aim of this paper is to control the properties of the fluid by acting on the exterior field $\vec{g}$. The difficulty is that the correspondence $\vec{g} \mapsto (\vec{v}, \vec{\omega})$ is multivalued in the three dimensional case.

For this reason, we formulate the control problem such that to every optimal control we can associate a strong solution. Since we cannot prove the existence of a strong solution, we have to choose a suitable functional.

We define

$$J : L^2(0,T;(L^2(\Omega))^3) \times L^2(0,T;V) \times L^2(0,T;(H_0^1(\Omega))^3) \mapsto \bar{I\!R}$$

$$J(\vec{g}, \vec{v}, \vec{\omega}) = \frac{1}{6} \int_0^T (|\vec{\omega}(t) - \text{curl } \vec{v}(t)|)^6 dt, \tag{3.2}$$

with $(\vec{v}, \vec{\omega})$ a solution of (2.2) corresponding to $\vec{g}$.

Since we cannot expect that we will be able to prove the coercivity of J, we take the exterior field $\vec{g} \in B_r$, with

$$B_r = \{\vec{\varphi} \in L^2(0,T;(L^2(\Omega))^3) \,/\, \|\vec{\varphi}\|_{L^2(0,T;(L^2(\Omega))^3)} \leq r\}.$$

We formulate the control problem in the following way:

$$\text{(CP)} \qquad \begin{cases} \text{Minimize } J(\vec{g}, \vec{v}, \vec{\omega}) \text{ when } \vec{g} \in B_r \text{ and the pair} \\ (\vec{v}, \vec{\omega}) \text{ verifies (2.2).} \end{cases}$$

We establish next the following important result:

Proposition 3.1. *Let* $(\vec{g},\vec{v},\vec{\omega})\in B_r\times L^2(0,T;V)\times L^2(0,T;(H_0^1(\Omega))^3)$ *with the properties:*

a) $J(\vec{g},\vec{v},\vec{\omega})<\infty$,

b) $(\vec{v},\vec{\omega})$ *is a solution of* (2.2), *corresponding to* $\vec{g}$.

Then $(\vec{v},\vec{\omega})$ *is the unique strong solution of* (2.2).

Proof. Since $\vec{v}$ is divergence free it follows that

$$\|\vec{v}\|_0=|\text{curl }\vec{v}|. \tag{3.3}$$

Using (3.3) we get the following inequality

$$J(\vec{g},\vec{v},\vec{\omega})\geq\frac{1}{6}\int_0^T(\|\vec{v}(t)\|_0^2-2\|\vec{v}(t)\|_0|\vec{\omega}(t)|+|\vec{\omega}(t)|^2)^3dt.$$

The hypothesis a) of the proposition together with the above inequality implies that $\vec{v}\in L^6(0,T;V)$. On the other hand, from the known inequality $\|u\|_{L^4(\Omega)}\leq\sqrt{2}|u|^{1/4}\|u\|_0^{3/4}\,\forall u\in H_0^1(\Omega)$ and from the regularity $\vec{v}\in L^\infty(0,T;H)$ given by Theorem 2.1, it follows

$$\|\vec{v}\|_{L^8(0,T;(L^4(\Omega))^3)}\leq\sqrt{2}\|\vec{v}\|_{L^\infty(0,T;H)}^{1/4}\|\vec{v}\|_{L^6(0,T;V)}^{3/4}. \tag{3.4}$$

Remark 3.2. The hypothesis a) from the previous proposition means that for fixed $\vec{f}$, $\vec{v}_0$, $\vec{\omega}_0$, we can find a function $\vec{g}\in B_r$ so that the system (2.2) has a strong solution.

Theorem 3.3. *If* $J\not\equiv\infty$, *then* (CP) *has at least a solution.*

Proof. We denote

$$m=\inf\{J(\vec{g},\vec{v},\vec{\omega})\,/\,\vec{g}\in B_r,\ (\vec{v},\vec{\omega})\text{ solution for (2.2)}\}.$$

Let $\{(\vec{g}_n,\vec{v}_n,\vec{\omega}_n)\}_n$ be a minimizing sequence. From Proposition 3.1. we obtain that $(\vec{v}_n,\vec{\omega}_n)$ is the unique strong solution of (2.2) corresponding to $\vec{g}_n$. It follows, from Corollary 2.3, that $\{\vec{v}_n\}_n$ and $\{\vec{\omega}_n\}_n$ are bounded in $(H^{2,1}(\Omega_T))^3$ by a constant depending on the fixed data and on $\|\vec{v}_n\|_{L^8(0,T;(L^4(\Omega))^3)}$, which is bounded, from (3.4), by a constant not depending on n. Since the embedding $H^{2,1}(\Omega_T)\subset L^2(\Omega_T)$ is compact, we can pass to the limit in (2.2) corresponding to $\vec{g}_n$ and we obtain that $(\vec{v},\,\vec{\omega})$ is the unique strong solution of (2.2) corresponding to $\vec{g}$, a weak limit point of $\{\vec{g}_n\}_n$ in $(L^2(0,T;(L^2(\Omega))^3)$. It follows that J is weakly lower semicontinuous, and, hence, the proof is achieved.

