

Moving schema

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ABSTRACT

Just as a Venn diagram can represent both a loop and a mathematical set, semi-concrete schema function as a bridge between the concrete and the abstract and help students to understand mathematical concepts. Using computers, schema can be moved in real time by the teacher to clarify the object in question, to make comparisons of the object before and after transformation, and to focus on invariants. In motion, schema take on greater strength as signifiers, and become harder to understand intuitively. Our task now is to learn how students can come to visualize moving schema for themselves.

Keywords: graphics, teaching methods, thinking

INTRODUCTION

This paper provides me with an opportunity to discuss the importance of moving schema in mathematics education and the role of computers in facilitating this concept. It is an area that has yet to receive the attention it deserves and one that I hope will be debated by more people in the future.

I use the word 'schema' here to denote a simple figure used to illustrate a mathematical concept. This figure is somewhere between a concrete object and an abstract object; it might reasonably be called a

semi-concrete object. It is similar to what in psychology is referred to as a 'mental model', but for this discussion 'schema' denotes both a mental image and a real figure.

As computers develop, it is becoming easier and easier to create, move and display such figures, and we can expect moving schema to continue growing in significance. Moving schema have had a place in the imagination of mathematicians for some time, but have yet to acquire a prominent role in education, perhaps because they are not easily taught to students. One thing is certain, however, and that is that schema are not primarily a private means of solving problems - they are communicative devices that can speak to many people.

THE ROLE OF MOVEMENT

Qualitative understanding

As Piaget said, 'Qualitative understanding precedes quantitative understanding', and moving figures created with computer graphics are a highly effective tool for leading students to a qualitative understanding of mathematical concepts. Students, for example, have difficulty when they are asked to obtain the eigen vector of two-dimensional linear transformation using defined expression and calculation. On the other hand, when they are shown with computer graphics the transformation of radiating lines passing through the origin, or the transformation of a number of points on a plane, students can sense the existence of the eigen vector even though they do not know its precise value. Offering students a qualitative understanding first leads them to an understanding of defined expression and gives them a motive to calculate answers for themselves.

This is especially true in the case of a special matrix, such as $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$,

where students can only find the eigen vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ after visualizing the transformation. This is shown in Figures A1 through B2.

This is a technique I have used successfully in many of my own classes on the eigen vector. Six years ago, however, my first attempt ended in failure. It was a summer class on basic linear transformation, and it failed because I mistakenly believed that every student would automatically perceive the relationship between the crossed stripes being transformed on the screen and the linear transformation itself. Not everyone did: some students saw in the image nothing more than figurative movement; they did not see the image as the signifier of a

