# Inter-Pixel Euclidean Paths for Image Analysis

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*Abstract.* Inter-pixel boundaries provide a robust and consistent description of segmented images but have a poor visual aspect, especially when being enlarged. Approximation curve are sometimes used to smooth discrete boundaries but they do not provide error free reconstruction and may be uneasy to use in this context. In this paper we show the advantages of using Euclidean paths in order to smooth inter-pixel boundaries and we demonstrate the interest of inter-pixel Euclidean paths for the purpose of image segmentation and analysis.

## 1 Introduction

The most common way to implicitly represent the regions of a 2D discrete image is the boundary representation. Two main approaches have been used. Roughly speaking, the first one [Fre61] consists in drawing the boundary "on" the pixels while the second one[BF70] consists in drawing it "between" the pixels. According to the terminology of *inter-pixel boundary* used by Fiorio[Fio95] we call this second approach the *inter-pixel oriented* approach, and thus the first one the *pixel oriented* approach.

Region boundaries present many interests from the point of view of both image synthesis and image analysis (extraction of geometrical features, image understanding, image compositing, etc.). Discrete boundaries provide an exact reconstruction of the domain of the image. On the other hand, by reason of their discrete nature, they have many geometrical drawbacks: aliased aspect, not differentiable, ill-adapted to geometrical transformations, etc. Several approaches have been developed in order to infer Euclidean boundaries from the discrete ones. Unfortunately these Euclidean boundaries have not a canonical form and do not provide an error free reconstruction of the region domain. In order to overcome this problem Braquelaire and Vialard have introduced *Euclidean paths* [BV95] which are a compromise between Euclidean and discrete boundaries.

The aim of this paper is to demonstrate the interest of mixing together both approaches of inter-pixel representation and Euclidean paths for the purpose of image segmentation and analysis. We consider images partitioned into

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4-connected regions. The boundary of a region is made of one or more inter-pixel 4-connected paths. The boundaries of the whole segmented image define a set of underlying topological maps [Dom92] providing a consistent topological framework. This model has been used as the kernel of a segmentation platform currently under development [BB96]. Euclidean paths have been implemented in this software, providing an interactive and automatic smoothing of boundaries. Such a smoothed boundary can be enlarged without staircase enhancement. Moreover since the discretization of the smoothed boundary produces exactly the discrete boundary, the smoothed boundary can be used to interactively extract geometric features and to drive segmentation steps.

In section 2 we briefly recall some advantages of the inter-pixel oriented approach and we describe the boundary model we have used. In section 3 we compare the Euclidean paths approach with the curve approximation ones. In section 4 we describe how to adapt Euclidean paths to a boundary graph. Finally several examples are given and discussed in the last section.

## 2 Pixel boundaries versus inter-pixel boundaries

The older approach used to represent the boundary of a discrete region is the *pixel oriented* approach where the boundary elements are defined in the domain of the image [Fre61, Mer73, Fre74, Pav82, RK82, CL90]. According to conventions the boundary points are either points of the region or of its complementary. The *inside pixel boundary* of a 4-connected region $r$ is the subset of its points having at least one 8-neighbor not belonging to $r$, and its *outside pixel boundary* is the set of pixels not belonging to $r$ but having a 4-neighbor in $r$. Finally, the boundary of a region induces one or more closed discrete curves (one plus the number of holes) also called *contours*.

A well known disadvantage of pixel oriented boundaries is that contours are not necessarily simple. Conversely, a simple curve drawn in the image space do not satisfy Jordan's theorem. Moreover two adjacent objects do not share the same boundary element. This drawback can be seen as a minor one when considering the restricted case of a foreground object laying on a background since it is generally useless to deal with the contour of the background. On the other hand it is a major drawback when considering an image decomposed into several adjacent objects.

An alternative to pixel boundaries is *inter-pixel* boundaries which has been introduced in 1970 by Brice and Fennema [BF70] as a data structure for implementing grouping segmentation. The pixels are considered as squares, two adjacent pixels sharing a vertical or an horizontal edge. A boundary is a sequence of pixel edges encoded by a sequence of points located at pixel corners. From the earlier eighties several theoretical approaches have been developed introducing *discrete topology*. The solution generally adopted is to extend the concept of pixel [KR89, KKM90b, KKM90a, Kov89, Bie90, Fra91, Fra95, AAF95] to allow a consistent definition of open and closed sets in $\mathbb{Z}^2$. These approaches provided a rigorous topological framework for digital boundaries. Another approach has been developed by Braquelaire and Domenger consisting in associating the inter-pixel boundaries of a segmented image with a set of topological maps [BD91, Dom92]. This approach is similar to the one proposed by Morse [Mor69] and Gangnet [GVT91].

