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Bilinear optimal control of a Kirchhoff plate via internal controllers

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Abstract

We consider the problem of optimal control of a Kirchhoff plate. Bilinear controls are used as forces acting on internal regions, to make the plate close to a desired profile, taking into the account a quadratic cost of control. We prove the existence of an optimal control and characterize it uniquely through the solution of an optimality system.

Keywords

bilinear control, Kirchhoff plate

1 INTRODUCTION

We consider bilinear optimal control of a Kirchhoff plate as modeled below. The controls act on small non-intersecting regions in the interior of the plate. These controls behave as an internal tension or "spring-like" control attached to the plate at specific locations.

In order to define admissible vector controls for our system, we begin by defining an admissible component controller. Let h_i be such that the support of $h_i \subset Q_i \equiv \Omega_i \times [0,T]$ and such that

$$h_i \in U_{M_i} = \{h_i \in L^\infty(Q_i) : \ \|h\|_{L^\infty(Q_i)} \leq M_i\},$$

where $0 < M_i$. We define our control vector, $\mathbf{h} \equiv (h_1, h_2, ..., h_k)$, where k is the number of controlled regions (and consequently the number of controllers) for the system, requiring that each $h_i \in U_{M_i}$, so that

$$\mathbf{h} \in U \equiv U_{M_1} \times ... \times U_{M_k}.$$

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For convenience later, we define $M = \max_{i=1}^{k} M_i$.

Concerning the regions which will be controlled, we require that $\Omega_i \subset\subset \Omega$, $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ for $i \neq j$ and that $\partial\Omega_i \cap \partial\Omega = \emptyset$.

Under these assumptions, the "displacement" solution $w = w(\mathbf{h})$ of our state equation, satisfies

$$\left.\begin{array}{ll} w_{tt} + \Delta^2 w + w = \sum_{i=1}^{k} h_i(x,y,t)w & \text{on } Q = \Omega \times (0,T) \\ w(x,y,0) = w_0(x,y), w_t(x,y,0) = w_1(x,y) & \text{when } t = 0 \\ \left.\begin{array}{l} \Delta w + (1-\mu)B_1 w = 0 \\ \frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w = 0 \end{array}\right\} \text{on } \Sigma = \Gamma \times (0,T) & \end{array}\right\} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ with C^2 boundary, $\partial\Omega = \Gamma$, $\vec{\nu} = \langle n_1, n_2 \rangle$ is the outward unit normal vector on $\partial\Omega$, and

$$\begin{aligned} B_1 w &= 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx} \\ B_2 w &= \frac{\partial}{\partial \tau}[(n_1^2 - n_2^2)w_{xy} + n_1 n_2 (w_{yy} - w_{xx})]. \end{aligned}$$

The direction τ in $B_2 w$ is the tangential direction along Γ. The plate has free vibrations along Γ. The constant $\mu, 0 < \mu < \frac{1}{2}$, represents Poisson's ratio.

We take as our cost functional

$$J(\mathbf{h}) = \frac{1}{2}\left(\int_Q (w-z)^2 dQ + \sum_{i=1}^{k} \beta_i \int_{Q_i} h_i^2 dQ_i\right), \tag{1.2}$$

where z is the desired evolution for the plate and the quadratic term in h_i represents the cost of implementing the controls. We seek to minimize the cost functional, i.e., find optimal control $\mathbf{h}^* \in U$ such that

$$J(\mathbf{h}^*) = \min_{\mathbf{h} \in U} J(\mathbf{h}).$$

The goal of this paper is to characterize the unique optimal control vector in system consists of the state equation coupled with an adjoint equation. We note that the solution $w = w(h)$ is a nonlinear function of the control, so that uniqueness of the optimal control becomes a delicate issue. We will show that the optimal control is unique, as the unique solution of the optimality system. However, due to the highly nonlinear structure of the optimality system, we obtain this uniqueness only for a small time interval. Consequently, we prove uniqueness of the optimal control for this same small time interval.

For background information on plate equations and control theory, the reader is referred to the classical works of Lagnese (1989), Lagnese and Lions (1988) and Lions (1971).

