COMPACTNESS AND LONG-TIME STABILIZATION OF SOLUTIONS TO PHASE-FIELD MODELS

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Abstract The compactness of trajectories of solutions to various phase-field models is proved. In some cases, the convergence of any strong solution to a single stationary state is also established.

1. Introduction

The aim of this note is to survey results on convergence of solutions of phase-field models to the stationary states, using a generalization of the Lojasiewicz theorem. We will consider models proposed by Caginalp [5], where the time evolution of the phase variable $\chi(t,x)$ and the temperature $\vartheta(t,x)$ is governed by the system of differential equations:

$$\tau \partial_t \chi = w, \tag{1}$$

$$\partial_t \Big(\vartheta + \lambda(\chi) \Big) + \operatorname{div} \mathbf{q} = 0, \tag{2}$$

where the so called chemical potential w is given by

$$w = \xi^2 \Delta \chi - W'(\chi) + \lambda'(\chi)\vartheta$$

W and λ are given functions, W is typically a double-well potential, and $\mathbf{q}$ denotes the heat flux. We shall also consider the conserved phase-field model, where (1) is replaced by

*The work of this author was supported by Grant A1019002 of GA AV ČR

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: 10.1007/978-0-387-35690-7_44
V. Barbu et al. (eds.), *Analysis and Optimization of Differential Systems*

$$\tau \partial_t \chi = -\xi^2 \Delta w \tag{3}$$

We shall treat the classical case

$$\mathbf{q} = -k_I \nabla \vartheta$$

and also the linearized Coleman–Gurtin [6] constitutive relation, where $\mathbf{q}$ is determined by

$$\mathbf{q} = -k_I \nabla \vartheta - k * \nabla \vartheta, \tag{4}$$

involving the memory effects, where the constant $k_I > 0$ is the instantaneous heat conductivity, k is a suitable dissipative kernel, and the symbol $*$ denotes the time convolution:

$$k * v(t) = \int_0^\infty k(s) v(t-s)\, \mathrm{d}s.$$

The material occupies a bounded regular domain $\Omega \subset R^3$ and the system (1)–(2) is complemented by the homogeneous Neumann boundary condition for χ, while ϑ obeys the homogeneous Dirichlet condition.

$$\nabla \chi \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \vartheta|_{\partial\Omega} = 0. \tag{5}$$

In the conserved model, we require Neumann boundary conditions for both χ, ϑ and also for the chemical potential w which can be expressed by

$$\nabla \vartheta \cdot \mathbf{n}|_{\partial\Omega} = \nabla \chi \cdot \mathbf{n}|_{\partial\Omega} = \nabla(\Delta \chi) \cdot \mathbf{n}|_{\partial\Omega} \quad \text{with } \mathbf{n} \text{ the outer normal vector.} \tag{6}$$

For the sake of simplicity, we set the constants τ, ξ, representing a relaxation time and a correlation length, respectively, equal to 1.

Systems of the same or similar type have been recently studied by many authors.(see Colli et al. [7], Giorgi et al. [11], Novick-Cohen [17] etc). The questions of well-posedness and existence of finite dimensional attractors for the conserved model were discussed by Grasselli et al. [12], and the dissipativity of the respective system was studied by Vegni [20]. In particular, the long-time behavior of solutions seems to be well understood and the equilibrium (stationary) solutions have been identified as the only candidates to belong to the $\omega-$limit set of each individual trajectory. If the stationary problem admits only a finite number of solutions, then any solution $\chi(t)$, $\vartheta(t)$ converges as $t \to \infty$ to a single stationary state. However, the structure of the set of stationary solutions for a general domain may be quite complicated, in particular, the set in question may contain a continuum of nonradial solutions if Ω is a ball or an annulus. If this is the case, it seems highly nontrivial to decide whether or not the solutions converge to a single stationary state. It is well-known that the convergence of any trajectory might not happen even for finite-dimensional dynamical systems (cf. Aulbach [4]), and similar examples for semilinear parabolic equations were derived by Poláčik and Rybakowski [18]. In 1983, Simon [19] developed a method to study the long-time behaviour of gradient-like dynamical systems based on deep results from the theory of analytic functions of several variables due to Lojasiewicz [16]. Roughly speaking, an analytic function behaves like a polynomial (of a sufficiently high degree) in a neighbourhood of any point where its gradient vanishes (critical points). More specifically, the following assertion holds (see [16, Theorem 4, page 88]):

