Vibration Modes for Nonrigid Motion Analysis in 3D Images

Chahab NASTAR

INRIA Rocquencourt
B.P. 105, 78153 Le Chesnay Cedex, France.
Email: Chahab.Nastar@inria.fr

Abstract. We concentrate on nonrigid motion analysis in 3D images using modal dynamics. Borrowing the solid state physics formulation, we develop the equations of motion and the analytical expression of vibration modes of a multidimensional elastically-deformable model. Thus, nonrigid motion of a 3D deformable object can be recovered in closed-form in *real time*. The power of the approach is demonstrated by a set of experimental results on 3D medical data.

1 Background

Following the theory of deformable models, physically-based modelling for nonrigid motion analysis has become extremely popular [5]. For purposes of deformation analysis, we make use of modal analysis [4, 2], a well-known mechanical engineering technique which consists in decomposing and approximating the motion in the free vibrations basis (modes) of the model.

2 Modal analysis for deformable models

Consider a discrete mass spring mesh of N nodes. Using the equations of dynamics, such a structure can be elastically deformed in 2D or 3D images to match the contour of an object of interest. In 3D, the system is governed by the $3N$-dimensional matrix equation :

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{F}(t) \tag{1}$$

where $\mathbf{U}$ is a vector storing nodal displacements $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the mass, damping and stiffness matrices of the system, and $\mathbf{F}$ is the image force which has the object attracted by image edges. Equation (1) is the finite element formulation of the deformation process.

Instead of solving directly the equilibrium equation (1), we transform it by a change of basis : $\mathbf{U} = \boldsymbol{\Phi}\tilde{\mathbf{U}}$ where $\boldsymbol{\Phi}$ is a matrix whose entries are the eigenvectors of the generalized eigenproblem $\mathbf{K}\boldsymbol{\phi} = \omega^2\mathbf{M}\boldsymbol{\phi}$:

$$\mathbf{U}(t) = \boldsymbol{\Phi}\tilde{\mathbf{U}} = \sum_{i=1}^{3N} \tilde{u}_i(t)\boldsymbol{\phi}_i \tag{2}$$

Lecture Notes in Computer Science, Vol. 800
Jan-Olof Eklundh (Ed.)
Computer Vision – ECCV '94

Equation (2) is referred to as the modal superposition equation. ϕ_i is the ith mode, $\tilde{u}_i$ its *amplitude*, and ω_i its frequency. The new modal basis simultaneously diagonalizes $\mathbf{M}$ and $\mathbf{K}$, and provided that the matrix $\tilde{\mathbf{C}} = \mathbf{\Phi}^T \mathbf{C} \mathbf{\Phi}$ is diagonal as well, the governing matrix-form equations decouple into $3N$ scalar equations :

$$\ddot{\tilde{u}}_i(t) + \tilde{c}_i \dot{\tilde{u}}_i(t) + \omega_i^2 \tilde{u}_i(t) = \tilde{f}_i(t) \qquad i = 1, \ldots, 3N. \tag{3}$$

Solving these equations at time t leads to $(\tilde{u}_i(t))_{i=1,\ldots,3N}$, and the displacement of the nodes is obtained by the modal superposition equation.

One of the major interests of modal analysis is that it provides an approximate but quite accurate a closed-form solution by selecting a few number p of low-frequency modes ($p \ll 3N$). The contribution of each mode to the motion is graded, providing a compact description of the motion.

But even as a precalculation, solving the generalized eigenproblem is very costly as soon as we consider 3D boundaries (surfaces). For instance, if we consider a mesh of 100×100 nodes, *a generalized eigenproblem where the size of the matrices is 30000 × 30000 has to be solved.* It is then clear that the analytical expression of the modes would noticeably reduce the computations. This leads us to consider the solid state physics theory, where similar types of problems are encountered at a microscopic level (ionic vibrations of a crystal lattice). If we parameterize our deformable curves by arc length, and similarly our deformable surfaces by natural coordinates, we get periodic boundary conditions which depend on surface topology. This allows the analogy between our deformable model and a crystal lattice.

