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Fuzzy integer sharing problem with fuzzy capacity constraints

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Abstract

The sharing problem is a method to find an equitable distribution of resources by maximizing the smallest value of all "tradeoff functions" where a tradeoff function is a function of the flux to a sink node. We generalize some researches about the problem and propose a fuzzy integer sharing problem with fuzzy capacity constraints. Our model has bicriteria, i.e., minimal satisfaction among all fuzzy capacity constraints and that among fluxes to all sink nodes, both of which are to be maximized.

Keywords

Sharing problem, integer flow, fuzzy capacity, bicriteria, nondominated flow pattern

1 INTRODUCTION

The sharing problem, originated by (Brown, 1979), is a method to find an equitable distribution of resources. Its objective is to maximize the smallest value of all "tradeoff functions" where a tradeoff function is a function of the flux to a sink node. In (Tada, Ishii, Nishida and Masuda, 1989), we considered a fuzzy version of the sharing problem by introducing a membership function designating a degree of satisfaction for the flux to each sink node and proposed an efficient algorithm for the problem. While, (Chanas and Kolodziejczyk, 1982) considered a fuzzy version of maximal flow problem by introducing the notion of fuzzy capacity constraint and derived an efficient algorithm for the problem by showing that the maximum flow minimum cut theorem also holds in this case. Further they extended this model to the integer flow case (Chanas and Kolodziejczyk, 1986). This paper combined both models, that is, our earlier model in (Tada, Ishii, Nishida and Masuda, 1989) and Chanas and Kolodziejczyk one and propose a fuzzy integer sharing problem with fuzzy capacity constraints. Our model is bicriteria one and this is another prominent feature of the model. Bicriteria are minimal satisfaction among all fuzzy

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capacity constraints and that among fluxes to all sink nodes, both of which are to be maximized.

Section 2 formulates our problem and defines nondominated flow patterns since generally speaking, there does not exist an optimal flow pattern maximizing both criteria. Section 3 proposes an algorithm for finding nondominated flow patterns and clarifies its validity. Section 4 discusses its complexity and improvement. Finally, section 5 summarize this paper and discusses further research problem.

2 PROBLEM FORMULATION

Let $G(N, A)$ be a distribution network where N is the set of nodes and A is the set of directed arcs connecting nodes. N includes special nodes, called source nodes and sink nodes. Let S, T be the set of source nodes and that of sink nodes, respectively. We attach to G a super source node s with directed arcs connecting from s to all source nodes and super sink node u with directed arcs connecting from all sink node to u. Let resulting network be $G'(N', A')$. Each arc $(i, j) \in A$ has a fuzzy capacity $\tilde{C}(i, j)$ with the membership function

$$\mu_{ij}(f_{ij}) = \begin{cases} 1 & (f_{ij} < c_{ij}) \\ \dfrac{\bar{c}_{ij} - f_{ij}}{\bar{c}_{ij} - c_{ij}} & (c_{ij} \le f_{ij} \le \bar{c}_{ij}) \end{cases} \tag{1}$$

where f_{ij} is a flow value in arc (i, j) and c_{ij}, $\bar{c}_{ij}$ are integers. We assume the capacity of each arc $(s, s_i), s_i \in S$ is ∞. While, the capacity of each arc $(t, u), t \in T$ is one of key points and it may be determined and updated as to maximize the second criterion under satisfaction function $\mu_t(f_t)$ of fluxes f_t of sink nodes $t \in T$ given as follows:

$$\mu_t(f_t) = \begin{cases} 1 & (f_t > b_t) \\ \dfrac{b_t - f_t}{b_t - a_t} & (a_t \le f_t \le b_t) \\ 0 & (f_t < a_t) \end{cases} \tag{2}$$

where a_t, b_t are integers. Further we restrict flow values f_{ij}, f_{ss_i}, f_t to be integer. Then we obtain the following bicriteria sharing problem **BSP**.

$$\begin{array}{lll} \textbf{BSP}: & & \\ \text{Maximize} & \min\limits_{(i,j)\in A} \mu_{ij}(f_{ij}), \qquad \text{Maximize} \ \min\limits_{t\in T} \mu_t(f_t) & \\ \text{subject to} & \sum\limits_{i\in N'-\{u\}} f_{ij} = \sum\limits_{k\in N'-\{s\}} f_{jk}, \ j \in N & \\ & f_{ij} : \textit{nonnegative integer}, \ (i,j) \in A & \end{array} \tag{3}$$

We define a flow pattern vector of flow pattern $\boldsymbol{f}$ to be vector

$$\boldsymbol{f} = (f^1, f^2) = (\min_{(i,j)\in A} \mu_{ij}(f_{ij}), \ \min_{t\in T} \mu_t(f_t)) \tag{4}$$

and nondominated flow patterns as follows.

