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# Exact controllability of anisotropic elastic bodies

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**Abstract**
The aim of this contribution is to study the exact controllability of linear, anisotropic elastic bodies by applying Lions' Hilbert Uniqueness Method.

**Keywords**
Unisotropic elasticity, exact controllability, Hilbert uniqueness method (HUM)

## INTRODUCTION

It seems that studies of exact controllability of solids and structures like plates and shells have so far been limited to isotropic materials, cf. Lions (1988), Lagnese (1991), Nicaise (1993). A comprehensive review of the relevant literature is out of scope of this limited in space contribution. However, two important aspects of materials properties: anisotropy and inhomogeneity still remain to be included into investigations on exact controllability and stabilization of solids and structures.

Our aim here is to examine exact controllability of linear elastic bodies made of homogeneous, anisotropic materials.

## 1 BASIC EQUATION AND FORMULATION OF THE PROBLEM OF EXACT CONTROLLABILITY

Let $\Omega \subset R^N$ be a bounded domain with sufficiently regular boundary $\Gamma = \partial\Omega$. Obviously, in physical situations $N = 2$ or 3. The linear elastic body in its undeformed state is identified with $\bar{\Omega}$, the closure of $\Omega$. The elasticity tensor $\mathbf{a} = (a_{ijkl})$ satisfies usual symmetry condition: $a_{ijkl} = a_{jikl} = a_{klij}$. Moreover, we assume that there exists a constant $C > 0$ such that

$$a_{ijkl}E_{ij}E_{kl} \geq CE_{ij}E_{ij}, \qquad \forall \mathbf{E} = (E_{ij}) \in E_s^N \tag{1.1}$$

Here $E_s^N$ is the space of symmetric $N \times N$ matrices. The material of the elastic body is homogeneous, i.e., $a_{ijkl}$ do not depend on $x \in \Omega$. It seems that the general case of exact controllability for $a_{ijkl} \in L^\infty(\Omega)$ still remains an open problem.

By $\mathbf{u} = (u_i), \mathbf{e} = (e_{ij})$ and $\boldsymbol{\sigma} = (\sigma_{ij})$ we denote the displacement vector, the strain tensor and the stress tensor, respectively. The constitutive equation has the form

$$\sigma_{ij} = a_{ijkl}e_{kl}. \tag{1.2}$$

The strain-displacement relation is linear

$$e_{ij}(\mathbf{u}) = u_{(i,j)} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})/2. \tag{1.3}$$

The structure of the elasticity tensor is studied in Chernykh (1988). Particularly, in the case of orthotropy only nine moduli are independent and we have

$$\mathbf{a} = \begin{bmatrix} a_{1111} & a_{1122} & a_{1133} & 0 & 0 & 0 \\ a_{1122} & a_{2222} & a_{2233} & 0 & 0 & 0 \\ a_{1133} & a_{2233} & a_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{1313} \end{bmatrix}$$

Here the following change of indices has been used

$$(11) \rightleftharpoons (1),\ (22) \rightleftharpoons (2),\ (33) \rightleftharpoons (3),\ (12) \rightleftharpoons (4),\ (23) \rightleftharpoons (5),\ (13) \rightleftharpoons (6).$$

In the absence of body forces we shall study the dynamic elasticity problem with Dirichlet control on a part of the boundary:

$$\mathbf{u}'' - div(\mathbf{a}\mathbf{e}(\mathbf{u})) = \mathbf{0} \text{ in } Q = \Omega \times (0,T); \quad \mathbf{u}(0) = \mathbf{u}^0, \mathbf{u}'(0) = \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}^1 \text{ in } \Omega, \tag{1.4}$$

$$\mathbf{u} = \begin{cases} \mathbf{v} & \text{on } \Sigma_0 \subset \Sigma = \Gamma \times (0,T), \\ \mathbf{0} & \text{on } \Sigma \setminus \Sigma_0. \end{cases} \tag{1.5}$$

Here $\mathbf{v} = (v_i)$ is a *control* through which the evolution of the solution is influenced. For the sake of simplicity, the density $\varrho$ is assumed to be equal to 1; moreover $\mathbf{u}'' = \frac{\partial^2 \mathbf{u}}{\partial t^2}$ and $[div(\mathbf{a}\mathbf{e}(\mathbf{u}))]_i = (a_{ijkl}e_{kl}(\mathbf{u}))_{,j}$.

