FACTORIZATION OF ELLIPTIC BOUNDARY VALUE PROBLEMS: THE *QR* APPROACH

Jacques Henry
INRIA
BP 105, 78153 Le Chesnay FRANCE
Jacques.Henry@inria.fr

1. Introduction

In this paper we describe and develop a method first proposed by Angel and Bellman ([1]) to factorize a second order elliptic boundary value problem in the product of two first order decoupled initial value problems by invariant embeding. For the sake of simplicity we consider a domain Ω of $I\!R^n$ which is a cylinder $]0,1[\times\mathcal{O}$ and the Laplacian Δ as elliptic operator. We denote x the coordinate along the first axis which is also the axis of the cylinder and y the $n-1$ other coordinates. The section $\mathcal{O}\subset I\!R^{n-1}$ is bounded and has a smooth boundary. We denote $\Sigma=]0,1[\times\partial\mathcal{O}$ the lateral boundary of the cylinder and $\Gamma_0=\{0\}\times\mathcal{O}$, $\Gamma_1=\{1\}\times\mathcal{O}$ the two faces of the cylinder. We consider the problem

$$(\mathcal{P}_0)\begin{cases} -\Delta u = f & \text{in } \Omega,\\ u|_\Sigma = 0, & -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0, \quad u|_{\Gamma_1} = u_1. \end{cases}$$

The case of a Dirichlet boundary condition on Γ_0 will also be considered in section 5. The problem is embedded in a family of similar problems in the subcylinders $]0,s[\times\mathcal{O}$. Let $Q(s)$ be the Dirichlet-to-Neumann map on the section $x=s$. We prove that the boundary value problem for the Poisson equation can be factorized as:

$$-\left(\frac{d}{dx}+Q\right)\left(\frac{d}{dx}-Q\right)u=f$$

each of the first order problem having an initial value given at $x=0$ or $x=1$. Furthermore the operator Q satisfies the Riccati equation

$$\frac{dQ}{dx}-Q^2=\Delta_y, \qquad Q(0)=0.$$

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: 10.1007/978-0-387-35690-7_44
V. Barbu et al. (eds.), *Analysis and Optimization of Differential Systems*

where Δ_y is the Laplacian on the section $\mathcal{O}$. A control problem equivalent to the Poisson problem whose time variable is the x-coordinate is presented. The previous Riccati equation yields the optimal feedback for this control problem.

The previous factorization of the Poisson problem can be viewed as an infinite dimensional extension of the Gauss LU block factorization. We also present a similar extension of the QR factorization. It first uses a factorization of the normal equation from which the orthogonal operator can be derived. The triangular part takes the form of a second order in x initial value problem.

Finally we present an optimal control problem with an elliptic state equation. We show that the factorization and uncoulping of both the state and adjoint state can be achieved together.

2. Factorization of the state equation

We briefly recall the factorization of the state equation from [2]. Using the technique of invariant embedding introduced by R. Bellman (see [1]), we embed problem $(\mathcal{P}_0)$ in a family of similar problems $(\mathcal{P}_{s,h})$ defined on $\Omega_s =]0,s[\times\mathcal{O}$ for $s\in]0,1]$. For each problem we impose the Dirichlet boundary condition $u|_{\Gamma_s} = h$, where $\Gamma_s = \{s\}\times\mathcal{O}$.

$$(\mathcal{P}_{s,h})\begin{cases} -\Delta u = f & \text{in } \Omega_s, \\ u|_\Sigma = 0, & -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0, \quad u|_{\Gamma_s} = h. \end{cases}$$

For every $s\in]0,1]$ we define the Dirichlet-to-Neumann (DtN) map $Q(s)$ by $Q(s)h = \frac{\partial u}{\partial x}|_{\Gamma_s}$, with f and u_0 set to zero. By linearity of $(\mathcal{P}_{s,h})$ we have $\frac{\partial u}{\partial x}|_{\Gamma_s} = Q(s)h + w(s)$.