Definition 1 *If $f_b^1 \leq f_a^1$, $f_b^2 \leq f_a^2$ and $(f_a^1, f_a^2) \neq (f_b^1, f_b^2)$ for $\boldsymbol{f}^a$ and $\boldsymbol{f}^b$, flow pattern $\boldsymbol{f}^a$ dominates flow pattern $\boldsymbol{f}^b$. And, if there exists no flow pattern vector $\boldsymbol{f}'$ that dominates $\boldsymbol{f}$, $\boldsymbol{f}$ is said to be nondominated flow pattern.*

Generally speaking, optimal flow pattern maximizing both criteria does not exist and so we seek nondominated flow patterns in the next section.

3 SOLUTION PROCEDURE FOR BSP

Since all flow values f_{ij}, f_{ss_i}, f_t are integer, we only need to consider integer capacity values. We first solve the following ordinary fuzzy sharing problem **BSP**(1) by the algorithm in (Tada, Ishii, Nishida and Masuda, 1989). That is, we first find the nondominated flow pattern $\boldsymbol{f}(1)$ whose corresponding flow pattern vector has the value 1 as a first component.

$$\begin{array}{ll} \textbf{BSP}(1): & \\ \text{Maximize} & \min_{t \in T} \mu_t(f_t) \\ \text{subject to} & \sum_{i \in N'-\{u\}} f_{ij} = \sum_{k \in N'-\{s\}} f_{jk},\ j \in N \\ & 0 \leq f_{ij} \leq c_{ij},\ (i,j) \in A \end{array} \tag{5}$$

Of course, we must find the maximum flow value $v(1)$ sent from s to u in network $G'(N', A')$ wit the capacities c_{ij}, $(i,j) \in A$ before solving **BSP**(1). Then let the optimal value of **BSP**(1) be $f(1)^2$. Second, we solve the following fuzzy sharing problem **BSP**(0).

$$\begin{array}{ll} \textbf{BSP}(0): & \\ \text{Maximize} & \min_{t \in T} \mu_t(f_t) \\ \text{subject to} & \sum_{i \in N'-\{u\}} f_{ij} = \sum_{k \in N'-\{s\}} f_{jk},\ j \in N \\ & 0 \leq f_{ij} \leq \bar{c}_{ij},\ (i,j) \in A \end{array} \tag{6}$$

Similarly, we must find the maximum flow value $v(0)$ from s to u in network $G'(N', A')$ with the capacities $\bar{c}_{ij}$, $(i,j) \in A$ before solving **BSP**(0). Let an optimal flow pattern and the optimal value of **BSP**(0) be $\boldsymbol{f}(0)$ and $f(0)^2$ respectively. Now sorting different $\mu_{ij}(k)$, $k \in (c_{ij}, \bar{c}_{ij})$, $(i,j) \in A$ and let result be $\mu^0 \equiv 1 > \mu^1 > \cdots > \mu^l > \mu^{l+1} \equiv 0$ (l is the number of different $\mu_{ij}(k)$). Then we obtain the following algorithm.

Algorithm

Step 0 : Set $q = 1$, $DS = \{\boldsymbol{f}(1)\}$ and $DV = \{(1, f(1)^2)\}$ and go to **Step 1**.

Step 1 : Solve the following fuzzy sharing problem **BSP** (μ^q) by the algorithm in (Tada, Ishii, Nishida and Masuda, 1989).

$$\begin{array}{ll} \textbf{BSP}(\mu^q): & \\ \text{Maximize} & \min_{t \in T} \mu_t(f_t) \\ \text{subject to} & \sum_{i \in N'-\{u\}} f_{ij} = \sum_{k \in N'-\{s\}} f_{jk},\ j \in N \\ & 0 \leq f_{ij} \leq C_{ij}^q,\ (i,j) \in A \end{array} \tag{7}$$

where $C_{ij}^q = [(1-\mu^q)\bar{c}_{ij} + \mu^q c_{ij}]$, $(i,j) \in A$. Let an optimal flow pattern and the optimal value of $\mathbf{BSP}(\mu^q)$ be $\boldsymbol{f}(\mu^q)$ and $f(\mu^q)^2$ respectively. If flow pattern $\boldsymbol{f}(\mu^q)$ is dominated by some flow pattern of DS, then go to **Step 2**. Otherwise, set $DS = DS \cup \{\boldsymbol{f}(\mu^q)\}$ and $DV = DV \cup \{(\mu^q, f(\mu^q)^2\}$ and go to **Step 2**.