*Exact Controllability Problem* reads: Given $T > 0$ and an "arbitrary" initial state $\{\mathbf{u}^0, \mathbf{u}^1\}$ of the system (1.4), find $\mathbf{v}$ in a suitable function space such that $\mathbf{u}(T) = \mathbf{u}'(T) = 0$.

The part $\Sigma_0$ of the boundary has to be suitably chosen. Our problem of exact controllability will be solved by applying Hilbert Uniqueness Method, cf. Lions (1988).

## 2 BASIC INEQUALITIES

Let $\mathbf{n} = (n_i)$ denote the outward unit normal vector to $\Gamma$. Further we set $H_0^1(\Omega)^N = [H_0^1(\Omega)]^N, L^2(\Omega)^N = [L^2(\Omega)]^N$. Essential role in applying HUM plays the system with the homogeneous boundary conditions on $\Gamma$:

$$\varphi'' - div(\text{ae}(\varphi)) = 0 \text{ in } Q; \qquad \varphi(0) = \varphi^0, \varphi'(0) = \varphi^1 \text{ in } \Omega, \tag{2.1}$$

$$\varphi = 0 \text{ on } \Sigma, \tag{2.2}$$

$$\varphi^0 \in H_0^1(\Omega)^N, \ \varphi^1 \in L^2(\Omega)^N. \tag{2.3}$$

In the derivation of the so called direct inequality we shall use

**Lemma 2. 1 (Lions, 1988)** *Let $\Omega$ be a bounded domain of $R^N$ with the boundary $\Gamma$ of class $C^2$. Then there exists a vector field $\mathbf{h} = (h_i) \in C^1(\bar{\Omega})^N$ such that $\mathbf{h}(x) = \mathbf{n}(x)$ on $\Gamma$.*

The total energy $E(t)$ of the system (2.1) - (2.3) is given by

$$E(t) = \frac{1}{2}[\|\varphi'(t)\|_{L^2}^2 + a(\varphi(t), \varphi(t))], \tag{2.4}$$

where $\|\varphi'(t)\|_{L^2}^2 = \int_\Omega |\varphi'(t)|^2 dx, a(\varphi(t), \varphi(t)) = \int_\Omega a_{ijkl} e_{ij}(\varphi(t)) e_{kl}(\varphi(t)) dx$. Since the system is conservative, therefore

$$E_0 := E(0) = E(t), \tag{2.5}$$

where

$$E(0) = \frac{1}{2} \int_\Omega [|\varphi^1|^2 + a_{ijkl} e_{ij}(\varphi^0) e_{kl}(\varphi^0)] dx. \tag{2.6}$$

More precisely to demonstrate (2.6) one has to show that $dE/dt = 0$.

### 2.1 Direct inequality

The aim of this subsection is to derive the following inequality

$$\int_\Sigma |\frac{\partial \varphi}{\partial \mathbf{n}}|^2 d\Sigma \le C_1 \int_\Sigma a_{ijkl} \frac{\partial \varphi_i}{\partial \mathbf{n}} n_j \frac{\partial \varphi_k}{\partial \mathbf{n}} n_l d\Sigma \le C_2(T+1)E_0, \tag{2.7}$$

where $C_1$ and $C_2$ are positive constants. In the first step we multiply $(2.1)_1$ by $h_m \frac{\partial \varphi_i}{\partial x_m}$ and integrate over $Q$, where $\mathbf{h}$ is a vector field of class $C^1(\bar{\Omega})^N$:

$$\int_Q [\varphi_i'' - (a_{ijkl} e_{kl}(\varphi))_{,j}] h_k \frac{\partial \varphi_i}{\partial x_k} dx dt = 0. \tag{2.8}$$