2 EXISTENCE OF THE OPTIMAL CONTROL

We begin by proving existence, uniqueness, and regularity results for the state equation (1.1). These results will provide the *a priori* estimates needed to prove the existence of an optimal control.

To define our notion of weak solution, we first define the following product Hilbert space: $\mathcal{H} = H^2(\Omega) \times L^2(\Omega)$. We note that the bilinear form

$$a(w,v) = \int_\Omega \{\Delta w \Delta v + (1-\mu)[2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}] + wv\}\, d\Omega \tag{2.3}$$

induces a norm on $H^2(\Omega)$ which is equivalent to the usual norm on $H^2(\Omega)$.

Definition. Given $\mathbf{h} \in U$, $\tilde{w} = \tilde{w}(\mathbf{h}) = (w, w_t)$ is a weak solution of (1.1) if $\tilde{w} \in C([0,T];\mathcal{H})$, $\tilde{w}(0) = (w_0, w_1)$, and $\tilde{w}$ satisfies

$$< w_{tt}, \phi > + a(w,\phi) = \sum_{i=1}^{k} \int_{\Omega_i} h_i w \phi \; d\Omega_i \quad \text{for all} \quad \phi \in H^2(\Omega).$$

Here, we interpret $< \cdot, \cdot >$ as the duality pairing between $H^2(\Omega)$ and $[H^2(\Omega)]'$.

Lemma 1 (Well-posedness and Regularity)

(i) Let $\tilde{w}(0) = (w_0, w_1) \in \mathcal{H}$ and $\mathbf{h} \in U$, then the state equation (1.1) has a unique weak solution $\tilde{w} = \tilde{w}(\mathbf{h}) = (w, w_t)$ with $(w, w_t) \in C([0,T];\mathcal{H})$.

(ii) If in addition, $(w_0, w_1) \in (H^4(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)$ with w_0 satisfying the homogeneous boundary conditions in (1.1), and $\mathbf{h} \in C^2(\bar{Q}_i) \cap U_{M_i}$, then the weak solution $\tilde{w} = \tilde{w}(h)$ satisfies

$$\begin{aligned} \tilde{w} &\in C([0,T]; (H^4(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)) \\ w_{tt} &\in C([0,T]; L^2(\Omega)) \end{aligned}$$

with $\tilde{w}(0) = (w_0, w_1)$. Also $\tilde{w}$ satisfies equation (1.1) in the L^2 sense.

Proof. We refer the reader to techniques used in Bradley and Lenhart (1994) where the authors used semigroup theory combined with a contraction mapping argument to obtain the desired well-posedness and regularity results. □

To prove the existence of an optimal control, we need the following *a priori* estimate.

Lemma 2 *Given $\tilde{w}_0 = (w_0, w_1) \in \mathcal{H}$ and $\mathbf{h} \in U$, the weak solution $\tilde{w} = \tilde{w}(\mathbf{h}) = (w, w_t)$ of (1.1) satisfies*

$$\|\tilde{w}\|_{C([0,T];\mathcal{H})} \leq C_1 e^{C_2 kMT} \tag{2.4}$$

where $C_1 = \|\tilde{w}_0\|_{\mathcal{H}}$ and k is the number of control regions.

Proof. The proof is obtained using "multipliers technique" on the smooth solutions guaranteed by Lemma 1 (i) and then passing with a limit for solutions in $\mathcal{H}$. For details, see Bradley and Lenhart (1994). □

We now prove the main result of this section.

Theorem 1 *There exists an optimal control vector* $\mathbf{h}^* \in U$ *which minimizes the cost functional* $J(\mathbf{h})$ *for* $\mathbf{h} \in U$.