3 Solid state physics formulation

The classical theory of vibration of a crystal lattice is based on the *harmonic approximation*, a theory which assumes that the first non-vanishing correction to the equilibrium potential energy is quadratic [1] :

$$V^{harm} = \frac{1}{2} \sum_{\substack{\mathbf{R},\mathbf{R}' \\ \mu,\nu}} u_\mu(\mathbf{R}) D_{\mu\nu}(\mathbf{R} - \mathbf{R}') u_\nu(\mathbf{R}') \tag{4}$$

where $u_\mu(\mathbf{R})$ is the displacement in the μ direction of the ion whose mean position is $\mathbf{R}$, and $\mathbf{D}$ is the Hessian matrix of the interaction energy. The motion in the μ direction ($\mu = x, y$ or z) of the point defined by $\mathbf{R}$ on the lattice is defined by :

$$M \ddot{u}_\mu(\mathbf{R}) + C \dot{u}_\mu(\mathbf{R}) + \frac{\partial V^{harm}}{\partial u_\mu(\mathbf{R})} = F_\mu(\mathbf{R})$$

In a three-order matrix form :

$$\mathbf{M}\ddot{\mathbf{u}}(\mathbf{R}) + \mathbf{C}\dot{\mathbf{u}}(\mathbf{R}) + \sum_{\mathbf{R}'} \mathbf{D}(\mathbf{R} - \mathbf{R}')\mathbf{u}(\mathbf{R}') = \mathbf{F}(\mathbf{R}) \tag{5}$$

For the whole lattice, N matrix equations of the above type are to be solved. Each one of these differential equations is *linear*, and nodal displacements are *coupled*

in space positions (the nodes) and directions (the axes). Note that the model does not recover angular variations, and rotational motion has to be linearized (similar limitations can be found in [4]). Thus, in equation (1), the stiffness matrix $\mathbf{K}$ is constituted by $\mathbf{D}$ submatrices. If these submatrices are 3×3 diagonal matrices, that is, if we assume that *a nodal motion in the μ direction exerts a force on the node's neighbors in the same μ direction*, the $3N$-order matrix equation above has the nice property of *decoupling* into three N-order matrix equations in each space direction, as it clearly appears in equation (5). From now on, we will keep this assumption which reduces computational cost as nodal vectors and matrices are of order N instead of $3N$ [2].

Free vibrations of a one-dimensional lattice : Consider a set of ions distributed along a chain at points separated by a distance a, so that the lattice vectors are $R = na$ for $n \in \{1, \ldots, N\}$. If only neighboring ions interact, we may take the harmonic potential energy to have the form :

$$V^{harm} = \frac{1}{2} K \sum_{n=1}^{N} [u(na) - u((n+1)a)]^2$$

where $K = v"(a)$ is the stiffness constant of the system, $v(x)$ being the interaction energy of two ions a distance x along the chain. The free vibrations of the lattice are governed by :

$$M\ddot{u}(na) = -\frac{\partial V^{harm}}{\partial u(na)} = K(u((n+1)a) + u((n-1)a) - 2u(na)) \qquad (6)$$

These are precisely the equations that would be obeyed if each ion were connected to its neighbors by perfect mass-less springs of stiffness K (and equilibrium length a, although the equations are in fact independent of the equilibrium length of the spring).

We seek solutions to equation (6) of the form : $u(na,t) = e^{-i\omega t}(Ae^{ikna} + Be^{-ikna})$. The periodicity of the chain leads to a discretization of the values of ka, depending on the chain being closed or open, as shown below :

	$k_p a$	$p \in$	$u(na,t)$
closed	$\frac{2p\pi}{N}$	$\mathcal{FBZ}$	$\sum_{p \in \mathcal{FBZ}} A_p e^{i(k_p na - \omega_p t)}$
open	$\frac{p\pi}{N}$	$\{0, \ldots, N-1\}$	$\sum_{p=0}^{N-1} A_p e^{-i\omega_p t} e^{i\frac{k_p a}{2}} cos(k_p na - \frac{k_p a}{2})$

$\mathcal{FBZ}$ is the "first Brillouin zone", equal to $\{-\frac{N}{2}+1, \ldots, \frac{N}{2}\}$ for N even, and $\{-\frac{N-1}{2}, \ldots, \frac{N-1}{2}\}$ for N odd. Substituting for u in (6) leads to the *dispersion equation* :

$$\omega_p^2 = \frac{4K}{M} \sin^2(\frac{k_p a}{2}) \qquad (7)$$

As the values of k are discrete, the vibration states of the crystal are discretized. This is the concept of *phonons* in solid state physics. The displacement expression $u(na,t)$ shows that the motion is fully determined by specifying N initial positions and N initial velocities of the ions. In mathematical terms, this equation can be seen as the Fourier expansion of the displacement in the basis of the complex exponential functions of period N. From a physics point of view, the solution describes waves propagating along the chain with phase velocity ω/k and group velocity $\partial\omega/\partial k$.