Integrating by parts we obtain

$$\int_Q \varphi_i'' h_k \frac{\partial \varphi_i}{\partial x_k} dxdt = (\varphi'(t), h_k \frac{\partial \varphi(t)}{\partial x_k})|_0^T + \frac{1}{2}\int_Q h_{k,k}|\varphi'|^2 dxdt, \tag{2.9}$$

$$\int_Q (a_{ijkl}e_{kl}(\varphi))_{,j} h_m \frac{\partial \varphi_i}{\partial x_m} dxdt = -\int_Q a_{ijkl}e_{kl}(\varphi)\frac{\partial h_m}{\partial x_j}\frac{\partial \varphi_i}{\partial x_m} dxdt +$$
$$+\frac{1}{2}\int_Q h_{m,m} a_{ijkl} e_{ij}(\varphi) e_{kl}(\varphi) dxdt - \frac{1}{2}\int_\Sigma a_{ijkl} e_{ij}(\varphi) e_{kl}(\varphi) h_m n_m d\Sigma +$$
$$+\int_\Sigma a_{ijkl} e_{kl}(\varphi) n_j h_m \frac{\partial \varphi_i}{\partial x_m} d\Sigma. \tag{2.10}$$

Here $|\varphi'|^2 = \varphi_i\varphi_i$, $\varphi' = 0$ on $\Sigma$ (because $\varphi = 0$ on $\Sigma$) and $(\varphi,\psi) = \int_\Omega \varphi_i(x)\psi_i(x)dx$ $\forall \varphi, \psi \in L^2(\Omega)^N$. We observe that in order to derive (2.10) we have exploited the fact that $a_{ijkl}$ do not depend on $x \in \Omega$. Substituting (2.9) and (2.10) into (2.8) we get

$$\int_\Sigma a_{ijkl} e_{kl}(\varphi) n_j h_m \frac{\partial \varphi_i}{\partial x_m} d\Sigma - \frac{1}{2}\int_\Sigma a_{ijkl} e_{ij}(\varphi) e_{kl}(\varphi) h_m n_m d\Sigma = (\varphi'(t), h_m \frac{\partial \varphi(t)}{\partial x_m})|_0^T +$$
$$+\frac{1}{2}\int_Q h_{m,m}[|\varphi'|^2 - a_{ijkl} e_{ij}(\varphi) e_{kl}(\varphi)] dxdt + \int_Q a_{ijkl} e_{kl}(\varphi) \frac{\partial h_m}{\partial x_j}\frac{\partial \varphi_i}{\partial x_m} dxdt. \tag{2.11}$$

Since $\varphi_i = 0$ on $\Sigma$ therefore we have

$$\frac{\partial \varphi_i}{\partial x_j} = n_j \frac{\partial \varphi_i}{\partial \mathrm{n}}. \tag{2.12}$$

Then the l.h.s of (2.11) takes the form

$$\int_\Sigma [a_{ijkl} e_{kl}(\varphi) n_j h_m \frac{\partial \varphi_i}{\partial x_m} - \frac{1}{2} a_{ijkl} e_{ij}(\varphi) e_{kl}(\varphi) h_m n_m] d\Sigma =$$
$$= \frac{1}{2}\int_\Sigma h_m n_m a_{ijkl} \frac{\partial \varphi_i}{\partial \mathrm{n}} n_j \frac{\partial \varphi_k}{\partial \mathrm{n}} n_l d\Sigma. \tag{2.13}$$

By virtue of (2.11) we write

$$\frac{1}{2}\int_\Sigma h_m n_m a_{ijkl} \frac{\partial \varphi_i}{\partial \mathrm{n}} n_j \frac{\partial \varphi_k}{\partial \mathrm{n}} n_l d\Sigma = (\varphi'(t), h_m \frac{\partial \varphi(t)}{\partial x_m})|_0^T +$$
$$+\frac{1}{2}\int_Q h_{m,m}\left[|\varphi'|^2 - a_{ijkl} e_{ij}(\varphi) e_{kl}(\varphi)\right] dxdt + \int_Q a_{ijkl} e_{kl}(\varphi) \frac{\partial h_m}{\partial x_j}\frac{\partial \varphi_i}{\partial x_m} dxdt. \tag{2.14}$$