Proposition 1.1 *Let $G : U(a) \to C$ be a real analytic function defined on an open neighbourhood $U(a)$ of a point $a \in R^n$. Then there exist $\theta \in (0, \frac{1}{2})$ and $\delta > 0$ such that*

$$|\nabla G(z)| \geq |G(z) - G(a)|^{1-\theta} \text{ for all } z \in R^n, \ |z - a| < \delta.$$

L. Simon succeeded in proving a generalized version of the above theorem applicable to analytic functionals on Banach spaces. Later on, Jendoubi [15], and Haraux and Jendoubi [13] simplified considerably Simon's original approach making it accessible for application to a broad class of semilinear problems with a variational structure. Related results in this direction were also obtained by Feireisl and Takáč [10], Hoffmann and Rybka [14] etc.

In some cases, Simon's approach can be used to deal with problems with only a partial variational structure. A typical example could be the system(1), (2) with the memory term omitted in (4) (i.e., for $k = 0$). Indeed the "elliptic" part of (1) is the variational derivative of the free energy functional with respect to χ while (2) is not. Since the temperature tends to zero when time is large, it is possible to modify Simon's method to prove convergence of the phase variable χ to a single stationary state, i.e. a solution of the problem

$$\Delta\chi_\infty - W'(\chi_\infty) = 0, \quad \nabla\chi_\infty \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \vartheta_\infty = 0, \tag{7}$$

under fairly general conditions imposed on λ and W. In the conserved case, the temperature satisfying the Neumann boundary conditions (6) also tends to zero provided that it has zero mean. Considering the model (3), (2), with λ linear and the boundary conditions (6), the quantities $\int_\Omega \chi(t)\mathrm{d}x$ and $\int_\Omega \vartheta(t)\mathrm{d}x$, are conserved, so we can normalize the initial functions $\chi(0), \vartheta(0)$ such that $\int_\Omega \chi(0)\mathrm{d}x = 0$, $\int_\Omega \vartheta(0)\mathrm{d}x = 0$ which leads to the convergence $\chi \to \chi_\infty$, $\vartheta \to \vartheta_\infty$, where

$$\Delta(\Delta\chi_\infty - W'(\chi_\infty)) = 0, \ \nabla\chi_\infty \cdot \mathbf{n}|_{\partial\Omega} = \nabla(\Delta\chi_\infty) \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \vartheta_\infty = 0, \tag{8}$$

or $\vartheta_\infty = const$ in the general case.

Using the *summed past history* of ϑ, introduced by Dafermos, we can obtain similar results when the memory effects are taken into account in (4), both for the conserved and non-conserved models.

We conclude this contribution with *a priori* estimates that imply compactness of trajectories of solutions of the most general conserved model with memory term and a nonlinear λ. These estimates can be used to prove the existence of strong global solutions emanating from sufficiently smooth data as well as the convergence of solutions satisfying the Poincaré inequality. The existence of global weak solutions vanishing on a nontrivial (positive measure) part of the boundary of Ω and, therefore, satisfying the Poincaré inequality is stated in [12].

2. Main results

In this section, we present a synthesis of some convergence results from the papers [1], [2], [3], [8].

Theorem 2.1 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\mu}$, $\mu > 0$. Suppose, moreover, that the nonlinearities λ, W satisfy the following hypotheses:*

$$\begin{array}{l} \textit{The function } \lambda \textit{ is of class } C^{1+\mu}(R),\ \lambda(0)=0,\ |\lambda'(z)|\leq\Lambda,\ z\in R; \\ \textit{The "free energy" function } W \textit{ is real analytic on } R. \end{array} \tag{9}$$

In addition, we assume that the instantaneous heat conductivity $k_I > 0$ is strictly positive and the kernel k satisfies:

$$\begin{array}{l} k \in L^1(0,\infty),\ k \textit{ is convex on } (0,\infty), \\ \mathrm{d}k'(s) + \delta k'(s)\ \mathrm{d}s \geq 0 \textit{ for a certain } \delta > 0. \end{array} \tag{10}$$

