RADIAL SYMMETRY OF CLASSICAL SOLUTIONS FOR BELLMAN EQUATIONS IN ERGODIC CONTROL

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1 INTRODUCTION

We are concerned with the d-dimensional Bellman equation of the form:

$$\lambda = \frac{1}{2}\Delta\phi(x) + F(D\phi(x)) + h(x), \qquad x \in \mathbf{R^d}, \tag{1.1}$$

where

$$F(\xi) = \min\{(\xi, p) : |p| \leq 1\} = -|\xi|, \tag{1.2}$$

and $|\cdot|, (,)$, and D denote the norm, the inner product of vectors, and the gradient respectively. We are given a convex function $h(x)$ with polynomial growth, and the unknown is the pair of a constant λ and a C^2-function $\phi(x)$ on $\mathbf{R^d}$.

The aim of the present paper is to study Bellman equation (1) without Lyapunov-type stability conditions and, in particular, the radial symmetry of $\phi(x)$ in the case that

$$h(x) \text{ is radial, i.e., } h(x) = f(r) \text{ for } r := |x|. \tag{1.3}$$

By the vanishing discount approach, we can show the existence of a unique classical solution of (1), for which the convexity and the polynomial growth property play essential roles. Further, for the radial symmetry, Bellman equation (1) turns out a 1-dimesional simple form which admits an explicit solution. It is shown that the radial symmetry of the Bellman equation is inherited from $h(x)$, and the optimal control is given by a feedback law $-x/|x|$.

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2 BELLMAN EQUATIONS OF DISCOUNTED COST CONTROL

We study the existence of a unique solution u_α with polynomial growth to the Bellman equation:

$$\alpha u_\alpha(x) = \frac{1}{2}\Delta u_\alpha(x) + F(Du_\alpha(x)) + h(x), \qquad x \in \mathbf{R^d}, \tag{2.1}$$

where $0 < \alpha < 1/2$. We assume:

$$h : \text{non-negative, convex on } \mathbf{R^d}, \tag{2.2}$$

h satisfies the polynomial growth condition, i.e.,

$$\exists C > 0, m \in \mathbf{N}_+; \quad h(x) \leq C(1 + |x|^m), \quad x \in \mathbf{R^d}, \tag{2.3}$$

$$h \in C^1(\mathbf{R^d}). \tag{2.4}$$

To simplify the notation, we make use of the following quantity:

$$[f]_{\delta,B_r} = \sup_{x\in B_r} |f(x)| + \sup_{x,y\in B_r\ x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\delta},$$

where $f(x)$ is the bounded Hölder continuous function with exponent δ on a ball $B_r = B_r(0)$ of $\mathbf{R^d}$. Let $t_n \in C_c^\infty(\mathbf{R^d})$ be a sequence such that $t_n = 1$ on B_n, 0 outside B_{2n} and $0 \leq t_n \leq 1, |Dt_n| \leq C/n$, and set $h_n = t_n h \in C_c^1(\mathbf{R^d})$. It is clear that $h_n \to h$ and $h_n \leq h$.

Now, let us consider the Bellman equation:

$$\alpha u_n(x) = \frac{1}{2}\Delta u_n(x) + F(Du_n(x)) + h_n(x), \qquad x \in \mathbf{R^d}. \tag{2.5}$$

Theorem 2.1. *Under (5), (6) and (7), equation (8) admits a unique solution* $u_n \in C_0(\mathbf{R^d}) \cap C^2(\mathbf{R^d})$, *which satisfies*

$$\sup_n [u_n]_{\delta,B_r} < \infty, \tag{2.6}$$

$$\sup_n \sum_i [D_i u_n]_{\delta,B_r} < \infty, \tag{2.7}$$

$$\sup_n \sum_{i,j} [D_{ij} u_n]_{\delta,B_r} < \infty, \tag{2.8}$$

for some $0 < \delta < 1$.

