21

A Trotter–type scheme for the generalized gradient of the optimal value function

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Abstract
An iterative formula for the generalized gradient of a Trotter–type approximation for the optimal value function associated with the control of a certain nonlinear parabolic system is established. This formula is useful in constructing suboptimal feedback controls.

Keywords
Trotter product formula, dynamic programming equation, parabolic variational inequality, (sub)optimal feedback control

1 INTRODUCTION

In this paper we shall reveal an interesting aspect of approximation of dynamic programming Hamilton–Jacobi equations via Trotter product formulas. Also, we shall explain how this aspect is reflected in constructing suboptimal feedback controls.

One of the main objectives of optimal control theory is the construction of optimal feedback controls. Simple heuristic considerations based on dynamic programming lead to feedback laws expressed by means of the optimal value function. In many significant situations, these laws can be rigorously justified (see, for example, Barbu (1984) and Popa (to appear)). However, it remains the question: how to compute (or approximate) the optimal value function in a reasonable manner (so that the feedback law should become effective)? Certainly, this function satisfies a Hamilton–Jacobi equation (called, in this context, the dynamic programming equation), but such an equation is a very complicated mathematical object. In author's opinion, the most promising way to compute solutions of Hamilton–Jacobi equations (satisfying certain initial or final conditions), is offered by a treatment of these equations by Trotter product formulas. These formulas are obtained by breaking Hamilton–Jacobi equations into two or several parts on small intervals (that is, by decomposing the associated Cauchy problem into two or several such problems). This decoupling leads to a better understanding of the making of solutions and, from a numerical viewpoint, to decentralization of calculus. Several convergent Trotter product formulas was proposed by V.Barbu and the author for dynamic programming equations associated with the control of parabolic variational inequalities (see Barbu (1988, 1991) and Popa (1991, 1995)). Numerical tests performed by V.Arnăutu and A.Niemistö (see Arnăutu (1995) and Niemistö (to appear)) show that the Trotter product formula approach is indeed an effective and realistic way in computing solutions of dynamic programming equations, and, implicitly, in constructing suboptimal feedback controls.

The generating idea of this paper consists in two simple observations. First, if we have a Trotter product formula approach to Hamilton–Jacobi equations in view, it is much easier to calculate the gradients of solutions than the solutions themselves. In other words, Trotter approximations for the gradients of solutions are much simpler than those for the solutions. On the other hand, the expression of feedback laws explicitly contains not only the solution of the dynamic programming equation but even its gradient. So, instead of computing solutions of dynamic programming equations by Trotter schemes and then differentiating them, it is preferable to directly compute gradients of solutions by schemes of the same kind.

Our aim here is to give an iterative formula for the generalized gradient of the Trotter approximation for the solution of the dynamic programming equation associated with the control of a certain nonlinear parabolic system. This formula seems to be much more effective in constructing suboptimal feedback controls. Also, it may be interpreted as a Trotter approximation for the vector variant of the conservation law equation obtained by formal differentiation of the dynamic programming equation.

2 AN EXAMPLE

Let us illustrate what we have asserted above by a simple example.

Consider the Hamilton–Jacobi equation of Hamiltonian mechanics corresponding to Hamiltonian $H(p,y) = \frac{1}{2}|p|^2 + U(y)$:

$$\begin{cases} S_t(t,y) + \frac{1}{2}|S_y(t,y)|^2 + U(y) = 0 \text{ in } [0,T] \times \mathcal{H}, \\ S(0,y) = S_0(y), \end{cases} \tag{1}$$

where $\mathcal{H}$ is the state space. Divide the interval $[0,T]$ into N subintervals of the same length $\varepsilon = T/N$. Let us decouple the terms corresponding to the two kinds of energy in (1) on each subinterval by decomposing Cauchy problem (1) into two elementary Cauchy problems. One obtains the following Trotter approximation:

$$\begin{cases} S_t^\varepsilon(t,y) + \frac{1}{2}|S_y^\varepsilon(t,y)|^2 = 0 \text{ in } ((i-1)\varepsilon, i\varepsilon] \times \mathcal{H}, \\ S^\varepsilon((i-1)\varepsilon + 0, y) = S^\varepsilon((i-1)\varepsilon, y) - \varepsilon U(y). \end{cases} \tag{2}$$