Step 2 : Set $q = q+1$. If $q \neq l+1$, then go to **Step 1**, else check whether $\boldsymbol{f}(0)$ is dominated by some flow pattern of DS. If dominated, terminate. Otherwise, set $DS = DS \cup \{\boldsymbol{f}(0)\}$ and $DV = DV \cup \{(0, f(0)^2)\}$.

Validity of the above algorithm is clear from the fact that it check all possibilities of the first component of nondominated flow pattern vectors and that the greater the flow value sent from s to u in network $G'(N', A')$ then so is $\min_{t \in T} \mu_t(f_t)$.

4 COMPLEXITY AND IMPROVEMENT OF THE ALGORITHM

The algorithm in the previous section is under the following theorem for the computation time.

Theorem 1 *The algorithm obtains nondominated flow patterns in at most*

$O(L \times \max(\log L,\ |T|^2 cf(n,m)))$ *computational time,*

where $L = \sum_{(i,j)\in A}(\bar{c}_{ij} - c_{ij})$, $n = |N|$, $m = |A|$ *and* $cf(n,m)$ *is a computation time of the maximum flow problem for a graph* (N, A).

Proof

Solving $\mathbf{BSP}(1)$: $O(|T|^2 cf(n,m))$ (Tada, Ishii, Nishida and Masuda, 1989).

Solving $\mathbf{BSP}(0)$: $O(|T|^2 cf(n,m))$.

For an arc $(i,j) \in A$, the number of different $\mu_{ij}(k)$ is $(\bar{c}_{ij} - c_{ij})$ because of treating integer flow. Then it holds

$$l \leq L. \tag{8}$$

Therefore, Sorting $\mu^0, \mu^1, \ldots, \mu^{l+1}$ takes

$$O(L \log L). \tag{9}$$

On **Step 1**, Solving $\mathbf{BSP}(\mu^q)$: $O(|T|^2 cf(n,m))$.

On **Step 2**, without going to **Step 1**, judging whether there exists a flow pattern in DS which dominates $\boldsymbol{f}(\mu^q)$ or not : $O(L+1)$.

As **Step 1** $\sim$ **Step 2** is repeated at most L times by (8), solving $\mathbf{BSP}(\mu^q)$ for all μ takes

$$O(|T|^2 Lcf(n,m)). \tag{10}$$

Therefore, the total complexity is

$$O(\max(L\log L,\ |T|^2Lcf(n,m))) = O(L \times \max(\log L,\ |T|^2cf(n,m))). \quad (11)$$

□

The computation time in above theorem is calculated under the condition that the maximum flow is obtained from scratch in the solution of **BSP**(μ^q) for every μ. However, current solution may be obtained from the previous solution by a little efforts, because capacities in the network do not decrease. Then the following improvement sophisticates the algorithm. That is, we insert the procedure

Step 0' : Solve **BSP**(μ^1). Set $q = 2$. If $\boldsymbol{f}(\mu^1)$ is dominated by $\boldsymbol{f}(1)$, then go to **Step 1**. Otherwise, set $DS = DS \cup \{\boldsymbol{f}(\mu^1)\}$ and $DV = DV \cup \{(\mu^1, f(\mu^1)^2\}$ and go to **Step 1**.

between **Step 0** and **Step 1**. Of course, "**Step 1**" in **Step 0** should change into "**Step 0'** ".

Theorem 2 *The sophisticated algorithm obtains nondominated flow patterns in at most*

$$O(\max(L\log L,\ |T|^2cf(n,m),\ |T|^2Lmn^2)).$$

5 CONCLUSION

There may be many nondominated flow patterns with same corresponding flow pattern vector but our algorithm only find one of them. Further though in the worst case there exist $O(l)$ nondominated flow patterns, we should refine the algorithm by taking not such a case into account since the algorithm check all possibilities in this case also. These are further research problems. Another is extension to the more general case that f_{ij}, f_{ss_i}, f_t are restricted to multiple of some integer d. Besides, we are going to investigate a fuzzy sharing problem by the modal optimization. In the model, we take a fuzzy weight for each sink node into account and formulate the problem as a possibility programming problem.

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