Let us give the following definitions [BD96a]. The **half-integer plane**, denoted by $P_{\frac{1}{2}}$, is the plane resulting from the translation of the usual discrete plane $\mathbb{Z}^2$ by $(\frac{1}{2}, \frac{1}{2})$:

$$P_{\frac{1}{2}} = \{(i + \tfrac{1}{2}, j + \tfrac{1}{2}), \text{ with } (i, j) \in \mathbb{Z}^2\}$$

The half-integer plane is also called the **boundary plane**. If $p = (x_p, y_p)$ is a point of the image plane $\mathbb{Z}^2$ and $p' = (x'_p, y'_p)$ is a point of the boundary plane then $p$ and $p'$ are **half-neighbors** if $|x_p - x'_p| = |y_p - y'_p| = \frac{1}{2}$. Each point of the image plane has four half-neighbors in the boundary plane and conversely. Two adjacent points of the boundary plane share exactly two half-neighbors in the image plane. Each point of the boundary plane having two or more half-neighboring points belonging to different regions of a segmented image is a **boundary point**. Two adjacent boundary points are **linked** if their common half-neighbors belong to different regions. A **contour** is a sequence of boundary points $b_1, b_2, \ldots, b_n$ with $n > 1$ and such that:

1. $b_i$ is linked to $b_i + 1$, $\forall i$ with $1 \leq i < n$.
2. $b_i \neq b_j$, $\forall i, j$ with $i \neq j$.

A **closed contour** is a sequence of boundary points $b_1, b_2, \ldots, b_n, b_1$, such that

1. $b_1, b_2, \ldots, b_n$ is a contour.
2. $b_n$ is linked to $b_1$.

Any boundary point linked to more than two other ones is a **natural node**. A **segment** is a maximal contour without node. An **augmented segment** is a segment augmented with both its linked nodes.

By embedding both the boundary plane and the image plane into the Khalimsky's plane [KKM90a] it can be verified that a closed single path of $P_{\frac{1}{2}}$ satisfy Jordan's theorem and that the boundaries of a segmented image define a unique set of planar maps augmented with an inclusion relation [BD96a]. These maps are canonically induced by the decomposition of the boundary into nodes, corresponding to map vertices, and segments, corresponding to map edges.

*Remark.* In order to fit with the usual definition of topological maps [Cor75] an *arbitrary node* has to be added on each closed contour without natural node. But this aspect can be ignored for the purpose of this paper. Thus, in order to simplify this presentation we shall consider in the following that a single loop without node is also a segment.

To sum up boundaries defined in $P_{\frac{1}{2}}$ have the following properties:

1. The boundaries of the segmented image are decomposed into nodes and segments. A segment is either a single loop or a single path joining two nodes.
2. Each connected set of boundaries defines a planar map.
3. The boundary of a region is a set of simple closed 4-connected paths.
4. Two adjacent regions share one or more augmented segments which constitute their common boundary.
5. This decomposition is canonical.

# 3 Smoothing discrete paths

As sketched above a discrete contour has many geometrical drawbacks and it is interesting to build a Euclidean curve as close as possible to the discrete one. Several criteria can be considered to evaluate the quality of a Euclidean boundary. We have retained the following ones:

1. The *visual aspect* (smoothness, preservation of lines and angles, etc.). This feature is of course subjective.
2. The *uniqueness* (does the method produce a unique Euclidean boundary from a discrete one ?). This feature is important for an automatic construction of the Euclidean curve.
3. The *reconstruction quality*. A reconstruction is *error free* if it produces exactly the same set of pixels than the one defined by the discrete boundary. An error free reconstruction is a strong condition of closeness of both Euclidean and discrete curves.
4. The *complexity*. It includes the computational complexity of the construction and the complexity of use of the method.

A fifth aspect which is the extraction of geometrical features such as perimeter or tangent orientations, can also be considered to estimate the quality of a Euclidean boundary. This aspect is developed in another work [Via96].