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

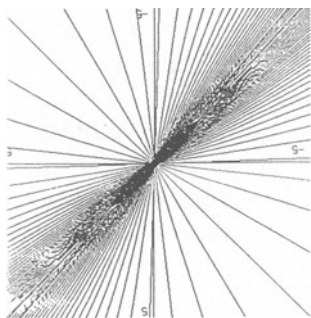


Figure A1

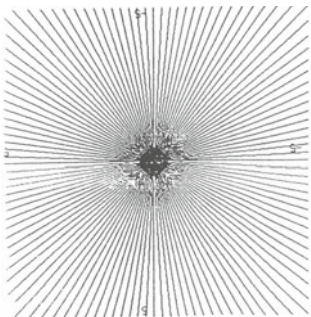


Figure A2

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

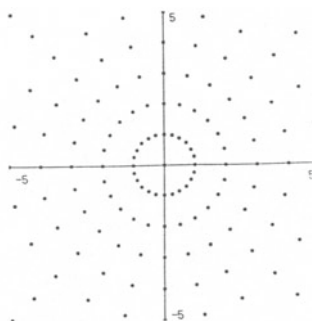


Figure B1

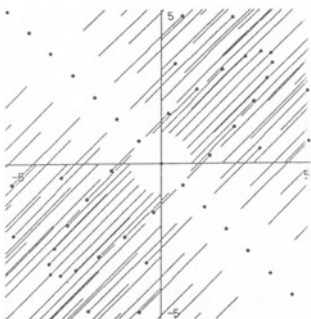


Figure B2

mathematical concept. To do this students must analyze the image and teachers have to help them in this regard. My solution after that experience was to put an indicator arrow in the picture that I could move around with the mouse. I find it beneficial to first have students consider the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and then to examine the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In this way, they come to understand the image as a whole and can get used to viewing that image as the signifier of a concept. (See Figures C1 and C2).

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$$

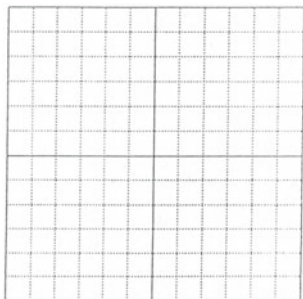


Figure C1

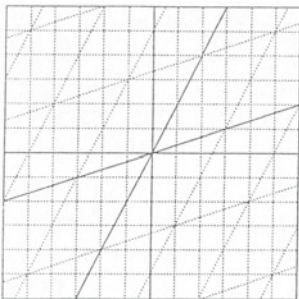


Figure C2

Inner product is another example. This is a difficult concept and students frequently have problems in understanding how to calculate it. In this case, I offer them near-concrete examples of rain and an umbrella, where both are seen as vectors because they have direction and size, and where the volume of rain striking the umbrella can be expressed using inner product. Here, students are shown that they do not need to concern themselves with concrete values. They see that changing the direction of each element changes the volume of rain that strikes the umbrella, and can accept the necessity of considering inner product as volume. Once students reach this stage, they are shown that inner product can be shown simply in terms of the area of a rectangle and come to a more complete understanding of the concept.

Semi-concrete objects

The schema of inner product I have just mentioned can be seen as halfway between a concrete object and an abstract figure. This can also be shown in another example, that of an explanation of $\sin \theta$. Students are asked to imagine a crane ship, floating on the water. They are told that the angle between the crane and the water's surface is θ , and that the distance separating the tip of the crane and the water's surface is $\sin \theta$. They are then asked to notice the changes in the crane's height as the angle is altered and, following the movement of the crane, they can estimate the value of $\sin \theta$ when the angle is between θ and 180° . Of course they see only a rough value, but their estimations lead them to an understanding of the meaning of $\sin \theta$ (see Figures D1 through D5).

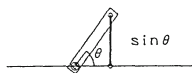


Figure D1

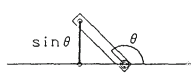


Figure D2

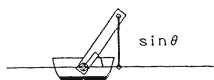


Figure D3

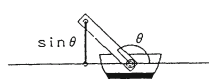


Figure D4

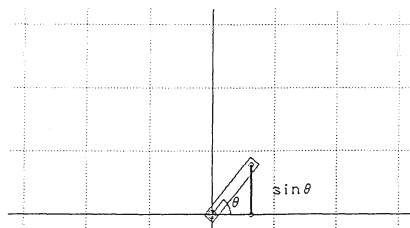


Figure D5

Following this, co-ordinates are added to the figure in the plane whose origin is the ship, and students are brought closer to an understanding of the abstract concept of $\sin \theta$. They are then taught the definition of a unit circle, and are shown an image of a unit circle in a black box. An angle is entered and the value of $\sin \theta$ is given by the unit circle. Students can see the output image of the value of $\sin \theta$ as a value of function. Certainly, we can regard the unit circle itself as the schema for trigonometric function,

but by moving the figure, teachers can focus more effectively on the process as a function.