We can express the solution of Cauchy problem (2) by using the well-known Lax representation:

$$S^\varepsilon(i\varepsilon, y) = \inf\{\frac{1}{2\varepsilon}|z-y|^2 + S^\varepsilon((i-1)\varepsilon, z) - \varepsilon U(z) : z \in \mathcal{H}\},\ i = 1,2,...,N. \tag{3}$$

Now, solving the minimization problem contained in (3), one easily finds the following alternative (but more elaborated) formula:

$$\begin{cases} S^\varepsilon(i\varepsilon,y) &= \frac{\varepsilon}{2}|(\nabla_y S^\varepsilon((i-1)\varepsilon,\cdot)-\varepsilon\nabla U)_\varepsilon y|^2 \\ & +S^\varepsilon((i-1)\varepsilon,(I+\varepsilon(\nabla_y S^\varepsilon((i-1)\varepsilon,\cdot)-\varepsilon\nabla U))^{-1}y) \\ & -\varepsilon U((I+\varepsilon(\nabla_y S^\varepsilon((i-1)\varepsilon,\cdot)-\varepsilon\nabla U))^{-1}y),\ i=1,2,...,N, \\ S^\varepsilon(0,y) &= S_0(y). \end{cases} \tag{4}$$

Next, it is not difficult to show (see Barbu and Precupanu (1986), Thm. 2.3, p.121) that the gradient of S^ε is given by the following iterative formula:

$$\begin{cases} \nabla_y S^\varepsilon(i\varepsilon,y)=(\nabla_y S^\varepsilon((i-1)\varepsilon,\cdot)-\varepsilon\nabla U)_\varepsilon y,\ i=1,2,...,N, \\ \nabla_y S^\varepsilon(0,y)=\nabla S_0(y). \end{cases} \tag{5}$$

In (4), (5) (and throughout in the sequel) the symbol ε subscript after an operator means passing to its Yosida approximation (for instance, $(\nabla S_0-\varepsilon\nabla U)_\varepsilon=\frac{1}{\varepsilon}(I-(I+\varepsilon(\nabla S_0-\varepsilon\nabla U))^{-1}))$.

Clearly, the iterative formula (5) for the gradient of S^ε is simpler than that for S^ε given by (4). To make this fact even more striking, apply (4) i times. One obtains

$$\begin{aligned} S^\varepsilon(i\varepsilon,y) &= \frac{\varepsilon}{2}|P_1Q_2Q_3...Q_iy|^2+\frac{\varepsilon}{2}|P_2Q_3...Q_iy|^2+\cdots+\frac{\varepsilon}{2}|P_iy|^2 \\ & -\varepsilon U(Q_1Q_2Q_3...Q_iy)-\varepsilon U(Q_2Q_3...Q_iy)-\cdots-\varepsilon U(Q_iy) \\ & +S_0(Q_1Q_2Q_3...Q_iy), \end{aligned} \tag{6}$$

where P_j,Q_j are the nonlinear operators defined by

$$\begin{cases} P_j=(P_{j-1}-\varepsilon\nabla U)_\varepsilon,\ j=1,2,...,N, \\ P_0=\nabla S_0, \end{cases} \tag{7}$$

$$Q_j=(I+\varepsilon(P_{j-1}-\varepsilon\nabla U))^{-1},\ j=1,2,...,N. \tag{8}$$

On the other hand, the expression of the gradient of S^ε at $i\varepsilon$ contains only operator P_i :

$$\nabla_y S^\varepsilon(i\varepsilon,y)=P_iy=\underbrace{(...((\nabla S_0-\varepsilon\nabla U)_\varepsilon-\varepsilon\nabla U)_\varepsilon...-\varepsilon\nabla U)_\varepsilon}_{i \text{ iterations}}\,y. \tag{9}$$