The usual approach used to build a Euclidean boundary from a discrete one is an approximation approach. The problem consists in finding the curve from a given family of curves which minimizes some error criterion. The vectorization approach is based on the construction of a sequence of real straight lines of which a possible discretization is the discrete contour. Several techniques exists but none of these methods provides a canonical vectorization and the result may highly depend on the boundary point on which the vectorization is initiated. It is also possible to use higher degree approximation curves such as splines. Each boundary point can be used as a control point of the spline curve. Experiments show us that to avoid oscillations it is necessary to use very high degree curves which does not preserve angular points. It is possible to reduce the set of control points to a subset of the discrete points but it is very difficult to do it automatically. Moreover splines cannot ensure an error free reconstruction.

Another model called *Euclidean paths* has been proposed [BV95] to smooth discrete contours. This approach is intermediate between a purely discrete approach and a continuous one. A Euclidean path associated with a discrete path $(p_1, \ldots, p_n)$ is a sequence of Euclidean points $(\pi_1, \ldots, \pi_n)$ which verifies the following property:

$$\forall k \in \{1, \ldots, n\}\ |x_k - i_k| < \frac{1}{2} \text{ and } |y_k - j_k| < \frac{1}{2} \tag{1}$$

with $\pi_k = (x_k, y_k)$ and $p_k = (i_k, j_k)$.

By rounding each coordinate of a point of a Euclidean path to the closest integer we get the related coordinate of the associated discrete path. Thus the reconstruction is obviously error free. We have shown that by using the definition of discrete lines given by Reveilles [Rev91] and by adapting the vectorization algorithm of Debled and Reveilles [DR92] it is possible to compute in a linear time a local tangent $\Delta(p)$ at each boundary point $p$ such that there exists a canonical projection of $p$ on $\Delta(p)$ which is a Euclidean point associated with $p$. This method provides a technique of construction of a family of Euclidean paths called *Tangent driven Euclidean paths* (see [BV95]).

To sum up Euclidean paths present the following advantages:

1. They provide a good visual aspect by removing jaggies and preserving angles (see for instance the example displayed in Figure 6).
2. The construction is canonical.
3. The reconstruction is error free.
4. The complexity is in $O(l.n)$ where $l$ is the average length of the maximal discrete lines around each discrete point and $n$ the number of discrete points on the processed contour.

Finally Euclidean paths provide good estimation of perimeter,local tangents and curvature [Via96].

## 4 Mixing both approaches

In previous sections we have pointed out the advantages

- of the inter-pixel oriented approach compared with the pixel-oriented one for the representation of segmented image,
- of Euclidean paths compared with approximation curves for the boundary smoothing of a discrete region.

Both approaches can be combined in order to smooth the boundaries of a segmented image. We call inter-pixel Euclidean path ( in short IPEP) the Euclidean path associated with an inter-pixel boundary.

Recall that the set of boundaries defines a partition of the segmented image into regions. The boundaries of these regions are made of nodes and segments defined in the $P_{\frac{1}{2}}$ plane. The set of nodes and segments which defines the boundary of a region $i$ is a closed curve $C_i$ defined in the $P_{\frac{1}{2}}$ plane. An IPEP $EC_i$

can be associated to each closed curve $C_i$. Thus, starting from a partition, of an image, we can build a set of closed curves $\{C_1, \ldots, C_n\}$, and a set of IPEP $\{EC_1, \ldots, EC_n\}$ based on these curves. The drawback of this method is that it does not preserve the consistency of the segmented image. As a matter of fact, let us suppose that two regions $i$ and $j$ are adjacent by a segment. Then we have $C_i \cap C_j = S_{ij}$, where $S_{ij}$ is a segment delimited by two nodes. Since $S_{ij}$ is shared by two regions, it has been traversed twice for computing $EC_i$ and $EC_j$. Thus two Euclidean points are associated with each point of $S_{ij}$. If all Euclidean points associated with the segment $S_{ij}$ are distinct we may have $EC_i \cap EC_j = \emptyset$ while $C_i \cap C_j = S_{ij} \neq \emptyset$. To overcome this inconsistency we associate with each boundary point $p$ the average of the Euclidean points associated to $p$. Since all Euclidean points associated to a boundary point verify equation 1, the mean of these points will verify the same equation and the set of points obtained by the averaging is still a set of Euclidean paths.