4. The optimality system

For obtaining the conditions of optimality, we define

$$A=\{\vec{g}\in L^2(0,T;(L^2(\Omega))^3)\,/\,(2.2)\text{ corresponding to }\vec{g}\text{ has a strong solution}\} \tag{4.1}$$

and $I : A \mapsto I\!R$,

$$I(\vec{g}) = J(\vec{g},\, F(\vec{g})) \quad \forall \vec{g} \in A, \tag{4.2}$$

where $F(\vec{g}) = (\vec{v}_g,\, \vec{\omega}_g)$, $(\vec{v}_g,\, \vec{\omega}_g)$ is the unique strong solution of (2.2) corresponding to $\vec{g}$.

Theorem 4.1. *Let $\vec{g}^*$ be an element of A. Then there exists a neighbourhood $\mathcal{U}$ of $\vec{g}^*$ with $\mathcal{U} \subset A$.*

Proof. We shall use the implicit function theorem. For this purpose we introduce the notation:

$$\begin{cases} M = L^2(0,T;(L^2(\Omega))^3), \\ Y = (H^{2,1}(\Omega_T))^3 \cap C([0,T];V) \times (H^{2,1}(\Omega_T))^3 \cap C([0,T];(H_0^1(\Omega))^3), \\ Z = L^2(0,T;H) \times L^2(0,T;(L^2(\Omega))^3) \times V \times (H_0^1(\Omega))^3 \end{cases}$$

and we define the operator $\vec{\Phi} : M \times Y \mapsto Z$ with

$$\begin{cases} \Phi_1(\vec{g},\, (\vec{v},\vec{\omega})) = \vec{v}\,' + b(\vec{v},\vec{v}) - (\mu+\chi) \triangle \vec{v} - \chi \text{curl}\, \vec{\omega} - \vec{f}, \\ \Phi_2(\vec{g},\, (\vec{v},\vec{\omega})) = j\vec{\omega}\,' + jb(\vec{v},\vec{\omega}) - \gamma\triangle\vec{\omega} - (\alpha+\beta)\nabla(\text{div}\, \vec{\omega}) \\ \qquad\qquad +2\chi\vec{\omega} - \chi\text{curl}\, \vec{v} - \vec{g} \\ \Phi_3(\vec{g},\, (\vec{v},\vec{\omega})) = \vec{v}(0) - \vec{v}_0, \\ \Phi_4(\vec{g},\, (\vec{v},\vec{\omega})) = \vec{\omega}(0) - \vec{\omega}_0. \end{cases} \tag{4.3}$$

It is obvious that $\vec{\Phi}(\vec{g}^*,\, (\vec{v}^*,\vec{\omega}^*)) = \vec{0}$. The boundedness and the uniform continuity of the operators $\vec{\Phi}$ and $\dfrac{\partial\vec{\Phi}(\vec{g},\, (\vec{v},\vec{\omega}))}{\partial(\vec{v},\vec{\omega})} : Y \mapsto Z$ being easy to obtain, it remains to prove the inversability of the operator $\dfrac{\partial\vec{\Phi}(\vec{g}^*,\, (\vec{v}^*,\vec{\omega}^*))}{\partial(\vec{v},\vec{\omega})}$. The computations give

$$\langle \frac{\partial\Phi_i(\vec{g}^*,\, (\vec{v}^*,\vec{\omega}^*))}{\partial(\vec{v},\vec{\omega})}, (\vec{v},\vec{\omega})\rangle = E_i((\vec{v}^*,\vec{\omega}^*),(\vec{v},\vec{\omega})),\ i = 1, ...4,$$

where

$$\begin{cases} E_1((\vec{v}^*,\vec{\omega}^*),(\vec{v},\vec{\omega})) = \vec{v}\,' + b(\vec{v},\vec{v}^*) + b(\vec{v}^*,\vec{v}) - (\mu+\chi) \triangle \vec{v} - \chi\text{curl}\, \vec{\omega}, \\ E_2((\vec{v}^*,\vec{\omega}^*),(\vec{v},\vec{\omega})) = j\vec{\omega}\,' + jb(\vec{v},\vec{\omega}^*) + jb(\vec{v}^*,\vec{\omega}) - \gamma \triangle \vec{\omega} \\ -(\alpha+\beta)\nabla(\text{div}\, \vec{\omega}) + 2\chi\vec{\omega} - \chi\text{curl}\, \vec{v}, \\ E_3((\vec{v}^*,\vec{\omega}^*),(\vec{v},\vec{\omega})) = \vec{v}(0), \\ E_4((\vec{v}^*,\vec{\omega}^*),(\vec{v},\vec{\omega})) = \vec{\omega}(0). \end{cases} \tag{4.4}$$