We can also show the composition of trigonometric function through the revolution of folded lines.

$$a \sin x + b \sin(x + \theta)$$

When $\theta = 90^\circ$, we can describe the composition of $a \sin x$ and $b \cos x$, and when $a = \cos \alpha$, $b = \sin \alpha$, we can show the additive theorem (Figures E and F). Moving schema are invaluable in areas like these: they allow teachers

$$\begin{aligned} y &= 2 \sin x + \sin(x + 90^\circ) \\ &= 2 \sin x + \cos x \end{aligned}$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

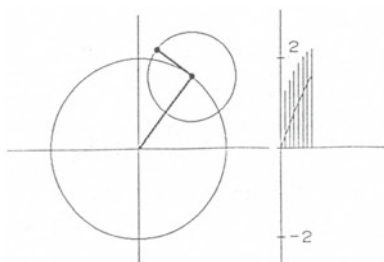


Figure E

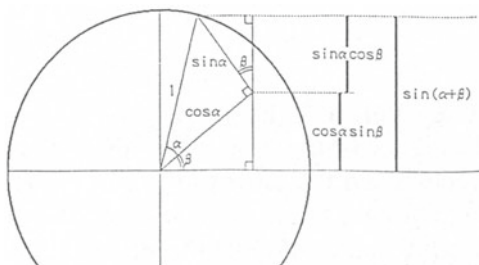


Figure F

to demonstrate difficult concepts with simple figures and movement. The following example (Figure G) uses the form of a snail shell to teach exponential function. The spiral of the shell is presented as a particularly elegant form of this function and is shown at first in natural section before being reduced to a simplified spiral. It is then used to explain logarithmic and exponential law.

The three cases $y = a^x$, $a = 3$ described here may be successful examples of the use of moving schema: their abstract renderings of concrete objects have certainly proved consistently popular with students.

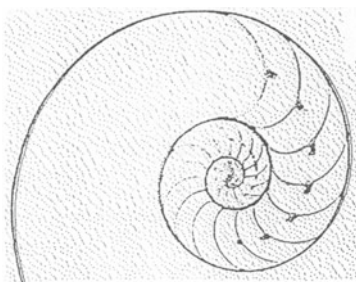


Figure G

Movement and preservation

Movement is useful in teaching laws and in focusing on elements or factors which remain constant. If we return to the previous example of inner product, we can consider, say, the inner product of a fixed vector and of a variable vector (see Figures H1, H2 and H3 opposite), and we can establish that only when \vec{x} moves perpendicularly to \vec{a} does the inner product remain constant. We can then see the equation of the straight line.

In other words, when $\vec{a} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, the equation of the straight line passing through the point (x_0, y_0) is

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ \Rightarrow ax + by = ax_0 + by_0$$

The whole and the parts

Using two arrows and the area of a rectangle, teachers can show two vectors and the inner product of \vec{a} and \vec{x} . In this way, when students are shown how to change the inner product by moving the arrows, they can readily appreciate that the value of the inner product affects each point of the plane. This is made clear by the shaded area of the rectangle, and is a clear illustration of how examining the part (the straight line) in relation to the whole leads to improved understanding overall.

In the same way, we can consider the example of linear transformation with a determinant of 0.

The nature of this linear transformation is clearly revealed when the straight line $x + y = c$ is shifted to the point (c, c) and the plane as a whole transforms into the straight line $y = x$. If students first look at the whole image and follow the transformation of each point on the plane, understanding how a part of the image, again a straight line, is transferred to a point becomes much simpler. Computer graphics aid understanding still further because they can also show the whole image in motion.