Proof. Let $\{\mathbf{h}^n\} \in U$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(\mathbf{h}^n) = \inf_{\mathbf{h}\in U} J(\mathbf{h}).$$

We denote the corresponding solution to (1.1) by $\tilde{w}^n = \tilde{w}(\mathbf{h}^n)$. By Lemma 2,

$$\|\tilde{w}^n\|_{C([0,T],\mathcal{H})} \leq C_1 e^{C_2 kMT}.$$

On a subsequence, we have

$$\begin{aligned}
&w^n \rightharpoonup w^* \text{ weakly in } L^2([0,T];H^2(\Omega)),\\
&w^n \to w^* \text{ strongly in } L^2(Q),\\
&w_t^n \rightharpoonup w_t^* \text{ weakly in } L^2(Q),\\
&w_{tt}^n \rightharpoonup w_{tt}^* \text{ weakly in } L^([0,T];[H^2(\Omega)]')\\
&\text{and}\\
&h_i^n \rightharpoonup h_i^* \text{ weakly in } L^2(Q_i).
\end{aligned}$$

We may now pass to the limit on (1.1) as $n \to \infty$, to obtain that $\tilde{w} = \tilde{w}(\mathbf{h}) = (w^*, w_t^*)$ solves the state equation (1.1) with control $\mathbf{h}^*$. Since the cost functional is lower semicontinuous with respect to weak convergence (basically Fatou's Lemma), we obtain $J(\mathbf{h}^*) \leq \underline{\lim}_{n\to\infty} J(\mathbf{h}^n) = \inf_{\mathbf{h}\in U} J(\mathbf{h})$. Hence $\mathbf{h}^*$ is an optimal control. □

3 CHARACTERIZATION OF THE OPTIMAL CONTROL

We now derive the optimality system by using the weak partial differentiability of the cost functional $J(\mathbf{h})$ with respect to the controllers h_i. In order to justify that such partial derivatives exist, we first must prove that the mapping $\mathbf{h} \to \tilde{w}(\mathbf{h})$ has the desired weak partial derivatives with respect to controllers h_i.

Lemma 3 *The mapping* $\mathbf{h} \in U \to \tilde{w}(\mathbf{h}) \in \mathcal{H}$ *is has weak partial derivatives in the following sense:*

$$\frac{\tilde{w}(h_1, ..., h_j + \varepsilon l, ..., h_k) - \tilde{w}(\mathbf{h})}{\varepsilon} \rightharpoonup \tilde{\psi}_j \text{ weakly in } L^2(0,T;\mathcal{H})$$

as $\varepsilon \to 0$, *for any* $h_j, h_j + \varepsilon\ell \in U_{M_j}$. *Moreover* $\tilde{\psi}_j = (\psi_j, \psi_{j,t})$ *is a weak solution of the following problem:*

$$\psi_{j,tt} + \Delta^2\psi_j + \psi_j - \sum_{i=1}^{k} h_i\psi_j = \ell w \text{ in } Q \tag{3.5}$$

$$\psi_j(x,0) = \psi_{j,t}(x,0) = 0 \text{ in } \Omega$$

$$\left.\begin{array}{l}\Delta\psi_j + (1-\mu)B_1\psi_j = 0 \\ \frac{\partial}{\partial\nu}\Delta\psi_j + (1-\mu)B_2\psi_j = 0\end{array}\right\} \text{ on } \Sigma$$

where $\tilde{w} = \tilde{w}(\mathbf{h}) = (w, w_t)$.

Proof. Denote $\tilde{w}^\varepsilon = \tilde{w}(h_1, ..., h_j + \varepsilon\ell, ..., h_k) = (w^\varepsilon, w_t^\varepsilon)$ and $\tilde{w} = \tilde{w}(\mathbf{h})$. (We note that w^ε will depend on both j and ε.) Then $\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon}$ is a weak solution of

$$\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_{tt} + \Delta^2\left(\frac{w^\varepsilon - w}{\varepsilon}\right) + \left(\frac{w^\varepsilon - w}{\varepsilon}\right) = \sum_{i=1}^k h_i\left(\frac{w^\varepsilon - w}{\varepsilon}\right) + \ell w^\varepsilon \text{ in } Q$$

$$\text{with } \left(\frac{w^\varepsilon - w}{\varepsilon}\right)(x,y,0) = \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t(x,y,0) = 0 \text{ in } \Omega$$

and satisfies zero boundary conditions on $\partial\Omega \times (0,T)$. Using a priori estimates like in Lemma 2, we obtain

$$\|\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon}\|_{C([0,T],\mathcal{H})} \leq \|\ell w^\varepsilon\|_{L^2(Q)} e^{CkMT} \leq C_3$$

where C_3 depends on the L^∞ bound on ℓ and the number of controlled regions, but is independent of ε, due to a bound on $\|\tilde{w}^\varepsilon\|_{L^2(Q)}$, independent of ε. Hence on a subsequence,

$$\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon} \rightharpoonup \tilde{\psi}_j \text{ weakly in } L^2(0,T;\mathcal{H}).$$