Thus, there is a one to one mapping between the boundaries and smoothed boundaries. Let us call *boundary cells* the open unit squares centered at boundary points. Since boundaries are 4-connected paths three adjacent boundary points may either be in line or make a right angle (see Figure 1). Each point of the IPEP boundary is located in the boundary cell centered at its associated boundary point. It follows that a boundary and its associated Euclidean path traverse exactly the same sequence of boundary cells. Thus if two boundary elements do not intersect then their associated IPEP do not intersect either. Conversely the construction described above ensures that intersection between boundary elements are preserved by smoothing. Hence the smoothing of the boundaries with Euclidean paths preserve the topology.

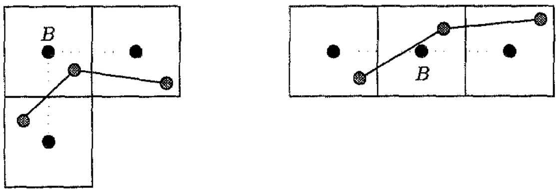


**Fig. 1.** The two possible neighboring configurations for a boundary point $B$ which is not a node.

Since Euclidean paths may also be defined for non-closed discrete curves a less time consuming method would be to compute the Euclidean path of each augmented segment. In this case, each segment will be traversed a single time and only one Euclidean path will be associated with each segment. The averaging would then be limited to nodes. This strategy has some drawbacks, which lead us to reject it. First, the object of interest of a segmented image are the regions which partition it. Thus a computation of IPEP based on regions is more meaningful than a computation of IPEP based on segments. Moreover, the computation of IPEP based on segments tends to deteriorate the smoothing effect around nodes.

## 5 Experiments and discussion

In this section we present some examples of inter-pixel Euclidean paths on *"real cases"*. All these images have been segmented and displayed by using an environment of segmentation that we are currently developing (this environment is described in several other works [BB96, BD96b]).

Consider the example displayed in Figure 2. The small square drawn on the upper right part corresponds to a pixel. There are two uniform grey regions with both the inter-pixel boundary and the associated Euclidean path. Remark first that both inter-pixel boundaries and Euclidean paths are symmetrical models of contour representation in the sense that a contour is never attached to a specific object of the image. An inter-pixel boundary separates equitably both objects sharing it. In the same way Euclidean paths provide a local best adjustment of all discrete lines composing the inter-pixel boundary. The inter-pixel Euclidean path boundary keeps a smooth appearance while keeping close to the inter-pixel boundary.

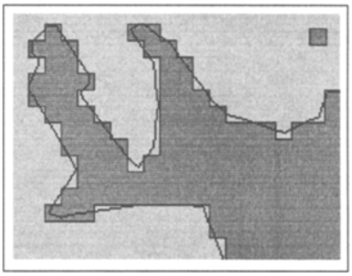

**Fig. 2.** The path drawn in black is the Euclidean path of the inter-pixel boundary separating both grey regions. The inter-pixel boundary is displayed in dark gray.

The left image displayed in Figure 3 shows a brain RMI on which is superimposed the inter-pixel boundaries resulting from a segmentation step. Of course inter-pixel boundaries cannot be displayed at scale 1 and the half-integer boundaries have been translated by $(\frac{1}{2}, \frac{1}{2})$ to be drawn in the image plane. Nevertheless, even when drawn at scale 1, IPEP improve the visualization since they can be displayed in 8-connexity as real polylines. Pixels will then be drawn either on the inside or on the outside pixel boundary (see the right image of the same Figure).

Since the construction of Euclidean paths is canonical the smoothing of the boundaries does not need any user intervention. Moreover the algorithm being linear (see section 3) the smoothed boundary of a region can be drawn interactively. It allows us to switch between discrete and smoothed boundaries. When

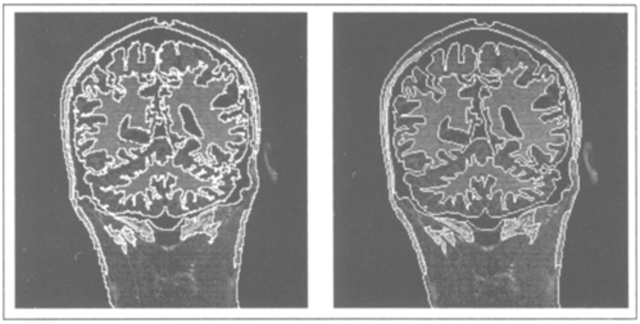

**Fig. 3.** This image results from a brain RMI. The size of the image is 230 × 230 pixels. On the left image are superimposed the discrete boundaries translated from the boundary plane and on the right one the discretization of the IPEP smoothed boundaries.

segmenting a small region it may be necessary to use an enlargement of the related area of the working image. In that case pixel boundaries can be displayed by drawing horizontal and vertical steps between pixels. Unfortunately the stair case effect is enhanced making the shape of the working region difficult to perceive. This drawback is removed when switching from discrete boundaries to IPEP smoothed boundaries (see Figure 4).