Proof. We shall give a brief sketch of the proof. It is well known [2] that, for every $n \in \mathbf{N}_+$, equation (8) has a unique solution u_n of the form:

$$u_n(x) = \inf\{E[\int_0^\infty e^{-\alpha t} h_n(x(t))dt] : |p(t)| \leq 1\}, \tag{2.9}$$

where $x(t)$ is a solution of the stochastic differential equation

$$dx(t) = p(t)dt + dw(t), \qquad x(0) = x \in \mathbf{R^d},$$

defined on some probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\})$ carrying a d-dimensional standard Brownian motion $w(t)$, and the infimum is taken over the class of all $\mathcal{F}_t$-progressively measurable processes $p(t)$ with $|p(t)| \leq 1$.

Multiplying both sides of (8) by $u_n^\nu \Theta \geq 0$ for $\nu \geq 1$ and integrating over $\mathbf{R^d}$, we have

$$2\alpha \int u_n^{\nu+1} \Theta dx + \int |Du_n|^2 \nu u_n^{\nu-1} \Theta dx \leq 2 \int h_n u_n^\nu \Theta dx, \qquad (2.10)$$

where $\Theta(x) = e^{-\theta|x|}$ and $0 < \theta < 2$. By (13) and Hölder's inequality,

$$\alpha(\int u_n^{\nu+1} \Theta dx)^{1/(\nu+1)} \leq (\int |h|^{\nu+1} \Theta dx)^{1/(\nu+1)}.$$

Taking $\nu = 1$ in (13), we get

$$\sup_n \int (u_n^2 + |Du_n|^2) \Theta dx < \infty,$$

and thus,

$$\sup_n (|u_n|_{L^2(B_r)} + |Du_n|_{L^2(B_r)}) < \infty \quad \text{for each } r > 0.$$

By the regularity result [5, Thm 8.8, p.183], there exists $C > 0$ such that

$$|u_n|_{W^{2,2}(B_r)} \leq C(|u_n|_{W^{1,2}(B_{r+1})} + |\Delta u_n|_{L^2(B_{r+1})}).$$

Therefore, we get by (8)

$$\sup_n |u_n|_{W^{2,2}(B_r)} < \infty.$$

By the Sobolev inequality [4, Thm IX.16, p.169], assuming $d > 2$, we have

$$\sup_n |Du_n|_{L^q(B_{r+1})} \leq \sup_n C|Du_n|_{W^{1,2}(B_{r+1})} < \infty, \qquad \frac{1}{q} = \frac{1}{2} - \frac{1}{d}.$$

We know by [1] that

$$|u_n|_{W^{2,q}(B_r)} \leq C(|u_n|_{W^{1,q}(B_{r+1})} + |\Delta u_n|_{L^q(B_{r+1})}).$$

Hence,

$$\sup_n |u_n|_{W^{2,q}(B_r)} < \infty.$$

By a bootstrap argument, we can obtain

$$\sup_n |u_n|_{W^{2,k}(B_r)} < \infty \qquad \text{for all } k > d.$$

Using the Sobolev inequality again, we have

$$\sup_n |u_n|_{L^\infty(B_r)} \leq \sup_n C|u_n|_{W^{1,k}(B_r)} < \infty.$$

Now, we apply the Morrey theorem [4, Thm IX.12, p.166 or p.169]:

$$|u_n(x) - u_n(y)| \leq C|u_n|_{W^{1,k}(B_r)}|x-y|^\delta, \qquad \forall x, y \in B_r, \qquad \delta = 1 - d/k,$$

to obtain (9). Similarly, we have (10). We can easily see by (7) that the derivative $D_i u_n$ satisfies

$$\alpha D_i u_n = \frac{1}{2}\Delta D_i u_n + (DF(Du_n), DD_i u_n) + D_i h_n.$$

Since $DF(\xi)$ is bounded, we have by virtue of [1]

$$\sup_n |Du_n|_{W^{2,k}(B_r)} < \infty.$$

Thus, by the same argument as above, we deduce (11).