It is now obvious that the inversability of $\dfrac{\partial\vec{\Phi}(\vec{g}^*,\, (\vec{v}^*,\vec{\omega}^*))}{\partial(\vec{v},\vec{\omega})}$ is equivalent with the existence and the uniqueness of the solution of the system (2.3) with $\vec{y} = \vec{u} = \vec{v}^*$, $\vec{\rho} = \vec{\omega}^*$. Since the functions $\vec{v}^*$, $\vec{\omega}^*$ have the regularity required by Theorem 2.2, we can apply this theorem and the inversability is obtained. Therefore we can use the implicit function theorem and we find a neighbourhood $\mathcal{U}$ of $\vec{g}^*$ and a function $\vec{F} : \mathcal{U} \mapsto Y$ so that $\vec{\Phi}(\vec{g},\, \vec{F}(\vec{g})) = \vec{0},\ \forall \vec{g} \in \mathcal{U}$. If we denote $\vec{F}(\vec{g}) = (\vec{v}_g,\, \vec{\omega}_g)$, it follows that $(\vec{v}_g,\, \vec{\omega}_g)$ is the strong solution of (2.2) corresponding to $\vec{g}$ and the proof is achieved.

The problem (CP) can be written in the form:

$$\begin{cases} \text{Find } \vec{g}^* \in A \cap B_r \text{ such that} \\ I(\vec{g}^*) = \min\{I(\vec{g}) \,/\, \vec{g} \in A \cap B_r\}. \end{cases} \tag{4.5}$$

Since we have proved that A is an open set, it follows that for every $\vec{g}_1, \vec{g} \in A \cap B_r$ there exists $\delta_0 \in (0,1)$ so that $\vec{g}_1 + \delta(\vec{g} - \vec{g}_1) \in A \cap B_r$, $\forall \delta \leq \delta_0$. We are now in a position to prove the differentiability of the functional I.

Proposition 4.2. *The functional I is G-differentiable on $A \cap B_r$ and $\forall\, \vec{g},\, \vec{g}_1$*

$$\begin{cases} (I'(\vec{g}_1),\, \vec{g} - \vec{g}_1)_{L^2(0,T;(L^2(\Omega))^3)} = \\ \displaystyle\int_0^T E_0^2(t)(\mathrm{curl}(\vec{v}^*(t) - \vec{v}_1(t)) - (\vec{\omega}^*(t) - \vec{\omega}_1(t)), \mathrm{curl}\vec{v}_1(t) - \vec{\omega}_1(t))dt, \end{cases} \tag{4.6}$$

where $E_0(t) = |curl\vec{v}_1(t) - \vec{\omega}_1(t)|^2$, $(\vec{v}_1,\, \vec{\omega}_1)$ is the unique strong solution of (2.2) *corresponding to $\vec{g}_1$ and $(\vec{v}^*,\, \vec{\omega}^*) \in (H^{2,1}(\Omega_T))^3 \cap C([0,T]; V) \times (H^{2,1}(\Omega_T))^3 \cap C([0,T]; (H_0^1(\Omega))^3)$ is the unique solution for the system:*

$$\begin{cases} (\vec{v}^{*\prime}(t), \vec{z}) + (b(\vec{v}^*(t), \vec{v}_1(t)), \vec{z}) + (b(\vec{v}_1(t), \vec{v}^*(t) - \vec{v}_1(t)), \vec{z}) \\ +(\mu + \chi)((\vec{v}^*(t), \vec{z}))_0 - \chi(\mathrm{curl}\ \vec{\omega}^*(t), \vec{z}) = (\vec{f}(t), \vec{z})\ \forall \vec{z} \in V, \\ j(\vec{\omega}^{*\prime}(t), \vec{\eta}) + j(b(\vec{v}^*(t), \vec{\omega}_1(t)), \vec{\eta}) + j(b(\vec{v}_1(t), \vec{\omega}^*(t) - \vec{\omega}_1(t)), \vec{\eta}) \\ +\gamma((\vec{\omega}^*(t), \vec{\eta}))_0 + (\alpha+\beta)(\mathrm{div}\ \vec{\omega}^*(t), \mathrm{div}\ \vec{\eta}) + 2\chi(\vec{\omega}^*(t), \vec{\eta}) \\ -\chi(\mathrm{curl}\ \vec{v}^*(t), \vec{\eta}) = (\vec{g}(t), \vec{\eta})\ \ \forall \vec{\eta} \in (H_0^1(\Omega))^3, \\ \vec{v}^*(0) = \vec{v}_0,\ \vec{\omega}^*(0) = \vec{\omega}_0. \end{cases} \tag{4.7}$$