4 Connection with the theory of modal analysis

For 2D curves, the eigenvalues are defined by the dispersion equation, and the eigenvectors are the real parts of the complex modes in the general expression of displacements :

	eigenvalue	eigenvector
closed	$\frac{4K}{M}\sin^2(\frac{p\pi}{N})$	$[\ldots,\cos\frac{2p\pi n}{N},\ldots]^T$
open	$\frac{4K}{M}\sin^2(\frac{p\pi}{2N})$	$[\ldots,\cos(\frac{p\pi(2n-1)}{2N}),\ldots]^T$

In order to generalize the analytical expressions of the modes to surfaces (3D), some topological properties have to be outlined. We use natural coordinates so that the parameters are submitted to different pairs of boundary conditions : closed and closed (torus), open and open (plane), open and closed (cylinder [1]). This means that we use quadrilateral elements.

	eigenvalue	eigenvector
planar	$\frac{4K}{M}(\sin^2\frac{p\pi}{2N}+\sin^2\frac{p'\pi}{2N'})$	$[\ldots,\cos\frac{p\pi(2n-1)}{2N}\cos\frac{p'\pi(2n'-1)}{2N'},\ldots]^T$
toric	$\frac{4K}{M}(\sin^2\frac{p\pi}{N}+\sin^2\frac{p'\pi}{N'})$	$[\ldots,\cos(\frac{2p\pi n}{N}+\frac{2p'\pi n'}{N'}),\ldots]^T$
cylindrical	$\frac{4K}{M}(\sin^2\frac{p\pi}{2N}+\sin^2\frac{p'\pi}{N'})$	$[\ldots,\cos\frac{p\pi(2n-1)}{2N}\cos\frac{2p'\pi n'}{N'},\ldots]^T$

In case of multiple eigenvalues (cylindrical and toric topology), a set of orthogonal eigenvectors may be found by using the corresponding "sine" functions of those described above. Note that $n=1,\ldots,N$ and $n'=1,\ldots,N'$.

5 Modal approximation

Figure 1 demonstrates the power of modal approximation in medical imaging. We make use of a set of 4D (3 space dimensions plus time) nuclear medicine data of the moving left ventricle. The left ventricle is extracted from the data, then tracked from the diastole to the systole (the ventricle is shown as a mesh of 4000 nodes moving from the diastole towards the systole, a rendered surface).

[1] The displayed shapes in this paper have cylindrical topology.

With a few modes, the result of the superposition has roughly the same shape as the original diastole (fig. 1.a and 1.b), thus illustrating low-pass filtering of the motion. As we increase the number of modes, we describe the motion more accurately (fig. 1.c), but we do not need to keep all modes, since a the final shape may be recovered in closed-form by superimposing a few modes (fig. 1.d). In other terms, the approximation error $\| \sum_{i=p+1}^{3N} \tilde{u}_i(t)\phi_i \|$ is a rapidly-decreasing function of p [3].

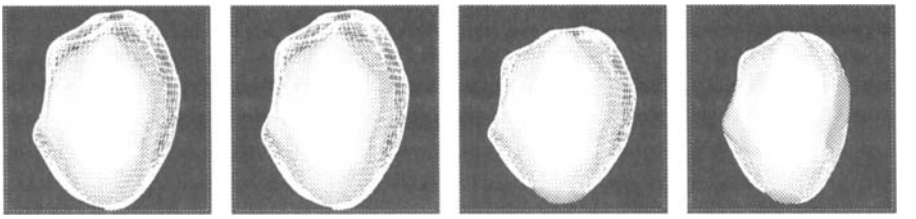

Fig. 1. Modal approximation. From left to right and top to bottom : **a.** Initial and final shapes. **b.** 3 modes, compression is 4000. **c.** 21 modes, compression is 571. **d.** 51 modes, compression is 235, and the final shape is recovered in closed form.