Furthermore, the solution u of $(\mathcal{P}_0)$ restricted to $]0,s[$ satisfies $(\mathcal{P}_{s,u|_{\Gamma_s}})$ for $s\in]0,1[$ so that

$$\frac{\partial u}{\partial x}(x,y) = (Q(x)u|_{\Gamma_x})(y) + (w(x))(y). \tag{1}$$

Then, by formally taking the derivative with respect to x of this formula, we obtain $\dfrac{\partial^2 u}{\partial x^2} = -\Delta_y u - f = \dfrac{dQ}{dx}u + Q\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial x}$, where Δ_y is the $(n-1)$–dimensional Laplacian on $\mathcal{O}$. Therefore substituting $\frac{\partial u}{\partial x}$ from equation (1)

$$0 = \left(\frac{dQ}{dx} + Q^2 + \Delta_y\right)u + \frac{\partial w}{\partial x} + Qw + f,$$

and then, since u is arbitrary, we obtain the decoupled system

$$\begin{cases} \dfrac{dQ}{dx} + Q^2 + \Delta_y = 0, & Q(0) = 0, \\ \dfrac{dw}{dx} + Qw = -f, & w(0) = -u_0, \\ -\dfrac{du}{dx} + Qu = -w, & u(1) = u_1. \end{cases} \quad (2)$$

The initial conditions for Q and w at $x = 0$ are obtained from the boundary conditions for u at Γ_0 and from (1) and similarly for the initial conditions for u at $x = 1$. Let us stress that Q is an operator on functions in y depending on x which satisfies a Riccati equation. The system (2) is decoupled because one can integrate the first two equations in x from 0 to 1 giving Q and w, then u is obtained by the integration backwards of the third equation. Formally, we have factorized $-\Delta u = f$ as

$$-\left(\frac{d}{dx} + Q\right)\left(\frac{d}{dx} - Q\right) u = f. \quad (3)$$

Since Q is self adjoint (see [2]), it is clear that the two factors are adjoint of each other. Also, as Q is coercive, the equations for w and u are of parabolic type. In the particular case of the Poisson equation in a cylinder it can be shown that Q and Δ_y commute.

3. Properties of Q

The precise properties of the DtN map Q and the meaning of the Riccati equation (2) are studied in [2] as continuous operator and in [3] in a Hilbert-Schmidt framework. Here we just briefly recall the functional framework used and the main results. We denote X

$$X = \{u \in H^1(\Omega) \quad | \quad u|_\Sigma = 0\} \equiv L^2(0,1;H_0^1(\mathcal{O})) \cap H^1(0,1;L^2(\mathcal{O})).$$

From [5], we introduce the 1/2 interpolate between $L^2(\mathcal{O})$ and $H_0^1(\mathcal{O})$

$$H_{00}^{1/2}(\mathcal{O}) = [L^2(\mathcal{O}), H_0^1(\mathcal{O})]_{1/2}.$$

Then from [5], we have $X \subset \mathcal{C}([0,1], H_{00}^{1/2}(\mathcal{O}))$, which allows to define the trace of $u \in X$, on Γ_s, $u|_s \in H_{00}^{1/2}(\mathcal{O})$. Assuming for the sake of simplicity $u_1 = 0$ (otherwise f and u are translated), we defin $X_0 = \{u \in X \mid u|_1 = 0\}$, and the variational formulation of $(\mathcal{P}_0)$ reads

$$\int_\Omega \nabla u.\nabla\varphi \, dx\, dy = \int_\Omega f\varphi \, dx\, dy + < u_0, \varphi|_0 >_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}$$

for all $\varphi \in X_0$. Then $Q \in L^\infty(0,1;\mathcal{L}(H_0^1(\mathcal{O}), L^2(\mathcal{O})))$ satisfies the Riccati equation(2) in the following sense

$$\frac{d}{dx}(Q(x)h,\overline{h}) + (Q(x)h, Q(x)\overline{h}) = (\nabla_y h, \nabla_y \overline{h}) \text{ in } \mathcal{D}'(]0,1[),\ \forall h, \overline{h} \in H_0^1(\mathcal{O}).$$