It is also possible to edit the contour by interactively addressing IPEP points with the mouse and working on the associated boundary points. The editing may be facilitated by superimposing both contours on the same image.

IPEP contours are very efficient for enlarging of discrete contours. In Figure 5 is shown an enlargement of a region of size 26 × 19 pixels in the original image ( it is the small dark area located in the middle of the right hemisphere). The image of the region is displayed at scale 10. The IPEP smoothed boundary shown in the right image of the picture still has a visual aspect very close to the shape of the original region. There is a small irregularity on the upper-right part of the smoothed boundary. By comparing the discrete boundary with the smoothed one it appears that this irregularity corresponds to the pixel located on the second row and on the eighth column. This pixel has been aggregated to the dark region. The irregularity can be removed simply by changing interactively the assignment of this pixel. This example shows how IPEP smoothed boundary can help the user during an interactive segmentation and editing.

The last example results from the segmentation of a small color image of size 139 × 200 pixels (see Figure 6). Our segmentation environment allows the user to perform any segment and region removing at any time during the segmentation process. The segmented image has been edited to keep only the contour of the meloidae. The upper right drawing is the inter-pixel boundary at scale 1. The smoothed boundary is the associated IPEP boundary at scale 4. In figure 7 is visualized the upper horn at scale 10 with the corresponding part of the image also at scale 10.

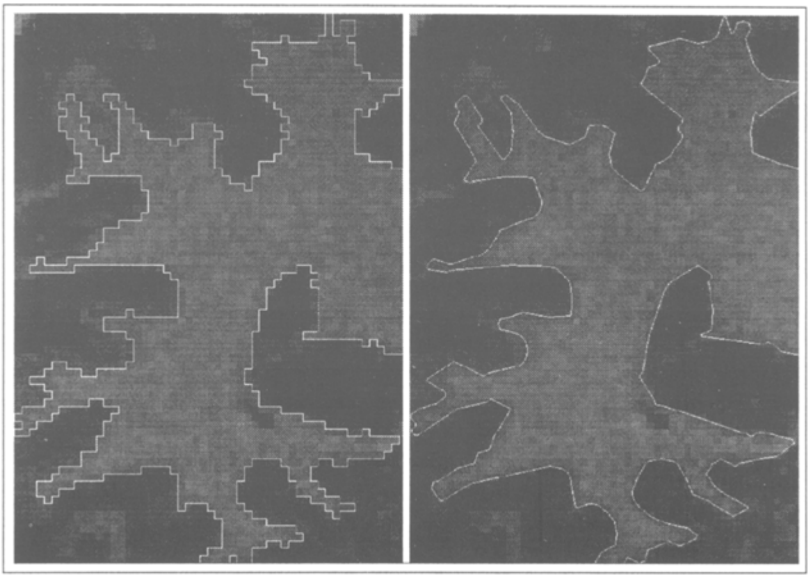

**Fig. 4.** The left picture shows an enlargement of the inter-pixel boundary of the upper part of the left hemisphere displayed in Figure 3. The right picture shows the associated IPEP smoothed boundary.

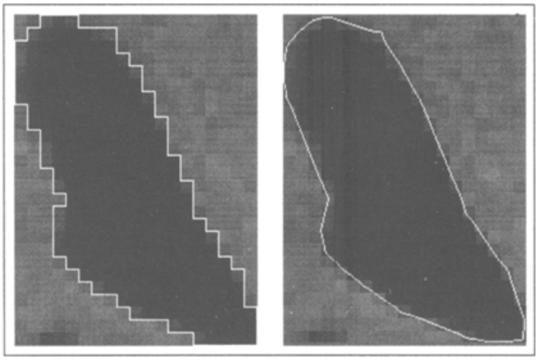

**Fig. 5.** This is an enlargement by 10 of the small dark area of the right hemisphere. The original size of the region is 26 $\times$ 19 pixels.