Inequality (1.1) and Lemma 2.1 yield

$$\frac{1}{2}\int_\Sigma h_m n_m a_{ijkl} \frac{\partial \varphi_i}{\partial \mathrm{n}} n_j \frac{\partial \varphi_k}{\partial \mathrm{n}} n_l d\Sigma \geq \frac{C}{2}\int_\Sigma \frac{\partial \varphi_i}{\partial \mathrm{n}}\frac{\partial \varphi_i}{\partial \mathrm{n}} d\Sigma = \frac{C}{2}\int_\Sigma |\frac{\partial \varphi}{\partial \mathrm{n}}|^2 d\Sigma, \tag{2.15}$$

because $h_m(x)n_m(x) = 1$ for $x \in \Gamma$. Consequently

$$\frac{C}{2}\int_\Sigma |\frac{\partial\varphi}{\partial \mathbf{n}}|^2 d\Sigma \le \frac{1}{2}\int_\Sigma a_{ijkl}\frac{\partial\varphi_i}{\partial \mathbf{n}}n_j\frac{\partial\varphi_k}{\partial \mathbf{n}}n_l d\Sigma = (\varphi'(t), h_m\frac{\partial\varphi(t)}{\partial x_m})|_0^T +$$
$$+\frac{1}{2}\int_Q h_{m,m}\left[|\varphi'|^2 - a_{ijkl}e_{ij}(\varphi)e_{kl}(\varphi)\right]dxdt + \int_Q a_{ijkl}e_{kl}(\varphi)\frac{\partial h_m}{\partial x_j}\frac{\partial \varphi_i}{\partial x_m}dxdt. \tag{2.16}$$

By using Korn's inequality ( Nečas and Hlavaček, 1981) we obtain

$$(\varphi'(t), h_m\frac{\partial\varphi(t)}{\partial x_m})|_0^T = (\varphi'(T), h_m\frac{\partial\varphi(T)}{\partial x_m}) - (\varphi^1, h_m\frac{\partial\varphi^0}{\partial x_m}) \le$$
$$\le \max_{x\in\Omega, m=1,\dots,N}|h_m(x)|(\|\varphi'(T)\|_{L^2(\Omega)}\|\nabla\varphi(T)\|_{L^2(\Omega)}+$$
$$+\|\varphi^1\|_{L^2(\Omega)}\|\nabla\varphi^0\|_{L^2(\Omega)}) \le C_3(E(T)+E(0)). \tag{2.17}$$

Taking account of (2.17) in (2.16), after standard estimations we arrive at

$$\int_\Sigma |\frac{\partial\varphi}{\partial \mathbf{n}}|^2 d\Sigma \le C_1\int_\Sigma a_{ijkl}\frac{\partial\varphi_i}{\partial \mathbf{n}}n_j\frac{\partial\varphi_k}{\partial \mathbf{n}}n_l d\Sigma \le C_4(2E_0 + 2TE_0).$$

## 2.2 Inverse inequality

First, we introduce new notations. Let $x^0 \in R^N$. We define, cf. Lions (1988,chap. I)

$$m_k(x) = x_k - x_k^0, \tag{2.18}$$

$$\Gamma(x^0) = \{x \in \Gamma | \mathbf{m}(x)\cdot\mathbf{n}(x) = m_k(x)n_k(x) > 0\}, \tag{2.19}$$

$$\Gamma^*(x^0) = \Gamma \setminus \Gamma(x^0) = \{x \in \Gamma | \mathbf{m}(x)\cdot\mathbf{n}(x) \le 0\}, \tag{2.20}$$

$$\Sigma(x^0) = \Gamma(x^0)\times(0,T), \qquad \Sigma^*(x^0) = \Gamma^*(x^0)\times(0,T), \tag{2.21}$$

$$R(x^0) = \max_{x\in\bar\Omega}|\mathbf{m}(x)| = \max_{x\in\bar\Omega}\left[\sum_{i=1}^N (x_i - x_i^0)^2\right]^{1/2}. \tag{2.22}$$

Geometrical interpretation of the set $\Gamma(x^0)$ is given by Lions (1988, pp.79-81).