Theorem 2.2. *Assume (5), (6), (7). Then there exists a unique solution* $u_\alpha \in C^2(\mathbf{R^d})$ *of equation (4) such that* u_α *is convex and satisfies*

$$0 \leq \alpha u_\alpha(x) \leq C(1 + |x|^{m+3}), \qquad x \in \mathbf{R^d}, \tag{2.11}$$

for some constant $C > 0$.

Proof. By Theorem 2.1, it is evident that the sequences $\{u_n\}, \{Du_n\}$ and $\{\Delta u_n\}$ are uniformly bounded and equi-continuous on every B_r. By the Ascoli-Arzelà theorem, we have

$$\begin{array}{lll} u_n & \to & u_\alpha \in C^2(\mathbf{R^d}), \\ Du_n & \to & Du_\alpha, \\ \Delta u_n & \to & \Delta u_\alpha \quad \text{uniformly on } B_r, \end{array}$$

taking a subsequence if necessary. Passing to the limit in (8), we can obtain (4).

Also, we can show (14) by (12) and

$$u_\alpha(x) = \inf\{E[\int_0^\infty e^{-\alpha t} h(x(t))dt] : |p(t)| \leq 1\}.$$

Thus the convexity of u_α is immediate.

Remark. In case of $d = 1$, the theorem is verified without (7), because h is Lipschitz continuous and (10) implies (11).

3 LIMIT AT INFINITY AND POLYNOMIAL GROWTH

We consider the limit of the solution u_α to Bellman equation (4) as $|x| \to \infty$ and also the polynomial growth property of $u_\alpha - \min u_\alpha$, denoted by v_α. We make the following assumption:

$$\text{There exists } C_0 > 0 \text{ such that } \; h(x) \geq C_0|x|. \tag{3.1}$$

Our objective in this section is to prove the following result.

Theorem 3.1.

We assume (5), (6), (7), (15). Then we have

$$u_\alpha(x) \to \infty \text{ as } |x| \to \infty \text{ uniformly in } \alpha,$$
$$v_\alpha(x) \le C(1+|x|^{m+1}),$$

for some constant $C > 0$.

Proof. By an elementary calculation, we can show that the assertions of Theorem 3.1 hold in case of d = 1. We prove the theorem, comparing (4) with

$$\alpha v_i = \frac{1}{2}\Delta v_i + F_i(Dv_i) + h_i, \quad i = 1, 2, \tag{3.2}$$

where

$$F_1(\xi) = \sum_{j=1}^{d} \min_{|p_j| \le 1} (p_j \xi_j), \qquad h_1(x) = \sum_{j=1}^{d} \eta_1 |x_j|,$$
$$F_2(\xi) = \sum_{j=1}^{d} \min_{|p_j| \le 1/d} (p_j \xi_j), \qquad h_2(x) = \sum_{j=1}^{d} \{\eta_2(|x_j|^m + 1) - \alpha \min u_\alpha\},$$

and each positive constant η_i will be chosen later. We can see that equation (16) admits a solution v_i of the form:

$$v_i(x) = \sum_{j=1}^{d} w_j^{(i)}(x_j),$$

for the solution $w_j^{(i)}(x_j)$ to (16) in the case d = 1. Hence, the following relations are fulfilled:

$$v_1(x) \to \infty \text{ as } |x| \to \infty \text{ uniformly in } \alpha,$$
$$v_2(x) - \min v_2 \le \sum_j (w_j^{(2)}(x_j) - \min w_j^{(2)}) \le C(1+|x|^{m+1}),$$

for sufficiently large $\eta_2 > 0$. By the same line as (12), we have

$$v_\alpha(x) = \inf\{E[\int_0^\infty e^{-\alpha t}(h(x(t)) - \alpha \min u_\alpha) dt] \ : \ |p(t)| \le 1\},$$
$$v_1(x) = \inf\{E[\int_0^\infty e^{-\alpha t} h_1(x(t)) dt] \ : \ |p_j(t)| \le 1\},$$
$$v_2(x) - \min v_2 = \inf\{E[\int_0^\infty e^{-\alpha t}(h_2(x(t)) - \alpha \min v_2) dt] \ : \ |p_j(t)| \le 1/d\}.$$