We also get the properties a.e.
$Q(x) \in \mathcal{L}(H^{3/2}(\mathcal{O}) \cap H^{1/2}_{00}(\mathcal{O}), H^{1/2}_{00}(\mathcal{O}))$ and
$Q \in \mathcal{L}(H^{1/2}_{00}(\mathcal{O}), H^{1/2}_{00}(\mathcal{O})') \cap \mathcal{L}(L^2(\mathcal{O}), H^{-1}(\mathcal{O}))$; $Q(x)$ is self-adjoint and coercive on $H^{1/2}_{00}(\mathcal{O})$ for $x > 0$. Then w and u are defined in X_0 by

$$\left(\frac{dw}{dx}, h\right) + (Qw, h) = -(f, h), \quad \forall h \in L^2(\mathcal{O}), \quad w(0) = -u_0,$$
$$\left(-\frac{du}{dx}, h\right) + (Qu, h) = -(w, h), \quad \forall h \in L^2(\mathcal{O}) \quad u(a) = 0.$$

Hence $\frac{dw}{dx} \in L^2(0,1; L^2(\mathcal{O}))$, $\frac{du}{dx} \in L^2(0,1; L^2(\mathcal{O}))$. These are variational formulation of parabolic type problems.

4. Optimal control problem associated to the boundary value problem.

In this section we show the relation with Riccati equations appearing in optimal control theory (see for instance [4]). In fact we show that problem $(\mathcal{P}_0)$ can be formulated as an optimal control problem. We use the operator Q and the function w defined in Section 2, with $u_0 = 0$ (for the sake of simplicity). Let us consider the control space $\mathcal{U} = L^2(\Omega)$. For every $v \in \mathcal{U}$ the state $u(v) \in H^1(0,1; L^2(\mathcal{O}))$ is solution of

$$\begin{cases} \dfrac{\partial u}{\partial x} = v & \text{in } \Omega, \\ u(1) = u_1. \end{cases} \tag{4}$$

We also denote $\mathcal{U}_{ad} = \{v \in \mathcal{U} : \; u(v) \in X_{u_1}\}$ the space of admissible controls, where $X_{u_1} = \{h \in L^2(0,1; H^1_0(\mathcal{O})) \cap H^1(0,1; L^2(\mathcal{O})) : h(1) = u_1\}$. The desired state u_d is given almost everywhere in x by the solution of the family of (n-1) dimensional problems

$$\begin{cases} -\Delta_y u_d(x) = f(x) & \text{in } \mathcal{O} \\ u_d|_{\partial\mathcal{O}} = 0. \end{cases} \tag{5}$$

Then u_d belongs to $L^2(0,1; H^1_0(\mathcal{O}))$. Now we look for $u \in \mathcal{U}_{ad}$ such that $J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v)$, where, for every $v \in \mathcal{U}_{ad}$,

$$J(v) = \int_0^1 \|\nabla_y u(v) - \nabla_y u_d\|^2_{L^2(\mathcal{O})}\, dx + \int_0^1\!\!\int_{\mathcal{O}} v^2 dx\, dy. \tag{6}$$

At this point we have the problem that $\mathcal{U}_{ad}$ is not a closed subset of $L^2(\Omega)$ and therefore we cannot use directly the classic techniques (see, for instance, [4]) in order to solve this problem. Nevertheless, since

$\mathcal{U}_{ad} = \{\frac{\partial h}{\partial x} : h \in X_{u_1}\}$, $J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v) = \inf_{h \in X_{u_1}} \overline{J}(h) = \overline{J}(u)$, where $\frac{\partial h}{\partial x} = u$ and

$$\overline{J}(h) = \int_0^1 \|\nabla_y h - \nabla_y u_d\|^2_{L^2(\mathcal{O})}\, dx + \int_0^1\!\!\int_{\mathcal{O}} \left|\frac{\partial h}{\partial x}\right|^2 dx\, dy. \qquad (7)$$