Now we are in position to derive the inverse inequality. Towards this end we set

$$X = (\varphi'(t), m_k\frac{\partial\varphi(t)}{\partial x_k})|_0^T = (\varphi_i'(t), m_k\frac{\partial\varphi_i(t)}{\partial x_k}), \tag{2.23}$$

$$Y = \int_Q |\varphi'(t)|^2 dxdt - \int_0^T a(\varphi(t),\varphi(t))dt. \tag{2.24}$$

Multiplying $(2.1)_1$ by $\varphi$ and integrating over $Q$ we infer that

$$Y = (\varphi'(t), \varphi(t))|_0^T. \tag{2.25}$$

Eq. (2.12), (2.14),(2.24) and (2.25) yield the relation

$$\frac{1}{2}\int_\Sigma m_k n_k a_{ijpq}\frac{\partial\varphi_i}{\partial \mathbf{n}}n_j\frac{\partial\varphi_p}{\partial \mathbf{n}}n_q d\Sigma = \frac{1}{2}\int_\Sigma m_k n_k a_{ijpq}e_{ij}(\varphi)e_{pq}(\varphi)d\Sigma =$$
$$= X + \frac{N-1}{2}Y + \frac{1}{2}\int_Q\left[|\varphi'|^2 + a_{ijkl}e_{ij}(\varphi)e_{kl}(\varphi)\right]dxdt \tag{2.26}$$

provided that $\boldsymbol{h} = \boldsymbol{m}$. We may write

$$X + \frac{N-1}{2}Y + \frac{1}{2}\int_Q\left[|\varphi'|^2 + a_{ijkl}e_{ij}(\varphi)e_{kl}(\varphi)\right]dxdt-$$
$$-\frac{1}{2}\int_{\Sigma\setminus\Sigma(x^0)} m_k n_k a_{ijpq}e_{ij}(\varphi)e_{pq}(\varphi)d\Sigma = \frac{1}{2}\int_{\Sigma(x^0)} m_k n_k a_{ijpq}e_{ij}(\varphi)e_{pq}(\varphi)d\Sigma, \tag{2.27}$$

and

$$|X + \frac{N-1}{2}Y| = |(\varphi'(t), m_k\frac{\partial\varphi(t)}{\partial x_k} + \frac{N-1}{2}\varphi(t))|_0^T \le$$
$$\le 2\|(\varphi'(t), m_k\frac{\partial\varphi(t)}{\partial x_k} + \frac{N-1}{2}\varphi(t))\|_{L^\infty(0,T)} \tag{2.28}$$

We shall prove that

$$|X + \frac{N-1}{2}| \le 2R(x^0)\sup_t\|\varphi'(t)\|_{L^2(\Omega)}\|\nabla\varphi(t)\|_{L^2(\Omega)}. \tag{2.29}$$

To corroborate this statement we find

$$\|(\varphi'(t), m_k\frac{\partial\varphi(t)}{\partial x_k} + \frac{N-1}{2}\varphi(t))\|_{L^\infty(0,T)} \le$$
$$\le \sup_t\|\varphi'(t)\|_{L^2(\Omega)}\|m_k\frac{\partial\varphi(t)}{\partial x_k} + \frac{N-1}{2}\varphi(t)\|_{L^2(\Omega)}. \tag{2.30}$$

Next we calculate

$$\|m_k\frac{\partial\varphi(t)}{\partial x_k} + \frac{N-1}{2}\varphi(t)\|^2_{L^2(\Omega)} = \|m_k\frac{\partial\varphi(t)}{\partial x_k}\|^2_{L^2(\Omega)} +$$
$$\left[\frac{(N-1)^2}{4} - \frac{N(N-1)}{2}\right]\|\varphi(t)\|^2_{L^2(\Omega)} \le \|m_k\frac{\partial\varphi(t)}{\partial x_k}\|^2_{L^2(\Omega)}, \tag{2.31}$$

since

$$(m_k\frac{\partial\varphi(t)}{\partial x_k}, \varphi(t)) = -\frac{N}{2}\|\varphi(t)\|^2_{L^2(\Omega)} \qquad \forall t\in[0,T]$$

and

$$\frac{\partial m_k}{\partial x_k} = N, \qquad \left[\frac{(N-1)^2}{4} - \frac{N(N-1)}{2}\right] \le 0.$$