In conclusion, from a theoretical point of view, the iterative formula (5) for the gradient of S^ε is more attractive than that for S^ε given by (4) at least because the final expression (9) for $\nabla_y S^\varepsilon(i\varepsilon,y)$ is much simpler than that for $S^\varepsilon(i\varepsilon,y)$ given by formulas (6)–(8) (it contains only P_i). Finally, let us point out that $\nabla_y S^\varepsilon(i\varepsilon,y)$ (given by (5)) can be interpreted as a Trotter approximation for the solution $\nabla_y S(t,y)$ of the vector variant of the conservation law equation obtained by formal differentiation of Hamilton–Jacobi equation (1). In other words, $\nabla_y S^\varepsilon(i\varepsilon,y)$ can be formally obtained like $S^\varepsilon(i\varepsilon,y)$, but by treating (this time) the conservation law equation in the same manner as above.

3 THE FRAMEWORK AND THE MAIN RESULT

All the preceding considerations can be repeated with the same effect in the more complex situation of dynamic programming equations associated with the control of nonlinear parabolic systems.

Let $\mathcal{U}$ be a real Hilbert space and set $\mathcal{H} = L^2(\Omega)$, where Ω is an open and bounded subset of $\mathbb{R}^n$ having a sufficiently smooth boundary. The control system we deal with is described by the following mixed boundary value problem:

$$\begin{cases} \dfrac{\partial y}{\partial t} + A_0 y + \beta(y) \ni Bu \ \text{ a.e. in } (0,T)\times\Omega, \\ y = 0 \ \text{ on } (0,T)\times\partial\Omega, \\ y(0,x) = y^0(x) \ \text{ in } \Omega. \end{cases} \tag{10}$$

Here A_0 is the elliptic differential operator defined by

$$A_0 y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial y}{\partial x_i}\right) + a_0(x)y,$$

where $a_{ij} \in C^1(\Omega)$, $a_0 \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$, $a_0(x) \geq 0$ a.e. $x \in \Omega$, and, for some $\omega > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \omega\sum_{i=1}^{n}|\xi_i|^2 \ \text{ for all } (\xi_1,\xi_2,...,\xi_n) \in \mathbb{R}^n, \ \text{ a.e. } x \in \Omega.$$

The nonlinear term β is a maximal monotone graph in $\mathbb{R}^2$ containing $(0,0)$. Take a convex function $j : \mathbb{R} \to (-\infty,+\infty]$ whose subdifferential is β, and define the convex function $\phi : \mathcal{H} \to (-\infty,+\infty]$ as $\phi(y) = \int_\Omega j(y(x))dx$. Operator B from $\mathcal{U}$ to $\mathcal{H}$ is linear and continuous, and $y^0 \in \overline{D(\phi)}$.

A standard existence result (see Barbu (1984), Thm. 4.3, p.131) states that under the above assumptions on A_0, β, B and y^0, problem (10) has a unique solution $y \in C([0,T];\mathcal{H})$ such that $\sqrt{t}\,y' \in L^2(0,T;\mathcal{H})$ and $\sqrt{t}\,y \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$. Let us also mention that problem (10) can be interpreted as an evolution equation in $\mathcal{H}$, that is, the solution y of (10) satisfies

$$y' + Ay + \beta(y) \ni Bu \ \text{ a.e. in } (0,T),$$

where A is the linear continuous operator from $\mathcal{V} = H_0^1(\Omega)$ to $\mathcal{V}' = H^{-1}(\Omega)$ defined by

$$(Ay,z) = \sum_{i,j=1}^{n}\int_\Omega a_{ij}\frac{\partial y}{\partial x_i}\frac{\partial z}{\partial x_j}\,dx + \int_\Omega a_0 yz\,dx \ \text{ for all } \ y,z \in H_0^1(\Omega).$$

Consider the following optimal control problem:

(P) Minimize

$$\int_0^T (h(u(t)) + g(y(t)))dt + \ell(y(T))$$

over all $u \in L^2(0,T;\mathcal{U})$, where $y \in C([0,T];\mathcal{H})$ satisfies (10).