These principles also apply in the area of functions. Students can better understand the graph of, say, $y = e^x \sin x$ by first considering the whole graph of $y = a^x \sin x$. Here, when $a = 1$, $y = \sin x$, as in the original form, and they can see that with successive changes to the value of a , there is a certain moment during the graph's movement when $y = e^{-x} \sin x$. After considering $y = xa^x$, they can accept that $y = xe^{-x}$ can be obtained from $y = x$. Viewing graphs of functions in this way, that is, as shifting curves on planes, may be an important means of improving understanding (see Figures I1, I2 and J).

$$\begin{pmatrix} 4 & 0 \\ 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 1 & 8 \\ 6 & 6 \end{pmatrix} = (13, 8)$$

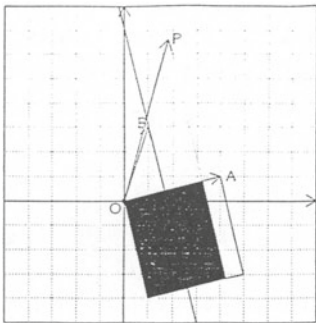


Figure H1

$$\begin{pmatrix} 4 & 0 \\ 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 0 & 2 \\ 6 & 2 \end{pmatrix} = (7, 0)$$

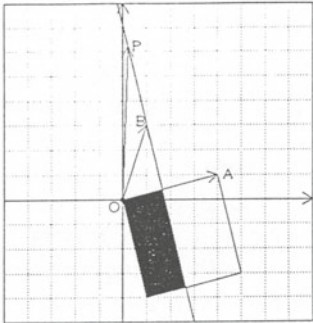


Figure H2

$$\begin{pmatrix} 4 & 0 \\ 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} -0 & 6 \\ 6 & 0 \end{pmatrix} = (3, 6)$$

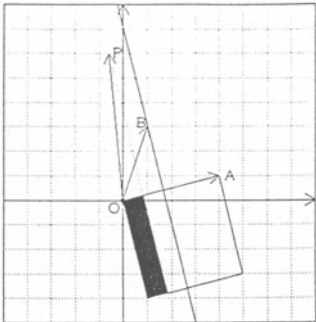
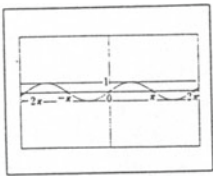
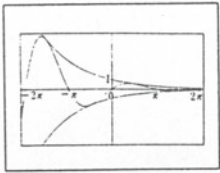


Figure H3



$$y = \sin x$$

Figure I1



$$y = a^{-x} \sin x$$

Figure I2

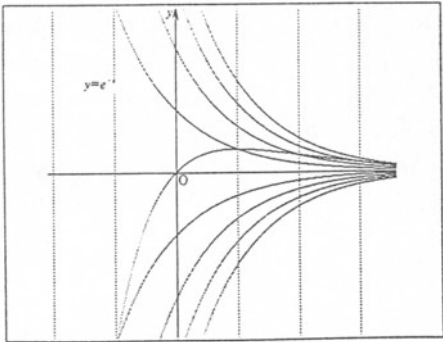


Figure J

Experience has shown that using examples like these in short teaching periods is ineffective. Perhaps students need time to establish the images in their own minds; certainly using these images does not mean that students will suddenly grasp the concepts involved. Symbolizing concepts brings creates its own difficulties, but immediate understanding is not necessarily a good thing in this area: students need time.

The point at issue

The ability to analyze is the key to seeing graphs as connections which carry meaning and as signifiers of concepts. Students do not perceive these images in the same way as teachers, and need to study a large number of them before they can acquire the necessary analytical skill to understand them correctly. Without it, they will continue to misunderstand the movements within figures and fail to distinguish certain movements from others. For the moment, I have little idea of how student analytical proficiency can best be developed.

CONCLUSION

Moving schema created with computers can enhance students' ability to form such schema in their own minds, and play an important part in understanding concepts. The most pressing task for the present is to order and arrange these figures in terms of movement so that students can understand them more easily. Moving schema, using the dynamic graphical capabilities of information technology are enabling mathematics students to tackle higher order conceptual thinking.



Ichiro Kobayashi has been a lecturer in mathematics at Japan's Kawaijuku Educational Institution since 1987. His publications include *Calculus for High School Students* (Sanseido) and *Linear Transformation of Second Degrees for High School Students* (Obunsha).