This convergence and the above *a priori* estimates are sufficient to guarantee that $\tilde{\psi}_j$ is a weak solution of (3.5). □

Finally, we derive our optimality system.

Theorem 2 *Given an optimal control* $\mathbf{h}$ *and corresponding solution* $\tilde{w} = \tilde{w}(\mathbf{h}) = (w, w_t)$, *there exists a weak solution* $\tilde{p} = (p, p_t)$ *in* $\mathcal{H}$ *to the adjoint problem,*

$$\begin{array}{l} p_{tt} + \Delta^2 p + p = \sum_{i=1}^k h_i p + w - z \text{ in } Q \\ \left.\begin{array}{l}\Delta p + (1-\mu)B_1 p = 0 \\ \frac{\partial}{\partial\nu}\Delta p + (1-\mu)B_2 p = 0\end{array}\right\} \text{ on } \Sigma \end{array} \tag{3.6}$$

and transversality conditions $p(x,y,T) = p_t(x,y,T) = 0$ *when* $t = T$. *Furthermore, each control element,* h_i, *satisfies*

$$h_i = \max(-M_i, \min(-\frac{wp}{\beta_i}, M_i)). \tag{3.7}$$

We note that, although we obtain dependence in of ψ_j on the particular partial derivative being taken, we obtain only one adjoint equation, since we have only one state equation.

Proof. Let $\mathbf{h} \in U$ be an optimal control vector and $\tilde{w} = \tilde{w}(\mathbf{h})$ be the corresponding optimal solution. Let $h_j + \varepsilon\ell \in U_{M_j}$ for $\varepsilon > 0$ and $\tilde{w}^\varepsilon = \tilde{w}(h_1, ..., h_j + \varepsilon\ell, ..., h_k)$ be the corresponding weak solution of the state equation (1.1). We compute the partial derivative of the cost functional $J(\mathbf{h})$ with respect to h_j in the direction of ℓ. Since $J(\mathbf{h})$ is a minimum value,

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0^+} \frac{J(h_1,...,h_j+\varepsilon\ell,...,h_k)-J(\mathbf{h})}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_Q ((w^\varepsilon - z)^2 - (w-z)^2)dQ + \frac{\beta_j}{2\varepsilon}\int_{Q_j}((h_j+\varepsilon\ell)^2 - h_j^2)dQ_j \\ &= \lim_{\varepsilon\to 0^+} \int_Q \left(\frac{w^\varepsilon - w}{\varepsilon}\right)\left(\frac{w^\varepsilon + w - 2z}{2}\right) dQ + \frac{\beta_j}{2}\int_Q (2h_j\ell + \varepsilon\ell^2)dQ \\ &= \int_Q \psi_j(w-z)dQ + \beta_j \int_Q h_j\ell \; dQ, \end{aligned} \tag{3.8}$$

where we have used the fact that the support of $h_j \subset\subset Q_i \subset Q$. Also, ψ_j is defined as in Lemma 3.