Impose on the functions h, g, ℓ the following hypotheses:

(H1) $h : \mathcal{U} \to (-\infty, +\infty]$ is convex, lower semicontinuous, not identically $+\infty$ and, for some $c_1 > 0$ and $c_2 \in I\!R$, satisfies

$$h(u) \geq c_1 |u|^2 - c_2 \ \text{ for all } u \in \mathcal{U}.$$

(H2) $g, \ell : \mathcal{H} \to I\!R$ are Lipschitz continuous on bounded subsets and bounded from below by affine functions.

We associate with problem (P) the corresponding optimal value function $V : [0,T] \times \overline{D(\phi)} \to I\!R$:

$$V(t,y) = \inf\{ \int_t^T (h(u(s)) + g(z(s)))ds + \ell(z(T)) : \\ z' + Az + \beta(z) \ni Bu \ \text{ a.e. in } \ (t,T), z(t) = y, u \in L^2(t,T;\mathcal{U})\}.$$

One knows that for every optimal pair of problem (P), the following feedback law holds (see Barbu (1984), Thm. 5.6, p.208, and Popa (to appear), Thm. 2.3):

$$u^*(t) \in \partial h^*(-B^* \partial_y V(t, y^*(t))) \ \text{ a.e. } \ t \in (0,T). \tag{11}$$

Here h^* is the convex conjugate of h, B^* is the adjoint of B, and $\partial_y V(t,y)$ is the generalized gradient (in Clarke's sense) of $y \mapsto V(t,y)$.

It is also well-known that the optimal value function V satisfies (in a certain generalized sense) the following Hamilton–Jacobi equation:

$$\begin{cases} D_t V(t,y) - h^*(-B^* D_y V(t,y)) - (Ay + \beta(y), D_y V(t,y)) + g(y) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathcal{H}, \\ V(T,y) = \ell(y), \ y \in \mathcal{H}. \end{cases} \tag{12}$$

One can obtain Trotter–type approximations for V by treating equation (12) in a similar manner as in the preceding section. Let $\varepsilon = T/N$. Decoupling the last three terms in the left–hand side of (12) on each subinterval $[i\varepsilon, (i+1)\varepsilon]$ and then using a Lax–type representation formula, we get (in a heuristic manner) the following Trotter scheme, proposed in Popa (1995) (for simplicity, we shall indicate it only for $t = i\varepsilon$):

$$\begin{cases} V^\varepsilon(i\varepsilon, y) = \inf\{ \ \varepsilon h(u) + \varepsilon g(y + \varepsilon Bu) \\ \qquad\qquad + V^\varepsilon((i+1)\varepsilon, (I + \varepsilon\beta)^{-1}(I + \varepsilon A)^{-1}(y + \varepsilon Bu)) : u \in \mathcal{U}\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad y \in \mathcal{H}, \ i = 0, 1, ..., N-1, \\ V^\varepsilon(T,y) = \ell(y), \ y \in \mathcal{H}. \end{cases} \tag{13}$$

The fact that V^ε really approximates V is not trivial in the case of nonlinear and infinite–dimensional control systems (see Popa (1995), Thm. 3.1).

The following question arises at this point: If we take V^ε instead of V in feedback law (11), does the new feedback law provide suboptimal controls (that is, approximately optimal controls) for problem (P)? All that we know at the moment is that the answer is positive in the following sense: Consider the following discrete approximation for (P) (see Popa (1995), Thm. 5.1):

(P^ε) Minimize
$$\sum_{i=1}^{N} \varepsilon(h(u_i) + g(y_i)) + \ell(y_N)$$
over all N-tuples $(u_1, u_2, ..., u_N) \in \mathcal{U}^N$, where $(y_1, y_2, ..., y_N) \in \mathcal{H}^N$ satisfies the scheme

$$\begin{cases} y_i = (I + \varepsilon\beta)^{-1}(I + \varepsilon A)^{-1}(y_{i-1} + \varepsilon B u_i), \ i = 1, 2, ..., N, \\ y_0 = y^0. \end{cases} \tag{14}$$