Let χ, ϑ be a globally defined strong solution of the problem (1), (2), (4), (5) *such that*

$$\sup_{t>0}\Big(\sup_{x\in\Omega}(|\chi(t,x)| + |\vartheta(t,x)|\Big) < \infty. \tag{11}$$

Then there exists χ_∞ - a solution of the stationary problem (7) *such that*

$$\chi(t) \to \chi_\infty,\ \vartheta(t) \to 0 \textit{ in } C(\overline{\Omega}) \textit{ as } t \to \infty.$$

Theorem 2.2 *Let the assumptions of the Theorem 1 be satisfied and*

$$\lambda(z) = cz \quad \textit{for some real} \quad c.$$

Let χ, ϑ *be a globally defined strong solution of the problem* (3), (2), (4), (6) *satisfying* (11). *Then there exists* χ_∞ *- a solution of the stationary problem* (8) *such that*

$$\chi(t) \to \chi_\infty,\ \vartheta(t) \to const\ in\ C(\overline{\Omega})\ as\ t \to \infty.$$

In the proof, using the fact that the integral means $\int_\Omega \chi(t,x)\mathrm{d}x$, $\int_\Omega \vartheta(t,x)\mathrm{d}x$ are conserved quantities, we normalize the initial functions $\chi(0)$, $\vartheta(0)$ to be of zero mean and work in the corresponding spaces, where the solution of the problem

$$-\Delta v = g\ (in)\ \Omega,\quad \nabla v\cdot \mathbf{n} = 0 \text{ on } \partial\Omega,\quad \int_\Omega v\mathrm{d}x = 0$$

is uniquely defined and denoted by $v = -\Delta_N^{-1}[g]$. In this space, the Poincaré inequality takes place. See [8] for the proof of Theorem 2.2 when $k = 0$ and [3] when the memory is included.

Remark 1. Here a globally defined strong solution means that χ_t, ϑ_t, $D_x^k\chi$, $D_x^2\vartheta$ are in the space $L_{loc}^r\Big(0,\infty; L^2(\Omega)\Big)$ for any $r \geq 1$, and the boundary conditions are satisfied for all $t \in (0,\infty)$. Moreover, $\chi(0)$ is supposed to belong to $W^{2,2}(\Omega)$ and the past values of ϑ are given for $t \in (-\infty, 0]$, and $\|\vartheta(t)\|_{W^{2,2}(\Omega)}$ are bounded uniformly for $t \in (-\infty, 0]$ and satisfy the boundary conditions (6) when we treat the conserved problem with memory.

Remark 2. A typical example of a kernel k satisfying (10) is $k(s) = s^{-\alpha}e^{-\beta s}$, $0 \leq \alpha < 1$, $\beta > 0$.

Remark 3. The assumption that χ, ϑ is a strong solution of the problem is not restrictive. It will be clear from the estimates presented in Section 3 that any weak solution emanating from smooth initial data will be globally defined and regular on the interval $(0,\infty)$. Moreover, those estimates also allow for more general energy functionals W than the ones considered in Grasselli, Pata and Vegni [12] and Vegni [20].

Sketch of proofs.

First, we derive necessary *a priori* estimates to show that the trajectories $\cup_{t\geq 1}\vartheta(t)$, $\cup_{t\geq 1}\chi(t)$ are precompact in $C(\overline{\Omega})$ and $\vartheta(t) \to 0$. From the strong maximum principle we deduce that the $\omega-$limit set $\omega[\chi]$ is contained in some interval $[-L, L]$. Accordingly, since we are interested in the $\omega-$limit set of one particular trajectory which is uniformly bounded with respect to $\chi-$component, we are allowed to suppose, without loss of generality, that W' has been modified outside of the interval $[-2L, 2L]$ in such a way that

$$W'(z) \text{ is real analytic on } (-L, L); \tag{12}$$

$$|W''(z)|, |W'(z)| \text{ are uniformly bounded for } z \in R. \tag{13}$$

The next step is to show that $\chi_t \in L_1(T,\infty;X)$ where X denotes a suitable space. To this end, we apply Simon's method to the functional

$$I(v) = \int_\Omega \left(|\nabla v|^2 + W(v)\right) \mathrm{d}x \tag{14}$$

to obtain the following generalization of Proposition 1.1.