We recall that $\varphi = 0$ on $\Gamma$. Consequently

$$\|m_k \frac{\partial \varphi(t)}{\partial x_k} + \frac{N-1}{2}\varphi(t)\|_{L^2(\Omega)} \leq \max_{x \in \bar{\Omega}} |\mathrm{m}(x)| \|\nabla \varphi(t)\|_{L^2(\Omega)} = R(x^0)\|\nabla\varphi(t)\|_{L^2(\Omega)}. \quad (2.32)$$

Thus we see that (2.28), (2.30) and (2.32) prove the inequality (2.29). By applying Korn's inequality (Nečas and Hlavaček, 1981), we conclude that there exists a constant $K > 0$ such that $\|\nabla\ \varphi(t)\|^2_{L^2(\Omega)} \leq Ka(\varphi(t), \varphi(t))$. Then (2.29) is estimated as follows

$$|X + \frac{N-1}{2}Y| \leq 2R(x^0)\sqrt{K} \sup_t \|\varphi'(t)\|_{L^2(\Omega)} \left[a(\varphi(t), \varphi(t))\right]^{1/2}. \quad (2.33)$$

Because

$$E_0 = E(t) = \frac{1}{2}\|\varphi'(t)\|^2 + \frac{1}{2}a(\varphi(t), \varphi(t)) \geq \|\varphi(t)\|_{L^2(\Omega)} \left[a(\varphi(t), \varphi(t))\right]^{1/2},$$

therefore

$$|X + \frac{N-1}{2}Y| \leq 2R(x^0)\sqrt{K}E_0. \quad (2.34)$$

Hence

$$-2R(x^0)\sqrt{K}E_0 \leq X + \frac{N-1}{2}Y \leq 2R(x^0)\sqrt{K}E_0. \quad (2.35)$$

We know that $m_k n_k \leq 0$ on $\Sigma \backslash \Sigma(X^0)$. From (2.27), by taking account of (2.35) we obtain

$$-2R(x^0)\sqrt{K}E_0 + TE_0 \leq \max_{x\in\bar{\Omega}} \frac{|\mathrm{m}(x)|}{2} \int_{\Sigma(x^0)} a_{ijkl} e_{ij}(\varphi(t)) e_{kl}(\varphi(t)) d\Sigma.$$

Thus we arrive at the final form of the inverse inequality:

$$\frac{R(x^0)}{2} \int_{\Sigma(x^0)} a_{ijkl} \frac{\partial \varphi_i}{\partial \mathrm{n}} n_j \frac{\partial \varphi_k}{\partial \mathrm{n}} n_l d\Gamma dt \geq \left[T - 2R(x^0)\sqrt{K}\right] E_0. \quad (2.36)$$

**Remark 2.1** For isotropic bodies straightforward calculation yield (Lions, 1988)

$$\frac{R(x^0)}{2} \int_{\Sigma(x^0)} \left[\mu(\frac{\partial \varphi}{\partial \mathrm{n}})^2 + (\lambda + \mu)(div\varphi)^2\right] d\Sigma \geq (T - \frac{2R(x^0)}{\sqrt{\mu}})E_0,$$

where $\lambda$ and $\mu$ are the Lamé constants.

## 3 EXACT CONTROLLABILITY: APPLICATION OF HUM

According to the Hilbert Uniqueness Method an important role is played by the adjoint system. By $\psi = (\psi_i)$ we denote the displacement field of the adjoint system. The system $\{\varphi, \psi\}$ of the HUM is now given by

$$\psi'' - div(\mathrm{ae}(\psi)) = 0 \text{ in } Q; \qquad \psi(T) = \psi'(T) = 0 \text{ in } \Omega, \quad (3.1)$$

$$\psi(t) = \begin{cases} (a_{ijkl}e_{kl}(\varphi(t)n_j) & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma \setminus \Sigma(x^0), \end{cases} \tag{3.2}$$

where $\varphi$ is the unique solution of the auxiliary system (2.1)-(2.3). In the case of isotropy relation $(3.2)_1$ reduces to, cf. Lions (1988,p.227)

$$\psi(t) = \mu \frac{\partial \varphi(t)}{\partial \mathbf{n}} + (\lambda + \mu)(div\varphi(t))\mathbf{n} \qquad \text{on } \Sigma(x^0),$$

where $div\varphi = \partial\varphi_i/\partial x_i = n_i\partial\varphi_i/\partial\mathbf{n}$, provided that $\varphi = 0$ on $\Gamma(x^0)$. The main result of this paper is formulated as