Now, X_{u_1} is a closed convex set in the Hilbert space $L^2(0,1;H^1_0(\mathcal{O})) \cap H^1(0,1;L^2(\mathcal{O}))$ and $\overline{J}(h)^{1/2}$ is a norm of that space. Then (see Theorem 1.3 of chapter I of [4]) there exists a unique $u \in X_{u_1}$ satisfying $\overline{J}(u) = \inf_{h \in X_{u_1}} \overline{J}(h)$, which is uniquely determined by

$$\overline{J}'(u)(h) = 0 \quad \forall\, h \in X_0. \qquad (8)$$

Let us show that u is solution of $(\mathcal{P}_0)$. Developping (7), one gets

$$\overline{J}(u) = \int_\Omega |\nabla u|^2\, dx - 2\int_\Omega \nabla_y u\, \nabla_y u_d\, dx + \int_\Omega |\nabla_y u_d|^2\, dx.$$

But from (5), u_d satisfies almost everywhere in x

$$\int_{\mathcal{O}} \nabla_y u_d(x) \nabla_y u(x)\, dy = \int_{\mathcal{O}} f(x) u(x)\, dy,$$

Then

$$\overline{J}(u) = \int_\Omega |\nabla u|^2\, dx - 2\int_\Omega f u dx + \int_\Omega |\nabla_y u_d|^2\, dx.$$

Now it is clear that $\overline{J}(u)$ is the energy functional associated to $(\mathcal{P}_0)$ up to a constant term. We introduce the adjoint state p by

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\Delta_y u - f & \text{in } \Omega, \\ p(0) = 0. \end{cases}$$

Then, since $-\Delta_y u - f \in L^2(0,1;H^{-1}(\mathcal{O}))$, we know (see Theorem 1.2 of chapter III of [4]) that $p \in H^1(0,1;H^{-1}(\mathcal{O}))$. Furthermore, since $u \in Y$, we also deduce that $\frac{\partial p}{\partial x} \in H^{-1}(0,1;L^2(\mathcal{O}))$ and therefore, $p \in L^2(\Omega)$. Now for every $h \in X_0$, we know that

$$\int_0^1 < -\Delta_y u - f, h >_{H^{-1}(\mathcal{O}) \times H^1_0(\mathcal{O})} dx = \int_0^1 < \frac{\partial p}{\partial x}, h >_{H^{-1}(\mathcal{O}) \times H^1_0(\mathcal{O})} dx$$
$$= -\int_0^1\!\!\int_{\mathcal{O}} p \frac{\partial h}{\partial x} dx\, dy.$$

Therefore, from optimality condition (8) we deduce that

$$\int_0^1 < -\Delta_y u - f, h >_{H^{-1}(\mathcal{O}) \times H^1_0(\mathcal{O})} dx \;+\; \int_0^1\!\!\int_{\mathcal{O}} \frac{\partial u}{\partial x}\frac{\partial h}{\partial x} dx\, dy =$$
$$\int_0^1\!\!\int_{\mathcal{O}} (-p + \frac{\partial u}{\partial x})\, u\, dx\, dy \;=\; 0, \qquad \forall\, u \in \mathcal{U}_{ad}. \qquad (9)$$

Then we have obtained the optimality system

$$\begin{cases} -\dfrac{\partial u}{\partial x} = -p, & u(1) = u_1, \\ \dfrac{\partial p}{\partial x} = -\Delta_y u - f, & p(0) = 0, \end{cases}$$

which has the same associated Riccati equation (see Section 4 of chapter III of [4]) that the system of equations for Q and w of Section 2.

5. Representation formula for the solution of the Riccati equation

Let $X(x) \in \mathcal{L}(H_{00}^{1/2}(\mathcal{O}), H_{00}^{1/2}(\mathcal{O})')$, $Y(x) \in \mathcal{L}(H_{00}^{1/2}(\mathcal{O}), H_{00}^{1/2}(\mathcal{O}))$ denote

$$\begin{array}{llll} X(x): & u(1) & \longrightarrow & \frac{du}{dx}(x), \\ Y(x): & u(1) & \longrightarrow & u(x), \end{array}$$

where u is solution of $(\mathcal{P}_0)$ assuming $f = 0$ and $u_0 = 0$. They satisfy $(' = \frac{d}{dx})$

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 & -\Delta_y \\ I & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

with $Y(1) = I$ and $X(0) = 0$. Furhtermore Q such that $Q(x) = X(x)Y(x)^{-1}$ satisfies the differential Riccati equation $Q' + Q^2 = -\Delta_y$.