(Note that scheme (14) can be viewed as a Trotter approximation for the state equation (10).) Then, for every optimal N-tuple $(u_1^\varepsilon, u_2^\varepsilon, ..., u_N^\varepsilon)$ of problem (P^ε), the following discrete version of feedback law (11) holds (see Popa (1995), Thm. 6.1):

$$\begin{cases} u_i^\varepsilon \in \partial h^*(-B^*\partial_y V^\varepsilon((i-1)\varepsilon, y_{i-1}^\varepsilon)), \\ y_i^\varepsilon = (I + \varepsilon\beta)^{-1}(I + \varepsilon A)^{-1}(y_{i-1}^\varepsilon + \varepsilon B u_i^\varepsilon), \ i = 1, 2, ..., N, \\ y_0^\varepsilon = y^0. \end{cases} \tag{15}$$

It is clear (if we regard scheme (15)) that to construct suboptimal feedback controls for problem (P), we must be able to compute $\partial_y V^\varepsilon(i\varepsilon, y)$. Our aim is to give an iterative formula for $\partial_y V^\varepsilon(i\varepsilon, y)$ analogous to that for $\nabla_y S^\varepsilon(i\varepsilon, y)$ established before.

The following additional hypotheses are needed:

(H3) g, ℓ are convex and bounded on bounded subsets.
(H4) If $y, z \in \mathcal{H}$ satisfy $y \leq z$ a.e. in Ω, then $g(y) \leq g(z)$ and $\ell(y) \leq \ell(z)$.

Set $V_-^\varepsilon((i+1)\varepsilon, y) = V^\varepsilon((i+1)\varepsilon, (I + \varepsilon\beta)^{-1}(I + \varepsilon A)^{-1} y) + \varepsilon g(y)$. (We have already met the above expression in the definition (13) of V^ε.)

The following theorem is our main result.

Theorem 1 *Under the above hypotheses on A_0, β, B, y^0 and h, suppose in addition that β is concave and g and ℓ satisfy (H3) and (H4). Then*

$$\partial_y V^\varepsilon(i\varepsilon, y) \subset \bigcup_{u_i} \partial_y V_-^\varepsilon((i+1)\varepsilon, y + \varepsilon B u_i), \tag{16}$$

where u_i runs over all solutions of the inclusion

$$u_i \in \partial h^*(-B^*\partial_y V_-^\varepsilon((i+1)\varepsilon, y + \varepsilon B u_i)), \ i = N-1, ..., 1, 0, \tag{17}$$

and

$$\partial_y V^\varepsilon(N\varepsilon, y) = \partial \ell(y).$$

If $\mathcal{U} = \mathcal{H}$ and $B = I$, then

$$\partial_y V^\varepsilon(i\varepsilon, y) \subset \partial_y V_-^\varepsilon((i+1)\varepsilon, (I - \varepsilon\partial h^*(-\partial_y V_-^\varepsilon((i+1)\varepsilon, \cdot)))^{-1} y). \tag{18}$$

Sketch of the proof. First of all, let us remark that the function $y \mapsto V^\varepsilon(i\varepsilon, y)$ is convex on $\mathcal{H}$. (Consequently, $\partial_y V^\varepsilon(i\varepsilon, y)$ will coincide with the subdifferential of $y \mapsto V^\varepsilon(i\varepsilon, y)$

in the sense of convex analysis.) Indeed, by convexity of ℓ and $r \mapsto (I + \varepsilon\beta)^{-1}(r)$ (do not forget that β is concave), using also (H4), we infer that the function $y \mapsto \ell((I + \varepsilon\beta)^{-1}(I + \varepsilon A)^{-1}y)$ is convex on $\mathcal{H}$. Also, this function is nondecreasing in the sense of (H4). (It suffices to apply the monotonicity of $r \mapsto (I + \varepsilon\beta)^{-1}(r)$ and the maximum principle to the elliptic operator $I + \varepsilon A_0$.) Using both these properties of $y \mapsto \ell((I+\varepsilon\beta)^{-1}(I+\varepsilon A)^{-1}y)$ (and g), it is not difficult to show that $y \mapsto V^\varepsilon((N-1)\varepsilon, y)$ is convex and nondecreasing (see also the proof of Lemma 4.1 in Popa (to appear)). Now we successively argue $N - i$ times as above to obtain that $y \mapsto V^\varepsilon(i\varepsilon, y)$ is convex on $\mathcal{H}$ (and nondecreasing). Consequently, $y \mapsto V_-^\varepsilon((i + 1)\varepsilon, y)$ is convex too.