Finally IPEP smoothed boundaries provide an efficient framework for image analysis. The underlying topological maps allow a very efficient extraction of topological features and the Euclidean paths produces good results for geometrical features extraction ([Via96]).

## 6 Conclusion

In this paper we have proposed to combine inter-pixel boundaries with Euclidean paths in order to improve image segmentation and we have described how to smooth a graph of boundaries. We have shown that IPEP smoothed boundaries present several important advantages: interactive and automatic construction, error free reconstruction, good visual aspect, efficiency for image enlargement. Moreover IPEP smoothed boundaries lay on the same unit cells as the discrete boundaries and preserve their topology. For these reasons IPEP smoothed boundaries are very efficient for editing and segmenting small areas of the image. For the same reason we think that IPEP can be integrated in the segmentation process in order to drive segmentation steps. We are currently working on this problem.

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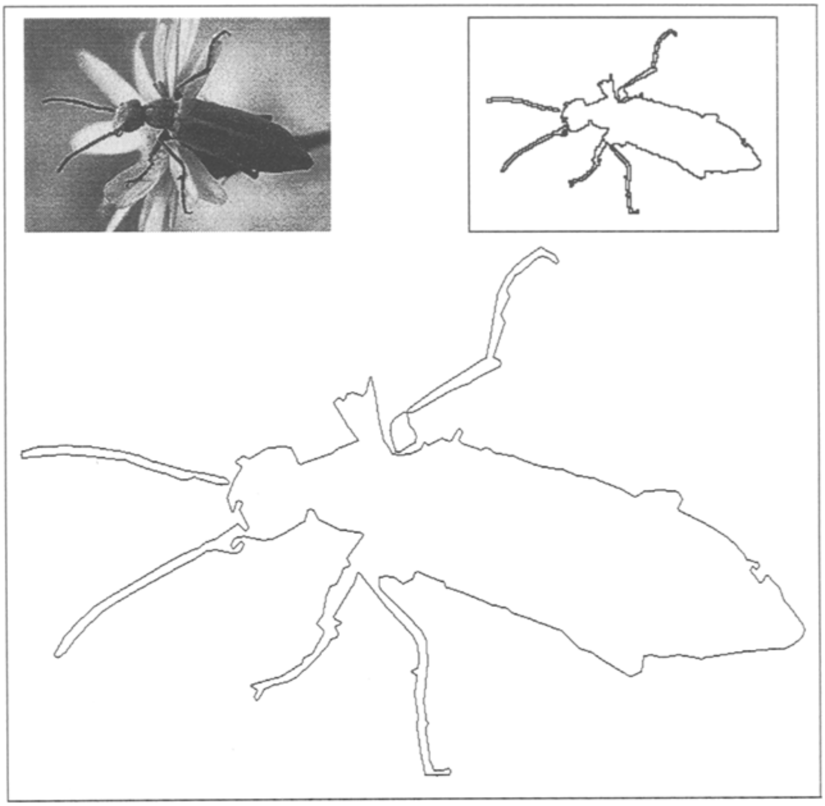

**Fig. 6.** The size of the original image is 139 × 200 pixels.

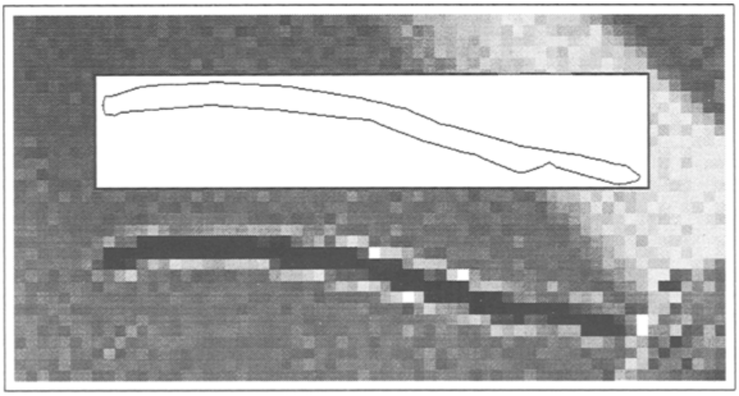

**Fig. 7.** In this image is displayed at scale 10 the upper horn of the insect with the drawing at the same scale of the IPEP smoothed boundary of the horn.