Proposition 2.1 *Let W satisfy the hypotheses* (12),(13). *Let $w \in W_N^{2,p}$,*

$$-L < w(x) < L \text{ for all } x \in \Omega.$$

Then for any $P>0$ there exist constants $\theta \in (0, 1/2)$, $M(P)$, $\varepsilon(P)$ such that

$$|I(v) - I(w)|^{1-\theta} \le M\| - \Delta v + W'(v)\|_{[W_N^{1,2}(\Omega)]^*} \tag{15}$$

holds for any $v \in W_N^{1,2}(\Omega)$ such that

$$\|v - w\|_{L^2(\Omega)} < \varepsilon, \ |I(v) - I(w)| < P. \tag{16}$$

The proof is identical with [9, Section 6, Proposition 6.1].

The energy equality, obtained by multiplying the equation (1) by χ_t, (2) by ϑ, ((3) by $-\Delta_N^{-1}[\chi_t]$ respectively), integrating the resulting expressions by parts and adding up, reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_\Omega (\frac{1}{2}|\nabla\chi(t)|^2 + \frac{1}{2}|\vartheta(t)|^2 + W(\chi(t))\mathrm{d}x + \frac{1}{2}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\mathrm{d}s\Big]$$
$$+\left\|\left((-\Delta)_N^{-\frac{1}{2}}\right)[\chi_t(t)]\right\|^2_{L^2(\Omega)} + k_I\|\nabla\vartheta(t)\|^2_{L^2(\Omega)} \tag{17}$$
$$+\frac{1}{2}\int_0^\infty \|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\ \mathrm{d}k'(s) = 0.$$

Here we used the summed past history of ϑ defined by

$$\eta(t,s,x) = \int_{t-s}^t \vartheta(z,x)\mathrm{d}z, \ s \ge 0,$$

and the relation

$$\int_\Omega (k * \vartheta)\vartheta \ \mathrm{d}x = \tag{18}$$

$$\frac{1}{2}\Big[\frac{d}{dt}\int_0^\infty (-k')(s)\|\eta(t,s)\|^2_{L^2(\Omega)}\ \mathrm{d}s + \int_0^\infty (-k)'(s)\frac{\partial}{\partial s}\|\eta(t,s)\|^2_{L^2(\Omega)}\ \mathrm{d}s\Big].$$

Denoting by E the "total energy",

$$E(t) = \frac{1}{2}\int_\Omega |\nabla\chi(t)|^2 + 2W(\chi(t))\ \mathrm{d}x + \frac{1}{2}\|\vartheta\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\ \mathrm{d}s,$$

and taking into account (10), we have $E(t) \to E_\infty$ as $t \to \infty$. Moreover, we can prove that

$$\|\vartheta(t)\|_{L^2(\Omega)} \to 0 \text{ as } t \to \infty, \tag{19}$$

and

$$\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\ ds \to 0 \text{ as } t \to \infty. \tag{20}$$

We have

$$I(\chi(t)) \to I_\infty = E_\infty = \frac{1}{2}\int_\Omega |\nabla\chi_\infty|^2 + 2W(\chi_\infty)\ \mathrm{d}x \text{ for any } \chi_\infty \in \omega[\chi].$$

In particular, the energy of solutions $\chi_\infty \in \omega[\chi]$ equals the same constant I_∞.

We integrate (17) with respect to t, and make use of Proposition 2.1 to conclude that there exists $T > 0$ such that

$$\int_T^\infty \|\chi_t(t)\|_X \mathrm{d}t < \infty,$$

which, together with the compactness of trajectories yields the assertions of Theorems 2.1 and 2.2.