Proof. The existence, the uniqueness and the regularity of $(\vec{v}^*,\, \vec{\omega}^*)$ follow from Theorem 2.2. The formula (4.6) is obtained with standard computations, so we shall skip the proof.

Corollary 4.3. *If $\vec{g}_1$ is a solution for the control problem* (4.5)*, then*

$$\int_0^T E_0^2(t)(\mathrm{curl}(\vec{v}^*(t) - \vec{v}_1(t)) - (\vec{\omega}^*(t) - \vec{\omega}_1(t)), \mathrm{curl}\vec{v}_1(t) - \vec{\omega}_1(t))dt \geq 0. \tag{4.8}$$

The last result of the paper states the optimality conditions satisfied by an optimal control.

Theorem 4.4. *Let $\vec{g}_1$ be an optimal control. Then, there exist the unique pairs: $(\vec{v}_1,\, \vec{\omega}_1)$, the strong solution of* (2.2) *corresponding to $\vec{g}_1$, $(\vec{u}_1, \vec{\rho}_1) \in (H^{2,1}(\Omega_T))^3 \cap C([0,T]; V) \times (H^{2,1}(\Omega_T))^3 \cap C([0,T]; (H_0^1(\Omega))^3)$,* *the unique solution of the adjoint system* (4.9), *writtem below which satisfy the following optimality system:*

the system (2.2) *written for* $\vec{g}_1$,

$$\begin{cases} -(\vec{u}_1'(t), \vec{z}) + (b(\vec{z}, \vec{v}_1(t)), \vec{u}_1(t)) - (b(\vec{v}_1(t), \vec{u}_1(t)), \vec{z}) \\ +j(b(\vec{z}, \vec{\omega}_1(t)), \vec{\rho}_1(t)) + (\mu + \chi)((\vec{u}_1(t), \vec{z}))_0 \\ -\chi(\text{curl}\vec{\rho}_1(t), \vec{z}) = E_0^2(t)(\text{curl}\vec{v}_1(t) - \vec{\omega}_1(t), \text{curl}\vec{z}) \ \forall \vec{z} \in V, \\ -j(\vec{\rho}_1'(t), \vec{\eta}) - j(b(\vec{v}_1(t), \vec{\rho}_1(t)), \vec{\eta}) + \gamma((\rho_1(t), \vec{\eta}))_0 \\ +(\alpha + \beta)(\text{div } \vec{\rho}_1(t), \text{div } \vec{\eta}) + 2\chi(\vec{\rho}_1(t), \vec{\eta}) - \chi(\text{curl } \vec{u}_1(t), \vec{\eta}) \\ = -E_0^2(t)(\text{curl } \vec{v}_1(t) - \vec{\omega}_1(t), \vec{\eta}) \ \forall \vec{\eta} \in (H_0^1(\Omega))^3, \\ \vec{u}_1(T) = \vec{0}, \ \vec{\rho}_1(T) = 0, \end{cases} \tag{4.9}$$

$$\int_{\Omega_T} \vec{\rho}_1 \cdot (\vec{g} - \vec{g}_1) dx dt \geq 0 \ \ \forall \vec{g} \in A \cap B_r. \tag{4.10}$$

Proof. The variational formulation of the adjoint system (4.9) being of the same type with the variational formulation of (2.3) we can apply Theorem 2.2. and we obtain the existence, the uniqueness and the regularity of ($\vec{u}_1$, $\vec{\rho}_1$). The inequality (4.10) is derived from (4.8), taking ($\vec{z}$, $\vec{\eta}$) = ($\vec{v}^*(t) - \vec{v}_1(t)$, $\vec{\omega}^*(t) - \vec{\omega}_1(t)$) in (4.9) and ($\vec{z}$, $\vec{\eta}$) = ($\vec{u}_1(t)$, $\vec{\rho}_1(t)$) in (4.7)-(2.2) corresponding to $\vec{g}_1$. Standard computations complete the proof.

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