Let $\tilde{p} = (p, p_t)$ be the weak solution of the adjoint problem (3.6). Existence and uniqueness of $\tilde{p}$ is proved by arguments similar to those in Section 2. Substituting the adjoint solution into (3.8) for $(w-z)$, we obtain

$$0 \le \int_0^T < p_{tt}, \psi_j > \, dt + \int_0^T a(p,\psi_j)dt - \alpha\int_Q \psi_j hp \; dQ + \int_Q \beta_j h_j \ell \; dQ.$$

Using the weak form of (3.5), we have

$$0 \le \int_Q \ell(wp + \beta_j h_j)dQ.$$

By a standard control argument concerning the sign of the variation ℓ depending on the size of h_j, we obtain the desired characterization of $h_j = \max(-M_j, \min(-\frac{wp}{\beta_j}, M_j))$. □

Substituting (3.7) for h_j into the state equation (1.1) and the adjoint equation (3.6), we obtain the optimality system:

$$\begin{array}{ll} w_{tt} + \Delta^2 w + w = \sum_{i=1}^k \max(-M_i, \min(-\frac{wp}{\beta}_i, M_i))w & \text{in } Q \\ p_{tt} + \Delta^2 p + p = \sum_{i=1}^k \max(-M_i, \min(-\frac{wp}{\beta}_i, M_i))p + w - z & \text{in } Q \\ \left.\begin{array}{l} \Delta w + (1-\mu)B_1 w = \Delta p + (1-\mu)B_1 p = 0 \\ \frac{\partial}{\partial\nu}\Delta w + (1-\mu)B_2 w = \frac{\partial}{\partial\nu}\Delta p + (1-\mu)B_2 p = 0 \end{array}\right\} \text{on } \Sigma & \\ w(x,y,0) = w_0(x,y), \quad w_t(x,y,0) = w_1(x,y) & \text{on } \Omega \\ p(x,y,T) = p_t(x,y,T) = 0. & \end{array} \tag{3.9}$$

Weak solutions of the optimality system exist by Lemma 1 and Theorems 1 and 2. However, the problem of uniqueness of solutions for this nonlinear optimality system (which implies the uniqueness of the optimal control vector) proves to be more difficult. We will now prove for small time T, that the optimality system (3.9) does, in fact, possess a *unique* solution and thereby show that the optimal control is in fact unique for a small time interval, $[0,T]$. This, then will give a characterization of the unique optimal control in terms of the solution of (3.9).

Theorem 3 *For T sufficiently small, weak solutions of the optimality system (3.9) are unique.*

Proof. Suppose we have two weak solutions,

$$\tilde{w} = (w, w_t),\ \tilde{p} = (p, p_t),\ \hat{w} = (\overline{w}, \overline{w}_t),\ \hat{p} = (\overline{p}, \overline{p}_t).$$

Since $w,\ \overline{w},\ p,\ \overline{p} \in C(0,T; H^2(\Omega))$, we have that $w,\ \overline{w},\ p,\ \overline{p}$ are bounded on $\overline{Q}$.

We change variables

$$w = e^{\lambda t}u, \quad p = e^{-\lambda t}q, \quad \overline{w} = e^{\lambda t}\overline{u}, \quad \overline{p} = e^{-\lambda t}\overline{q}.$$

Then u (and respectively, q) satisfies in a weak sense

$$\begin{aligned} &u_{tt} + 2\lambda u_t + (\lambda^2+1)u + \Delta^2 u + u = \textstyle\sum_{i=1}^k \max(-M_i, \min(-\frac{uq}{\beta_i}, M_i))u \\ &-q_{tt} + 2\lambda q_t - (\lambda^2+1)q - \Delta^2 q - q = \textstyle\sum_{i=1}^k \max(-M_i, \min(-\frac{uq}{\beta_i}, M_i))(-q) \\ &\qquad\qquad\qquad\qquad - e^{2\lambda t}u + e^{\lambda t}z. \end{aligned}$$

One can check that u, q satisfy similar boundary and initial/terminal conditions as before, so that $u - \overline{u}$ and $q - \overline{q}$ satisfy equations as above (modulo $e^{\lambda t}z$ term in q equation), with homogeneous data.