Since $\{p : |p_j| \le 1/d\} \subset \{p : |p| \le 1\} \subset \{p : |p_j| \le 1\}$, we can obtain

$$v_1 \le u_\alpha, \qquad v_\alpha \le v_2 - \min v_2,$$

for a convenient choice of each η_i such that $h_1 \le h$ and $h - \alpha \min u_\alpha \le h_2 - \alpha \min v_2$. Thus the theorem is established.

4 A-PRIORI ESTIMATES FOR APPROXIMATION

For the approximation problem of (1), we consider here the gradient estimates of v_α satisfying

$$\alpha v_\alpha = \frac{1}{2}\Delta v_\alpha + F(Dv_\alpha) + (h - \alpha \min u_\alpha). \tag{4.1}$$

Theorem 4.1. *Assume (5), (6), (7), (15). Then there exists* $0 < \delta < 1$ *such that*

$$\sup_{0<\alpha<1/2} [v_\alpha]_{\delta, B_r} < \infty, \tag{4.2}$$

$$\sup_{0<\alpha<1/2} \sum_i [D_i v_\alpha]_{\delta, B_r} < \infty, \tag{4.3}$$

$$\sup_{0<\alpha<1/2} \sum_{i,j} [D_{ij} v_\alpha]_{\delta, B_r} < \infty, \tag{4.4}$$

for every $r > 0$.

Proof. Multiplying both sides by $v_\alpha \Theta$ and integrating over $\mathbf{R^d}$, we have

$$2\int \alpha v_\alpha^2 \Theta dx - \int (\Delta v_\alpha) v_\alpha \Theta dx - 2\int F(Dv_\alpha) v_\alpha \Theta dx = 2\int (h - \alpha \min u_\alpha) v_\alpha \Theta dx.$$

where Θ is as in the proof of Theorem 2.1. The second term of the left-hand side can be rewritten as

$$\int (Dv_\alpha, D(v_\alpha \Theta)) dx \geq \int [|Dv_\alpha|^2 - \theta v_\alpha |Dv_\alpha|] \Theta dx.$$

By the choice of θ with $0 < \theta < 2$, we get

$$\int |Dv_\alpha|^2 \Theta dx \leq 2\int (h - \alpha \min u_\alpha) v_\alpha \Theta dx.$$

Since

$$v_\alpha(\gamma_\alpha) - v_\alpha(x) \geq (Dv_\alpha(x), \gamma_\alpha - x)$$
$$h(\gamma_\alpha) \leq \alpha v_\alpha(\gamma_\alpha) + \alpha \min u_\alpha \leq \alpha u_\alpha(0)$$

for $\gamma_\alpha :=$ arg min v_α, we have

$$0 \leq v_\alpha(x) \leq C|Dv_\alpha(x)|(|x| + 1). \tag{4.5}$$

Then, by the Schwarz inequality

$$\int |Dv_\alpha|^2 \Theta dx \leq 2C\Big(\int (h - \alpha \min u_\alpha)^2 (|x| + 1)^2 \Theta dx\Big)^{1/2} \Big(\int |Dv_\alpha|^2 \Theta dx\Big)^{1/2}.$$

Therefore, we deduce

$$\sup_\alpha \int |Dv_\alpha|^2 \Theta dx < \infty.$$

Now, we can find a constant $C > 0$ independent of α such that

$$|Dv_\alpha|_{L^2(B_r)} \leq C,$$

and, by (21)

$$|v_\alpha|_{L^2(B_r)} \leq C.$$

Again, using the regularity result and (17), we can obtain

$$\sup_\alpha |v_\alpha|_{W^{2,2}(B_r)} < \infty.$$

Finally, by the same bootstrap argument as the proof of Theorem 2.1,

$$\sup_\alpha |v_\alpha|_{W^{2,k}(B_r)} < \infty \qquad \text{for all } k > d.$$

Further, repeating the proof of Theorem 2.1, we can deduce (18), (19), (20).