**Theorem 3. 1** *Let $x^0 \in R^N$ and define $\Sigma(x^0)$ and $R(x^0)$ by (2.21) and (2.22), respectively. If $T > 2R(x^0)\sqrt{K}E_0$ then for $\mathbf{u}^0 \in L^2(\Omega)^N, \mathbf{u}^1 \in H^{-1}(\Omega)^N$ there exists $\mathbf{v} \in L^2(\Sigma(x^0))^N$ such that the solution of the system (1.4)-(1.5) satisfies $\mathbf{u}(T) = \mathbf{u}'(T) = 0$, i.e. this system is exactly controllable.*

*Proof.* According to HUM we define

$$\Lambda\{\varphi^0, \varphi^1\} = \{\psi'(0), -\psi(0)\}, \tag{3.3}$$

a linear and continuous operator. Multiplying Eq. $(3.1)_1$ by $\varphi$, the solution of (2.1)-(2.3) and performing integration by parts we obtain

$$-(\psi'(0), \varphi^0) + (\psi(0), \varphi^1) + \int_{\Sigma(x^0)} a_{ijkl} \frac{\partial \varphi_i}{\partial \mathbf{n}} n_j n_k \psi_l d\Sigma = 0,$$

since $\psi = 0$ on $\Sigma \setminus \Sigma(x^0)$. Hence, by virtue of $(3.2)_1$

$$\langle \Lambda\{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\} \rangle = \int_{\Sigma(x^0)} a_{ijkl} n_j \frac{\partial \varphi_k}{\partial \mathbf{n}} n_l \psi_i d\Sigma =$$
$$= \int_{\Sigma(x^0)} a_{ijkl} \frac{\partial \varphi_k}{\partial \mathbf{n}} n_j n_l a_{impq} \frac{\partial \varphi_p}{\partial \mathbf{n}} n_q n_m d\Sigma = \int_0^T \|\psi(t)\|^2_{L^2(\Gamma(x^0))} dt. \tag{3.4}$$

In the Appendix we demonstrate that the matrix $A_{klpq} = a_{ijkl}n_j a_{impq} n_m$ satisfies ($\lambda_0$ -a positive constant)

$$A_{klpq}\xi_k n_l \xi_p n_q \geq \lambda_0 a_{ijkl} \xi_i n_j \xi_k n_l \qquad \forall \boldsymbol{\xi} \in R^N. \tag{3.5}$$

Taking account of (2.36) and (3.5) in (3.4) we obtain

$$\langle \Lambda\{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\} \rangle \geq \lambda_0 \int_{\Sigma(x^0)} a_{ijkl} \frac{\partial \varphi_i}{\partial \mathbf{n}} n_j \frac{\partial \varphi_k}{\partial \mathbf{n}} n_l d\Sigma \geq \frac{2\lambda_0}{R(x^0)} \left[T - 2R(x^0)\sqrt{K}\right] E_0. \tag{3.6}$$

Thus $\Lambda$ is an isomorphism of $H_0^1(\Omega)^N \times L^2(\Omega)^N$ on $H^{-1}(\Omega)^N \times L^2(\Omega)^N$ and may apply HUM. □

**Remark 3.1** The r.h.s. of $(3.2)_1$ is the stress vector whilst on the l.h.s. the displacement vector occurs. Therefore we must assume that relevant quantities are non-dimensional.