3. A priori estimates

In this section, we prove *a priori* estimates for solutions of the problem (3), (2), (4) with a nonlinear function λ satisfying

$$\lambda \in C^3(R),\quad |\lambda'(r)| \le \Lambda,\quad |\lambda''(r)| \le \Lambda \text{ for a certain } \Lambda > 0 \tag{21}$$

The free energy functional $W : R \mapsto R$ will be supposed to satisfy the following hypotheses:

- $$W(z) \ge 0 \text{ for all } z \ge 0, \tag{22}$$
- $$W'(z)z > 0 \text{ for } |z| > 1, \tag{23}$$
- $$W'(z)z \ge c_1 W(z) - c_2 \text{ for all } z \in R, \tag{24}$$

- $$W''(z) \geq -c_3, \tag{25}$$

- $$W \in C^{3+\mu}(R),\ |W'(z)| \leq c_4(1+|z|^{p-1}),\ p<5. \tag{26}$$

The energy estimate (17) gives

Lemma 3.1 *Under the hypotheses of Theorem 2.2, there exists E_0 depending only on the quantities*

$$\sup_{t\in(-\infty,0]} \|\nabla\vartheta(t)\|_{L^2(\Omega)},\ \|\nabla\chi(0)\|_{L^2(\Omega)},\ \|\chi(0)\|_{L^\infty(\Omega)}$$

such that

$$\sup_{t>0}\|\vartheta(t)\|_{L^2(\Omega)} + \sup_{t>0}\|\nabla\chi(t)\|_{L^2(\Omega)} \leq E_0, \tag{27}$$

$$\int_0^\infty \|(-\Delta_N)^{-\frac{1}{2}}[\chi_t(t)]\|^2_{L^2(\Omega)} + \|\nabla\vartheta(t)\|^2 s\mathrm{d}t \leq E_0 \tag{28}$$

Next, we multiply (3) by χ to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\chi\|^2_{L^2(\Omega)} + \|\Delta\chi\|^2_{L^2(\Omega)} + \int_\Omega W''(\chi)|\nabla\chi|^2\mathrm{d}x = -\int_\Omega \lambda'(\chi)\vartheta\Delta\chi\mathrm{d}x,$$

Consequently, (27), (28), (25) and the Poincaré and Young inequalities imply

$$\int_t^{t+1} \|\Delta\chi\|^2_{L^2(\Omega)}\ \mathrm{d}\tau \leq E_0 \ \text{ for any } t\geq 0, \tag{29}$$

To improve the estimates on χ, we write (3) as an evolutionary equation

$$\frac{\partial\chi}{\partial t} + \Delta^2\chi = \Delta[W'(\chi)] - \Delta[\lambda'(\chi)\vartheta]. \tag{30}$$

Let p be as in (26). We prove first that

$$\chi \in L^r(t,t+1;W^{2,q_1}(\Omega)),\ t\geq 0,\ \text{for any } 1\leq r<\infty,\ q_1=\min\{2,6/p\}.$$

For this, for all $1<q<\infty$, we define a linear operator $\Delta_{N,q}$ on the Banach space $L^q(\Omega)$ by

$$\mathcal{D}(\Delta_{N,q}) = \{v\in W^{2,q}(\Omega)\mid \nabla v\cdot\vec{n}=0 \text{ on } \partial\Omega\},\ \Delta_N v=\Delta v,$$

and rewrite (30) in the abstract form

$$\chi_t + \Delta^2_{N,q}.$$

From (27), (21), we know that h_2 is bounded in $L^\infty(t, t+1; [W^{2,2}(\Omega)]^*)$ uniformly for all $t \geq 0$. On the other hand, using (27) and the Sobolev embedding $W^{1,2}(\Omega) \subset L^6(\Omega)$, we have $\chi \in L^\infty(0, \tau; L^6(\Omega))$ for all $\tau > 0$. From (26) we get $W'(\chi) \in L^\infty(0, \tau; L^{\frac{6}{p}}(\Omega))$.

Also, $\Delta_{N,q}^{-1}(\Delta_{N,q})f = f - \int_\Omega f(x)\mathrm{d}x$ for $f \in L^q(\Omega)$. Hence

$$\|h\|_{\mathcal{D}(\Delta_{N,q}^{-1})} = \|[W'(\chi) - \lambda'(\chi)\vartheta] - \frac{1}{|\Omega|}\int_\Omega [W'(\chi) - \lambda'(\chi)\vartheta]\ \mathrm{d}x\|_{L^q(\Omega)}$$
$$\leq C\Big(\|W'(\chi)\|_{L^q(\Omega)} + \|\vartheta\|_{L^q(\Omega)}\Big).$$

This implies that $\chi \in L^r(t, t+1; W^{2,q_1}(\Omega))$ where $q_1 = \min\{2, \frac{6}{p}\}$, $r \geq 1$. Consequently, by the Sobolev embedding theorem.