Let P_0 be the positive solution of the algebraic Riccati equation

$$-P_0 \Delta_y P_0 = I, \qquad P_0 = (-\Delta_y)^{-1/2}$$

By the change of variable

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} I & -P_0^{-1} \\ P_0 & I \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$$

equations for X and Y are diagonalized in

$$\begin{pmatrix} \Phi' \\ \Psi' \end{pmatrix} = \begin{pmatrix} P_0^{-1} & 0 \\ 0 & -P_0^{-1} \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$$

$$W = \Phi\Psi^{-1} = (X + P_0^{-1}Y)(Y + P_0 X)^{-1} = (Q + P_0^{-1})(I - P_0 Q)^{-1} = \mathcal{H}(Q)$$

satisfies the linear equation

$$W' = P_0^{-1} W + W P_0^{-1}$$

with $W(0) = \mathcal{H}(Q(0)) = \mathcal{H}(0) = P_0^{-1}$

$$\begin{array}{ccc} Q(0) & \longrightarrow & Q(x) \\ \downarrow \mathcal{H} & & \uparrow \mathcal{H}^{-1} \\ W(0) & \longrightarrow & W(x) \end{array}$$

$$0 \leq W \leq P_0^{-1} \Rightarrow Q = (W - P_0^{-1})(I + P_0 W)^{-1}$$

is well defined.

5.1. Dirichlet boundary condition at $x = 0$

It corresponds to a singularity of

$$Q(0) = \mathcal{H}^{-1}(W(0)) = (W(0) - P_0^{-1})(I + P_0 W(0))^{-1}$$

i.e. $W(0) = -P_0^{-1}$.

$$0 > W > -P_0^{-1} \text{ for } 0 < x \leq 1 \quad \Rightarrow \quad Q = (W - P_0^{-1})(I + P_0 W)^{-1}$$

is well defined for $0 < x \leq 1$.

6. The QR factorization

As the factorization (3) is viewed as an infinite dimensional generalization of the LU block triangular factorization, we now turn to the QR factorization, i.e. as the product of an orthogonal operator Q and an upper triangular part R. We begin by writing the normal equation for $(\mathcal{P}_0)$. We assume that f is regular enough and is null in a neighbourhood of Σ in order to avoid non-homogeneous boundary conditions.

$$\left\{ \begin{array}{l} \Delta^2 u = -\Delta f \quad \text{in } \Omega, \\ u|_\Sigma = \Delta u|_\Sigma = 0, \quad -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0, \quad \dfrac{\partial \Delta u}{\partial x}|_{\Gamma_0} = -\dfrac{\partial f}{\partial x}|_{\Gamma_0} \\ u|_{\Gamma_1} = u_1, \quad \Delta u|_{\Gamma_1} = -f|_{\Gamma_1}. \end{array} \right. \tag{10}$$

Of course $(\mathcal{P}_0)$ and (10) have the same solution. Similarly to section 2 we embed (10) in a family defined on Ω_s with additional boundary condition on Γ_s :

$$u|_{\Gamma_s} = h, \quad \Delta u|_{\Gamma_s} = k,$$

and we set

$$\frac{\partial u}{\partial x}|_{\Gamma_s} = Q(s)h + P(s)k + r(s). \tag{11}$$

If we choose $k = -f$, u is the solution of $(\mathcal{P}_{s,h})$, then Q (which should not be confused with the orthogonal part of the factorization) is the same operator as the one defined in section 2 and it satisfies (2). We also deduce that $r - Pf = w$ satisfies (2). Furthermore, from (10), Δu can be viewed as satisfying $(\mathcal{P}_{s,h})$ with right hand side Δf. Hence it admits the following factorization

$$\begin{cases} \dfrac{dt}{dx} + Qt = -\Delta f, & t(0) = -\frac{\partial f}{\partial x}|_{\Gamma_0}, \\ -\dfrac{d\Delta u}{dx} + Q\Delta u = -t, & u(1) = -f(1). \end{cases} \tag{12}$$