6 Animation of medical data

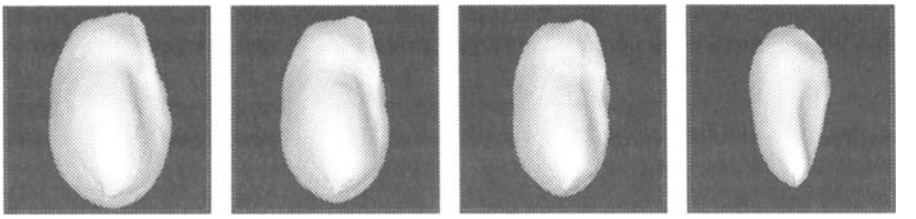

Fig. 2. Animation of the left ventricle between diastole and systole (6400 nodes)

We make use of modal dynamics for animation of our medical data. Indeed, one can deform any complex 3D object through its modes by using the modal superposition equation.

Canine heart data from the dynamic spatial reconstructor (DSR, a high speed X-ray CT scanner) was used as an input to our deformable model, which has reconstructed the left ventricle and recovered its motion in closed form during a cardiac cycle (figure 2). One can observe that the papillary muscles show up clearly. In figure 3, the human head segmented by our model from 3D MR data is animated by adding arbitrary modes.

Note that these animations are *real time irrespective of the size of the structure* since we make use of an analytical expression of mode shapes.

7 Conclusion

We have presented modal dynamics from a new point of view : *the computation of the analytical expression of vibration modes.* The solid state physics formulation

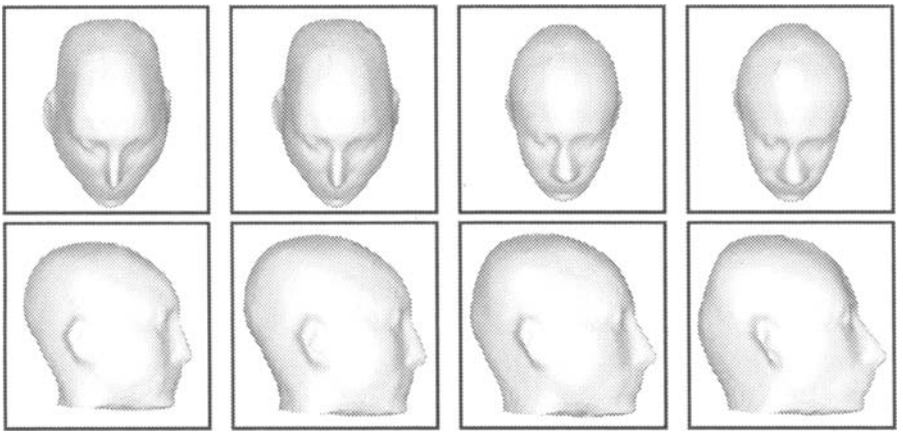

Fig. 3. Animating the human head by adding directional modes (11130 nodes)

is indeed a sound framework for deriving the equations of motion of deformable models. We have shown the power of modal approximation for a compact description, smoothing and *real time* animation of a complex deformation (e.g. 3D left ventricle motion, 3D human head). Current applications of our computations include *nonrigid motion classification* [3].

Acknowledgements : we wish to thank Nicholas Ayache for helpful discussions and advices. Thanks are also due to Siemens France and Sopha Medical France for providing the head and the heart data, and to Dr. Robb and Mr. Hanson at Biomedical Imaging Resource, Mayo Foundation/Clinic, for the DSR data. This work was supported in part by a grant from Digital Equipment Corporation.

References

1. Neil W. Ashcroft and M. David Mermin. *Solid State Physics.* Saunders College Publishing International Edition, 1976.
2. Chahab Nastar and Nicholas Ayache. Fast segmentation, tracking, and analysis of deformable objects. In *Proceedings of the Fourth International Conference on Computer Vision (ICCV '93)*, Berlin, May 1993.
3. Chahab Nastar and Nicholas Ayache. Classification of nonrigid motion in 3D images using physics-based vibration analysis. Technical report, INRIA, 1994. Submitted to the IEEE Workshop on Biomedical Image Analysis, Seattle, June 1994.
4. Alex Pentland and Stan Sclaroff. Closed-form solutions for physically based shape modelling and recognition. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-13(7):715–729, July 1991.
5. Demetri Terzopoulos, Andrew Witkin, and Michael Kass. Constraints on deformable models: recovering 3-D shape and nonrigid motion. *AI Journal*, 36:91–123, 1988.