$$\chi \in L^r(t, t+1; L^{q_2}(\Omega)) \ \text{ with } \ q_2 = \frac{3q_1}{3 - 2q_1} \text{ if } 2q_1 < 3, \ q_2 = \infty \text{ otherwise.}$$

Next we argue by induction (bootstrap argument). We deduce from (26) that

$$W'(\chi) \in L^{\frac{r}{p}}(t, t+1; L^{\frac{q_2}{p}}(\Omega)).$$

Remark that we have

$$\frac{q_2}{p} - q_1 \geq \frac{6}{p(p-4)} - \frac{6}{p} > 0$$

provided $p \in (4, 5)$, $q_2 = \infty$ if $p \leq 4$. Hence, after a finite number of steps we arrive at the estimate

$$\chi \in L^r(t, t+1; W^{2,2}(\Omega)) \subset L^r(t, t+1; L^\infty(\Omega)), \ t \geq 0 \tag{31}$$

for any $1 \leq r < \infty$. Also, $\chi_t \in L^r(t, t+1; [W^{2,2}(\Omega)]^*)$ which implies

$$\chi \in C\Big(t, t+1; (W^{2,2}(\Omega), [W^{2,2}(\Omega)]^*)_\theta\Big),$$

with θ satisfying $\theta(1 - \frac{1}{r}) > \frac{1-\theta}{r}$, where $(X, Y)_\theta$ denotes the interpolation space. As $r > 1$ is arbitrary, we can choose θ such that $(W^{2,2}(\Omega), [W^{2,2}(\Omega)]^*)_\theta \hookrightarrow C(\overline{\Omega})$.

$$\sup_{t>0} \|\chi(t)\|_{C(\overline{\Omega})} \leq C_\infty. \tag{32}$$

This implies that $W''(\chi)$, $W'''(\chi)$ are bounded, and $\nabla\chi$ is bounded in $L^r(t, t+1; L^6(\Omega))$ for all r, independently of $t > 0$. Then

$$\int_t^{t+1} \|\Delta W'(\chi(s))\|^2_{L^2(\Omega)}\mathrm{d}s < C \ \text{ for all } t > 0. \tag{33}$$

Moreover, by (27), (28),

$$\vartheta \in L^\infty(t,t+1;L^2(\Omega)) \cap L^2(t,t+1;W^{1,2}(\Omega)) \hookrightarrow L^s(t,t+1;L^3(\Omega)) \text{ for } s<4.$$

By (31),

$$\nabla\chi \in L^r(t,t+1;L^6(\Omega)) \text{ for all } 1 \le r < \infty. \tag{34}$$

It follows that $\nabla(\lambda'(\chi)\vartheta) \in L^2(t,t+1;L^2(\Omega))$ and, by the same reasoning as above

$$\chi \in L^2(t,t+1;W^{3,2}(\Omega)),\ t \ge 0. \tag{35}$$

This yields

$$\lambda(\chi) \in L^2(t,t+1;W^{3,2}(\Omega)), \quad t \ge 0. \tag{36}$$

In fact, $\lambda'''(\chi)$ is bounded because of (32) and

$$\Delta\chi \in L^2(t,t+1;L^6(\Omega)) \cap L^r(t,t+1;L^2(\Omega)) \Rightarrow \Delta\chi \in L^s(t,t+1;L^3(\Omega)),\ s<4,$$

which, together with (34) gives

$$|\nabla\chi \cdot \Delta\chi| \in L^2(t,t+1;L^2(\Omega)), \quad |\nabla\chi|^3 \in L^r(t,t+1;L^2(\Omega)).$$

Now, we multiply (2) by $\Delta(\vartheta+\lambda(\chi))$, integrate by parts and use (18) with $\Delta\vartheta$ in place of ϑ, to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla(\vartheta+\lambda(\chi))\|^2_{L^2(\Omega)} + \int_0^\infty (-k')(s)\|\Delta\eta(t,s)\mathrm{d}s\|^2_{L^2(\Omega)}\Big]$$
$$+\|\Delta\vartheta\|^2_{L^2(\Omega)} + \int_\Omega \Delta\vartheta\Delta\lambda(\chi)\mathrm{d}x + \int_0^\infty \|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\mathrm{d}k'(s) =$$
$$\int_\Omega k * \nabla\vartheta \cdot \nabla\Delta\lambda(\chi)\mathrm{d}x.$$