Deriving (11) along a trajectory $u(x)$ and by identification we get for Q and P

$$\frac{dQ}{dx} + Q^2 + \Delta_y = 0, \quad Q(0) = 0, \tag{13}$$

$$\frac{dP}{dx} + PQ + QP = I, \quad P(0) = 0. \tag{14}$$

Both operators P and Q are self-adjoint and positive for $x > 0$. For that particular problem they commute with Δ_y. The term independent of u and Δu yields

$$Pt + \frac{dr}{dx} + Qr = 0.$$

Then, using (12), we get the decoupled form of (10) as

$$P^{-1}\frac{d^2r}{dx^2} - \left(P^{-2} - 2P^{-1}Q - 2QP^{-1}\right)\frac{dr}{dx} \tag{15}$$
$$- \left(P^{-2} - 2QP^{-1}Q + P^{-1}\Delta_y\right) r = \Delta f,$$

$$r(0) = -u_0, \quad \frac{\partial r}{\partial x}(0) = 0, \tag{16}$$

where the initial conditions are derived from (11) and its derivative written on Γ_0 and

$$\frac{d^2u}{dx^2} - P^{-1}\frac{du}{dx} + \left(P^{-1}Q + \Delta_y\right) u = -P^{-1}r, \tag{17}$$

$$u(1) = u_1, \quad \frac{du}{dx}(1) = Q(1)u_1 - P(1)f(1) + P(1)r(1). \tag{18}$$

Let us denote $\mathcal{Q}$ the mapping $f \longrightarrow P^{-1}r$ defined by (15) from $Y = \{u \in L^2(\Omega) | \Delta u \in L^2(\Omega)\}$ into itself, and $\tilde{r} = P^{-1}r$. Then $\tilde{r}$ satisfies

$$\frac{d^2\tilde{r}}{dx^2} + P^{-1}\frac{d\tilde{r}}{dx} + \left(2QP^{-1} + P^{-1}Q - P^{-2} + \Delta_y\right)\tilde{r} = \Delta f. \tag{19}$$

Now it can be checked using (13) that the differential operators applied to $\tilde{r}$ and u in (19), (17) respectively are adjoint of each other. Then

$$\mathcal{Q}^\star\mathcal{Q} \ : f \longrightarrow -\Delta u,$$

and $\mathcal{Q}$ is orthogonal. Denoting $\mathcal{R}$ the operator defined by the Cauchy problem (17) ($\mathcal{R}u = \tilde{r}$), we have obtained the factorization of $(\mathcal{P}_0)$ as $\mathcal{Q}^\star\mathcal{R}$. It can be checked that $P^{-1}Q + \Delta_y$ is positive and (17) is an hyperbolic problem with damping.

7. Joint factorization for a control problem

7.1. Statement of the control problem and optimality system

Now we consider a control problem for an elliptic state equation. The state equation is

$$(\mathcal{P}_c)\begin{cases} -\Delta u = f + Bv \quad \text{in } \Omega, \\ u|_\Sigma = 0, \quad \dfrac{\partial u}{\partial x}|_{\Gamma_0} = -u_0, \quad \dfrac{\partial u}{\partial x}|_{\Gamma_1} = 0, \end{cases}$$

where the control v lies in a Hilbert space V identified to its dual and $B \in \mathcal{L}(V; L^2(\Omega))$. Define the cost function

$$J(v) = \int_{\Gamma_1} |u|_{\Gamma_1} - u_d|^2\, dy + \nu \|v\|_V^2 .$$

Then defining the adjoint state p by

$$\begin{cases} -\Delta p = 0 \quad \text{in } \Omega, \\ p|_\Sigma = 0, \quad \dfrac{\partial p}{\partial x}|_{\Gamma_0} = 0, \quad \dfrac{\partial p}{\partial x}|_{\Gamma_1} = u|_{\Gamma_1} - u_d, \end{cases}$$

the minimum of J is characterized by $\overline{v} = -\frac{1}{\nu}B^\star p$.