Then, we interpret the minimization problem (13) defining V^ε as a discrete optimal control problem with *convex* performance criterion, where the state variable takes only two values: the initial value y and the final one $y + \varepsilon Bu$. For this problem, it is easy to derive the following discrete Pontryagin–type maximum principle (see also formulas (6.17), (6.18) in Popa (1995)): For any optimal element $u_i \in \mathcal{U}$, there exists $p_i \in \mathcal{H}$ such that

$$\begin{cases} p_i \in -\partial_y V_-^\varepsilon((i+1)\varepsilon, y + \varepsilon B u_i), \\ u_i \in \partial h^*(B^* p_i). \end{cases} \tag{19}$$

(The characteristic feature of the above optimality conditions is that the discrete costate is constant, that is, the initial costate coincides with the final one.)

Since $y \mapsto V_-^\varepsilon((i + 1)\varepsilon, y)$ is convex, we can use the same argument of the proof of Proposition 2.2 from Barbu and Precupanu (1986), p. 317, to prove that

$$\partial_y V^\varepsilon(i\varepsilon, y) = \{-p_i \in \mathcal{H} : \text{there exists } u_i \in \mathcal{U} \text{ such that } (p_i, u_i) \text{ satisfies } (19)\}. \tag{20}$$

But (20) in conjunction with (19) gives (16) and (17). □

A result similar to the above theorem was formulated by Barbu but for convex control problems governed by *linear* parabolic equation (see Barbu (to appear)).

Theorem 1 says that to compute $\partial_y V^\varepsilon(i\varepsilon, y)$, we only need to know $z \mapsto \partial_y V_-^\varepsilon((i+1)\varepsilon, z)$. However, to express the gradient of V_-^ε (with respect to y) in terms of the gradient of V^ε, we need some adequate chain rules. So, one can successively compute $\partial_y V^\varepsilon(i\varepsilon, y)$ $(i = N - 1, ..., 1, 0)$ by starting with $\partial_y V^\varepsilon(N\varepsilon, y) = \partial\ell(y)$. Here is an example.

Corollary 1 *If $\beta \in C^1(\mathbb{R})$, $\mathcal{U} = \mathcal{H}$ and $B = I$, then* (18) *can be written in the form*

$$\partial_y V^\varepsilon(i\varepsilon, y) \subset -\partial h(-(-\partial h^*(-\partial_y V_-^\varepsilon((i+1)\varepsilon, \cdot)))_\varepsilon y), \tag{21}$$

where

$$\begin{aligned} \partial_y V_-^\varepsilon((i+1)\varepsilon, y) \;=\; & (I + \varepsilon A)^{-1}(1 + \varepsilon\beta'((I + \varepsilon\beta)^{-1}(I + \varepsilon A)^{-1}y))^{-1} \\ & \partial_y V^\varepsilon((i+1)\varepsilon, (I + \varepsilon\beta)^{-1}(I + \varepsilon A)^{-1}y) + \varepsilon\partial g(y). \end{aligned} \tag{22}$$

(Here, as before, ε subscript means passing to Yosida approximation.)

Proof. One easily derives (22) by using Theorem 2.3.10 from Clarke (1983). □

Let us point out that the iterative formulas (21), (22) for the generalized gradient of V^ε (with respect to y) are a substantial generalization of the iterative formula (5) for the gradient of S^ε. (We obtain a retrograde version of (5) if we take $h(\cdot) = \frac{1}{2}|\cdot|^2$, $g \equiv U$, $A \equiv 0$ and $\beta \equiv 0$ in (21), (22).)

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