5 BELLMAN EQUATION OF ERGODIC CONTROL

We shall show the existence of a unique solution $(\lambda, \phi) \in \mathbf{R} \times C^2(\mathbf{R^d})$ to the Bellman equation

$$\lambda = \frac{1}{2}\Delta\phi(x) + F(D\phi(x)) + h(x), \qquad x \in \mathbf{R^d}. \tag{5.1}$$

Theorem 5.1.
We assume (5), (6), (7), (15). Then there exists a subsequence $\alpha \to 0$ such that

$$\begin{aligned} \alpha \min u_\alpha \quad &\to \quad \lambda \in \mathbf{R}_+, \\ v_\alpha(x) \quad &\to \quad \phi(x) \in C^2(\mathbf{R^d}) \quad \textit{uniformly on each } B_r. \end{aligned}$$

The limit (λ, ϕ) satisfies Bellman equation (22), and furthermore

$$\phi : \textit{convex on } \mathbf{R^d}, \tag{5.2}$$

$$\phi(x) \quad \to \quad \infty \quad \text{as} \quad |x| \quad \to \quad \infty, \tag{5.3}$$

$$0 \leq \phi(x) \leq C(1 + |x|^{m+1}), \qquad (C > 0 : \textit{const.}). \tag{5.4}$$

Proof. The proof follows from Theorem 4.1 and the Ascoli-Arzelà theorem.

Remark. We can show the uniqueness of the solution (λ, ϕ) (ϕ up to a constant) of (22) with (23), (24) and (25). For the proof, we refer to [6].

6 RADIAL SYMMETRY

In this section, we study the radial symmetry of $\psi(x)$ of the Bellman equation

$$\lambda = \frac{1}{2}\Delta\psi(x) + F(D\psi(x)) + f(|x|), \quad x \in \mathbf{R^d}, \tag{6.1}$$

raplacing (22) in case of (3). Without loss of generality, by the assumptions on h, we may assume

$$\begin{aligned} &f(r) : \text{ non-decreasing on } [0,\infty), \quad f(\infty) = \infty, \\ &|f(r)| \le C(1 + r^m), \quad r \ge 0, \\ &f \in C^1([0,\infty)). \end{aligned} \tag{6.2}$$

We are involved in a reduction to the 1-dimensional equation:

$$\mu = \frac{1}{2}(\varphi''(r) + \frac{d-1}{r}\varphi'(r)) - |\varphi'(r)| + f(r), \quad r > 0. \tag{6.3}$$

Lemma 6.1. *Assume (27). Then there exists a unique solution $(\mu,\varphi) \in \mathbf{R} \times C^2([0,\infty))$ (φ up to a constant) of (28) such that*

$$\varphi'(r) : \textit{ polynomial growth}, \quad \varphi'(0+) = 0, \quad \varphi'(r) > 0 \textit{ for } r > 0. \tag{6.4}$$

Moreover, the solution (μ,φ) is given by

$$\mu = \int_0^\infty f(s)s^{d-1}e^{-2s}ds \Big/ \int_0^\infty s^{d-1}e^{-2s}ds, \tag{6.5}$$

$$\varphi'(r) = \frac{2e^{2r}}{r^{d-1}}\int_0^r (\mu - f(s))s^{d-1}e^{-2s}ds. \tag{6.6}$$

Proof. Define (μ,φ) by (30) and (31). By an elementary manipulation, we see that $\varphi'(r) > 0$, $\varphi'(0+) = 0$ and

$$\lim_{r\to\infty}\frac{\varphi'(r)}{r^m} = \lim_{r\to\infty}\frac{2(\mu - f(r))r^{d-1}}{(m+d-1)r^{m+d-2} - 2r^{m+d-1}} \le C. \tag{6.7}$$

Thus (29) is verified. Now, multiplying (31) by $r^{d-1}e^{-2r}$ and differentiating both sides, we can get (28).