7.2. Factorization of the optimality system

The factorization via dynamic programming applied to the state equation in the previous section can be applied to the coupled system of state and adjoint state equations. Setting $u|_{\Gamma_s} = \varphi$ and $p|_{\Gamma_s} = \psi$ we define a family of problems depending on s, φ, ψ by

$$\begin{cases} -\Delta u = f - \dfrac{1}{\nu}BB^\star p \quad \text{in } \Omega_s, \\ u|_\Sigma = 0, \quad \dfrac{\partial u}{\partial x}|_{\Gamma_0} = 0, \quad u|_{\Gamma_s} = \varphi, \\ -\Delta p = 0 \quad \text{in } \Omega_s, \\ p|_\Sigma = 0, \quad \dfrac{\partial p}{\partial x}|_{\Gamma_0} = 0, \quad p|_{\Gamma_s} = \psi. \end{cases} \tag{20}$$

Then the mapping $(\varphi, \psi) \to (u, p)$ is affine but p depends linearly on ψ and not on φ. Furthermore the mapping $\psi \to \frac{\partial p}{\partial x}|_{\Gamma_s}$ as well as the mapping $\varphi \to \frac{\partial u}{\partial x}|_{\Gamma_s}$ if $\psi = 0$ are exactly the DtN mapping Q satisfying (2). So there exists a linear mapping $\overline{P}(s)$ such that

$$\begin{aligned}\frac{\partial u}{\partial x}|_{\Gamma_s} &= \overline{P}(s)\psi + Q(s)\varphi + w(s),\\ \frac{\partial p}{\partial x}|_{\Gamma_s} &= Q(s)\psi.\end{aligned} \tag{21}$$

The equation satisfied by $\overline{P}$ is obtained in a similar fashion. Let us derive the first equation (21) along a solution of (20) and substituting the derivatives of u and p from (21)

$$\frac{\partial^2 u}{\partial x^2} = \frac{d\overline{P}}{dx}p + \overline{P}Qp + \frac{dQ}{dx}u + Q(\overline{P}p + Qu + w) + \frac{dw}{dx} = -\Delta_y u - f + \frac{1}{\nu}BB^\star p.$$

Using the equation for Q from (2) and the fact that p is arbitrary we obtain the equation for $\overline{P}$

$$\frac{d\overline{P}}{dx} + \overline{P}Q + Q\overline{P} = \frac{1}{\nu}BB^\star \quad \overline{P}(0) = 0. \tag{22}$$

$$\frac{dw}{dx} + Qw = -f, \quad w(0) = -u_0 \tag{23}$$

Knowing Q from (2), equation (22) is linear. One can show that it is well posed and its solution is self-adjoint.

Now, equations for Q, $\overline{P}$ and w being integrated once for all, for any new measurement u_d the optimal control $\overline{v}$ is obtained in the following way

- find an initial condition for p at $x = 1$ from the system

$$\begin{aligned}\overline{P}(1)p(1) + Q(1)u(1) + w(1) &= 0,\\ Q(1)p(1) &= u(1) - u_d,\end{aligned}$$

  which gives $p(1) = -(\overline{P}(1) + Q(1)^2)^{-1}(Q(1)u_d + w(1))$.
- integrate the equation for p backwards from Γ_1 to Γ_0 $\frac{dp}{dx} - Qp = 0$
- the optimal control $\overline{v}$ is then given by $\overline{v} = -\frac{1}{\nu}B^\star p$.

References

[1] Angel E., Bellman R. *Dynamic Programming and Partial Differential Equations* Academic Press 1972.

[2] Henry J., Ramos A. *Factorization of Second Order Elliptic Boundary Value Problems by Dynamic Programming*, submitted to Journal Math. Ana. Appl.

[3] Henry, J., Ramos, A. *A Direct Study in a Hilbert-Schmidt Framework of the Riccati Equation Appearing in a Factorization Method of Second Order Elliptic Boundary Value Problems* Rapport de Recherche INRIA 4451.

[4] Lions, J.L. (1968). *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles.* Dunod.

[5] Lions, J.L., and Magenes E. (1968). *Problèmes aux Limites Non Homogènes et Applications*, vol 1. Dunod.