Using multiplier $(u - \overline{u})_t$ on the $u - \overline{u}$ equation and multiplier $(q - \overline{q})_t$ on the $q - \overline{q}$ equation, and combining, we have the following estimate:

$$\begin{aligned} &\frac{1}{2}\int_\Omega ((u-\overline{u})_t)^2(x,T)d\Omega + \frac{1}{2}\int_\Omega ((q-\overline{q})_t)^2(x,0)d\Omega \\ &\quad + \frac{\lambda^2}{2}\int_\Omega ((u-\overline{u})^2(x,T) + (q-\overline{q})^2(x,0))d\Omega \\ &\quad + a(u-\overline{u}, u-\overline{u})(T) + a(q-\overline{q}, q-\overline{q})(0) \\ &\quad + 2\lambda\int_Q [((q-\overline{q})_t)^2 + ((u-\overline{u})_t)^2]dQ \\ &= \sum_{i=1}^k \int_Q \left[(h_i u - \overline{h}_i\overline{u})(u-\overline{u})_t - (h_i q - \overline{h}_i\overline{q})(q-\overline{q})_t - e^{2\lambda t}(u-\overline{u})(q-\overline{q})_t\right] dQ \end{aligned} \tag{3.10}$$

where $h_i = \max(-M_i, \min(-\frac{uq}{\beta_i}, M_i))$ and $\overline{h}_i = \max(-M_i, \min(-\frac{\overline{uq}}{\beta_i}, M_i))$. It can be shown by direct computation that

$$|h_i - \overline{h}_i| \le \frac{1}{\beta_i}|\overline{uq} - uq| \le \frac{1}{\beta_i}\left(|\overline{u} - u||\overline{q}| + |\overline{q} - q||u|\right),$$

so that we can estimate the right hand side of (3.10) and obtain,

$$\begin{aligned} &2\lambda\int_Q [((q-\overline{q})_t)^2 + ((u-\overline{u})_t)^2]dQ \\ &\quad \le \int_Q [((q-\overline{q})_t)^2 + ((u-\overline{u})_t)^2]dQ \quad + (C_1 e^{C_2(kM+\lambda)T})\int_Q [(u-\overline{u})^2 + (q-\overline{q})^2]dQ, \end{aligned} \tag{3.11}$$

where C_1, C_2 are independent of λ and T but do depend on the number of controlled regions, k and on the L^∞ bounds on u and $\overline{q}$. Noting that

$$\begin{aligned}
\int_Q (u-\overline{u})^2 dQ &= \int_\Omega \int_0^T \left(\int_0^t (u-\overline{u})_t(x,y,s)ds \right)^2 dt d\Omega \\
&\leq \int_\Omega \int_0^T t \left(\int_0^t ((u-\overline{u})_t)^2 ds \right) dt d\Omega \\
&\leq \int_0^T t dt \int_Q ((u-\overline{u})_t)^2 ds d\Omega \\
&\leq \frac{T^2}{2} \int_Q ((u-\overline{u})_t)^2 dQ,
\end{aligned}$$

and using a similar argument for the variable q, we obtain,

$$\begin{aligned}
(2\lambda - 1) \int_Q [((q-\overline{q})_t)^2 &+ ((u-\overline{u})_t)^2] dQ \leq \\
T^2(C_1 e^{C_2(kM+\lambda)T}) &\int_Q [((q-\overline{q})_t)^2 + ((u-\overline{u})_t)^2] dQ.
\end{aligned}$$

We now fix λ such that $2\lambda - 1 > 0$ and choose T sufficiently small so that

$$2\lambda - 1 > T^2(C_1 e^{C_2(kM+\lambda)T}),$$

and thus $(q-\overline{q})_t = (u-\overline{u})_t \equiv 0$ in Q. Due to agreement of $q, \overline{q}$ and $u, \overline{u}$ at top and bottom of the cylinder Q respectively, we obtain $q = \overline{q}$ and $u = \overline{u}$, as desired. □

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REFERENCES

Bradley, M. E. and Lenhart, S. M. (1994) Bilinear optimal control of a Kirchhoff plate, *Systems and Control Letters*, **22**, 27-38.

Lagnese, J. E. (1989) *Boundary Stabilization of Thin Plates,* Society for Industrial and Applied Mathematics, Philadelphia.

Lagnese, J. E. and Lions, J. L. (1988) *Modelling Analysis and Control of Thin Plates,* Masson, Paris.

Lions, J. L. (1971) *Optimal Control of Systems Governed by Partial Differential Equations,* Springer-Verlag, Berlin.