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## APPENDIX

Our aim now is to show that there exists a positive constant $\lambda_0$ depending on $(a_{ijkl})$ such that (3.5) is satisfied. We are considering the case $N = 3$ only; the case $N = 2$ is obviously simpler. We set $B_{ik} = a_{ijkl}n_jn_l$. The matrix $\mathbf{B}$ is symmetric and positive definite (p.d.), what follows from the properties of the elasticity tensor $\mathbf{a} = (a_{ijkl})$. Then (3.5) takes the form $B_{ij}B_{jk}\xi_i\xi_k \geq \lambda_0 B_{ij}\xi_i\xi_j$. Hence $B_{ij}(B_{jk} - \lambda_0\delta_{jk})\xi_i\xi_j \geq 0$. It is thus sufficient to show that the matrix $\mathbf{C} = (C_{ij}), C_{ik} = B_{ij}(B_{jk} - \lambda_0\delta_{jk})$ is p.d. for some $\lambda_0 > 0$. $\mathbf{C}$ is p.d. iff all its principal minors are p.d. The principal minor of $\mathbf{C}$ of rank 1 is $C_{11} = B_{1j}(B_{j1} - \lambda_0\delta_{j1}) = B_{11}^2 + B_{12}^2 + B_{13}^2 - \lambda_0 B_{11}$. Since $B_{11} > 0$, therefore $C_{11} > 0$ if $0 < \lambda_0 < B_{11}$, say $\lambda_0 = 1/2B_{11}$. Consider now the principal minor of C of rank 2. We have $C_{\alpha\beta} = B_{\alpha j}(B_{j\beta} - \lambda_0\delta_{j\beta}),\quad j = 1,2,3;\ \alpha,\beta = 1,2$, and $\det(C_{\alpha\beta}) = a\lambda_0^2 - b\lambda_0 + c \equiv g(\lambda_0)$, where $a = B_{11}B_{22} - B_{12}^2 > 0$, $b = (B_{11} + B_{22} + \frac{B_{23}^2}{B_{22}})a + B_{22}(B_{13} - \frac{B_{12}B_{23}}{B_{22}})^2 > 0$, $c = a^2 + (B_{11}B_{23} - B_{13}B_{12})^2 + (B_{12}B_{23} - B_{13}B_{22})^2 > 0$, because $(B_{ij})$ is p.d. The following cases are possible: (i) The determinant of the quadratic function $g(\lambda_0)$ is negative ($b^2 - 4ac < 0$) and then $g(\lambda_0) > 0$ for all $\lambda_0$, (ii) The determinant is nonnegative and then there exists two positive roots ($\lambda_0^{(1)} \leq \lambda_0^{(2)}$) of the equation $g(\lambda) = 0$. Thus $g(\lambda_0) > 0$ for $0 < \lambda_0 < \lambda_0^{(1)}$, for instance $\lambda_0 = 1/2\lambda_0^{(1)}$. Consider now the determinant of the matrix $\mathbf{C} : \det\mathbf{C} = \det(\mathbf{B}(\mathbf{B} - \lambda_0\mathbf{I})) = (\det\mathbf{B})\det(\mathbf{B} - \lambda_0\mathbf{I})$. Since $\det\mathbf{B} > 0$ therefore $\det\mathbf{C} > 0$ iff $\det(\mathbf{B} - \lambda_0\mathbf{I}) > 0$. We have $f(\lambda_0) = \det(\mathbf{B} - \lambda_0\mathbf{I}) = -\lambda_0^3 + \lambda_0^2 I_B - \lambda_0 II_B + III_B$, where $I_B, II_B, III_B$ are principal invariants of $\mathbf{B}$. We see that $f(0) = III_B = \det\mathbf{B} > 0$ and $f(\lambda) < 0$ for large $\lambda$. Thus there exists at least one positive root $\lambda_1$, i.e. $f(\lambda_1) = 0$. Let $\lambda_1 > 0$ be the smallest of such roots. Then $f(\lambda) > 0$ for $0 < \lambda < \lambda_1$. Finally a good candidate for $\lambda_0$ is $\lambda_0 = \frac{1}{2}\min\{B_{11}, \lambda_1\}$ in case (i) or $\lambda_0 = \frac{1}{2}\min\{B_{11}, \lambda_0^{(1)}, \lambda_1\}$ in case (ii).