Conversely, equation (28) turns out a linear differential eqution under (29). Hence we can solve this equation to obtain (30) and (31).

Theorem 6.2. *We assume (3), (27). Then equation (26) admits a solution $(\lambda,\psi) \in \mathbf{R} \times C^2(\mathbf{R^d})$ given by*

$$\lambda = \mu, \tag{6.8}$$

$$\psi(x) = \varphi(|x|). \tag{6.9}$$

Further, ψ fulfills (25).

Proof. Define (λ, ψ) by (33) and (34). By a simple calculation, we have

$$\Delta\psi(x) = \varphi''(r) + \frac{d-1}{r}\varphi'(r),$$
$$|D\psi(x)| = |\varphi'(r)|.$$

Hence (λ, ψ) satisfies

$$\lambda = \frac{1}{2}\Delta\psi(x) + F(D\psi(x)) + f(|x|), \quad x \in \mathbf{R}^d \setminus \{0\}. \tag{6.10}$$

By (28) and (29), we see that the limit of $|D\psi(x)|$ exists as $|x| \to 0$, and then $|D\psi(x)|$ and $\Delta\psi(x)$ are bounded on every B_r. By the regularity result [1]

$$|\psi|_{W^{2,k}(B_r)} \le C(|\psi|_{W^{1,k}(B_{r+1})} + |\Delta\psi|_{L^k(B_{r+1})}) < \infty \quad \text{for} \quad k > d.$$

Moreover, differentiating (35), we get

$$0 = \frac{1}{2}\Delta D_i\psi + (DF(D\psi), DD_i\psi) + f'(|x|)D_i|x|,$$

which implies $\Delta D_i\psi \in L^k(B_r)$. Applying the regularity result again, we have

$$\psi \in W^{3,k}(B_r), \qquad k > d.$$

Therefore, by the imbedding theorem [5], we deduce $\psi \in C^2(B_r)$ and hence (26). We remark by (32) that ψ of (34) fulfills the polynomial growth condition.

7 AN APPLICATION TO ERGODIC CONTROL

We shall study the ergodic control problem to minimize the cost

$$J(p) = \limsup_{T\to\infty} \frac{1}{T} E[\int_0^T h(x(t))dt]$$

over all $p \in \mathcal{P}$ subject to the state equation

$$dx(t) = p(t)dt + dw(t), \qquad x(0) = x,$$

where $\mathcal{P}$ denotes the set of all progressively measurable $\mathcal{F}_t$-adapted processes $p(t)$ such that

$$|p(t)| \le 1, \qquad \lim_{T\to\infty} \frac{1}{T} E[|x(t)|^{m+1}] = 0 \quad \text{for the response } x(t) \text{ to } p(t).$$

Now, let us consider the stochastic differential equation

$$dx^*(t) = G(D\psi(x^*(t)))dt + dw(t), \qquad x^*(0) = x,$$

where

$$G(z) = \begin{cases} -z/|z| & \text{if } z \in \mathbf{R^d}\backslash\{0\}, \\ 0 & \text{if } z = 0. \end{cases}$$

Lemma 7.1. *For any $n \in \mathrm{N}_+$, there exists $C > 0$ such that*

$$E[|x^*(t)|^{2n}] \leq C(1+t).$$

Proof. The proof follows from the relation:

$$E[|x^*(t)|^{2n}] = |x|^{2n} + E[\int_0^t \{-2n|x^*(s)|^{2n-1} + n(2n+d-2)|x^*(s)|^{2n-2}\}dt].$$

Theorem 7.2. *We make the assumptions of Theorem 6.2. Then the optimal control $p^*(t)$ is given by*

$$p^*(t) = G(D\psi(x^*(t))) = G(x^*(t)),$$

and the value by

$$J(p^*) = \lambda.$$

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