If we denote

$$F(t) = \|\nabla(\vartheta+\lambda(\chi))\|^2_{L^2(\Omega)} + \int_0^\infty (-k')(s)\|\Delta\eta(t,s)\mathrm{d}s\|^2,$$

employ (10), the Poincaré and Young inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) + aF(t) \le C\Big(1 + \|\lambda(\chi(t))\|^2_{W^{3,2}(\Omega)} + k * \|\nabla\vartheta\|^2_{L^2(\Omega)}(t)\Big),$$

for some small $a > 0$, which yields the estimate

$$F(t) \le C\Big(1 + \sup_{t>0}\int_t^{t+1} \|\lambda(\chi(s))\|^2_{W^{3,2}(\Omega)} + k * \|\nabla\vartheta\|^2_{L^2(\Omega)}(s)\mathrm{d}s\Big). \tag{37}$$

We thereby arrive at

Lemma 3.2 *Under the hypotheses of Theorem* 2.2, *there exists* E_0 *depending only on the quantities* $\sup_{t\in(-\infty,0]} \|\Delta\vartheta(t)\|_{L^2(\Omega)}$, $\|\Delta\chi(0)\|_{L^2(\Omega)}$, *such that*

$$\sup_{t\geq 0} \|\nabla\vartheta(t)\|_{L^2(\Omega)} \leq E_0, \tag{38}$$

$$\int_t^{t+1} \|\Delta\vartheta(s)\|^2 \mathrm{d}s \leq E_0 \quad \text{for all } t \geq 0. \tag{39}$$

We continue the bootstraping, and use Lemma 3.2 together with (31) and (35) to deduce

$$\Delta[\lambda'(\chi)\vartheta] \in L^2(t, t+1; L^2(\Omega)),\ t \geq 0. \tag{40}$$

By virtue of (33), (40), the phase field variable χ satisfies the equation (30) with the right hand side bounded in $L^2(t, t+1; L^2(\Omega))$ independently of t. Therefore, we obtain

$$\chi \in L^2(t, t+1; W^{4,2}(\Omega)),\ \chi_t \in L^2(t, t+1; L^2(\Omega)),\ t > 1. \tag{41}$$

The convolution term in (2) is bounded in $L^2(\Omega)$ according to (10), (39), and $\Delta\lambda(\chi)$ is bounded in $L^r(t, t+1; L^2(\Omega))$ for any r independently of $t > 0$ by (31), (33). Hence the equation (2) can be written as a parabolic equation for $e = \vartheta + \lambda(\chi)$

$$e_t - \Delta e = -\Delta\lambda(\chi) + k * \Delta\vartheta$$

with the right-hand side bounded in $L^r(t, t+1; L^2(\Omega))$ for any $r \geq 1$ and any $t > 0$. This gives $\Delta\vartheta \in L^r(t, t+1; L^2(\Omega))$, and, consequently

$$\|\chi_t\|_{L^r(t,t+1;L^2(\Omega))} \leq C, \quad \text{for any } \ r \geq 1,\ t > 0, \text{ and then also}$$

$$\|\vartheta_t\|_{L^r(t,t+1;L^2(\Omega))} \leq C, \quad \text{for any } \ r \geq 1,\ t > 0$$

In particular we have obtained the following result:

Proposition 3.1 *Let* $\Omega \subset R^N$, $N \leq 3$ *be a bounded domain with boundary of class* $C^{2+\mu}$, $\mu > 0$. *Let* λ, $W \in C^{3+\mu}(R)$ *satisfy hypotheses* (21)–(26). *Then for any strong solution* χ, ϑ *of the problem* (3), (2), (4), (6) *on the time interval* $(0, \infty)$, *the trajectories* $\bigcup_{t\geq 1} \chi(t)$, $\bigcup_{t\geq 1} \vartheta(t)$ *are precompact in the space* $C(\overline{\Omega})$. *Moreover,* $\bigcup_{t\geq 1} \chi(t)$ *is precompact in* $W^{1,2}(\Omega)$.

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