(n, m)-Cubes and Farey Nets for Naive Planes Understanding

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Abstract. A digital naive plane can be represented by repetition of specific elements, called (n, m)-cubes, composed of $n \times m$ adjacent voxels. The aim of this paper is to study the class of (n, m)-cubes appearing in a plane in relation with the parametric representation based on the normal vector. Planes are ordered using Farey series coding and we prove the relationship between the segmentation issued from the Farey net and configurations of (n, m)-cubes. This is an original contribution.

1 Introduction

The topological development as well as the geometrical development of the discrete space ($\mathbb{Z}^n$) theory is continually increasing. Algorithmic problems which are difficult to solve by the Euclidean geometry are easier to solve in the discrete context as it is the case for object representation or squelettization problems [3]. Moreover, the arithmetic definitions of discrete lines and planes [11][1] allow the entire discrete analysis of objects.
The problem of recognizing digital lines [4] is now solved but the generalization to discrete planes [4][5] is not optimal. It would be interesting to recognize digital planes by their geometry. The study of the coexistence of tricubes in a discrete plane has been studied by different authors either by Fourier algorithm [7][13] or by a syntaxic analysis [14]. Nevertheless the characterization of planes by pieces with different sizes could reduce the set of corresponding planes.
In this paper our interest is to generalize the representation of naive planes by (n, m)-cubes. For this, we will see the links between the parameters of naive planes and their position in Farey nets via a two-dimensional continued fraction algorithm [9][2][8]. We will illustrate the way how (n, m)-cubes generators of the naive plane can be identified to the parameters of the naive plane. Some results on 2D digital lines [6][10] are here extended to 3D digital planes.

2 Definitions

A **digital naive plane** of normal vector (a, b, c) and translation parameter r is defined as the set of points $M(x, y, z) \in \mathbb{Z}^3$ satisfying the double-inequality:

$$0 \le ax + by + cz + r < \max(|a|, |b|, |c|)$$

G. Bertrand, M. Couprie, L. Perroton (Eds.): DGCI'99, LNCS 1568, pp. 76–87, 1999.

where all parameters a, b, c, r are integers and a, b, c are not null all together and verify $\gcd(a,b,c) = 1$. We limit our study to naive planes in the 24^{th} part of space such that $0 \le a \le c$, $0 \le b \le c$ and $c \ne 0$. These planes will be noted by $\mathcal{P}(a,b,c,r)$.
We call **remainder at point** $M(x,y,z)$ of the naive plane $\mathcal{P}(a,b,c,r)$ the value $\mathcal{R}(x,y) = (ax + by + r) \bmod (c)$.
The **lower leaning points** (resp. **upper leaning points**) of the naive plane $\mathcal{P}(a,b,c,r)$ are the points $M(x,y,z)$ satisfying $\mathcal{R}(x,y) = 0$ (resp. $\mathcal{R}(x,y) = c-1$) (fig.1(a)).
For $n \ge 2$ and $m \ge 2$, the (n,m)**-cube at point** (i,j) of the naive plane $\mathcal{P}$ is defined as the set $\mathcal{C}(i,j,n,m) = \{(x,y,z) \in \mathcal{P} \,/\, i \le x < i+n,\ j \le y < j+m\}$ (fig.1(b)).

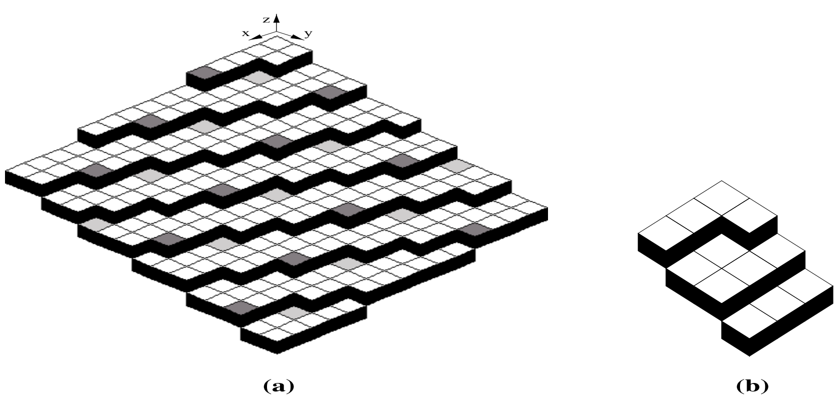


Fig. 1. *(a) Part of the naive plane* $\mathcal{P}(3,7,19,0)$*; in dark gray we have the upper leaning points and in light gray, the lower leaning points. (b) A* $(3,4)$*-cube of* $\mathcal{P}(3,7,19,0)$*.*

Previous works [12] have shown that a naive plane satisfies the two following properties:

Property 1. A naive plane contains at most nm different configurations of (n,m)-cubes.
Property 2. Each $(2n-1, 2m-1)$-cube centered on a leaning point (upper or lower) of a naive plane contains all the different configurations of (n,m)-cubes that can be encountered in this plane.

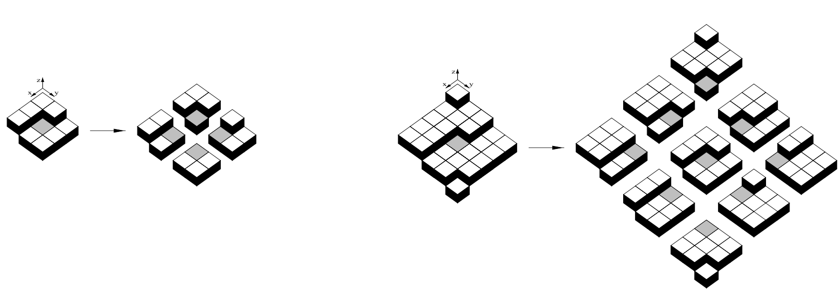


Fig. 2. 3×3 *and* 5×5 *blocs centered on a leaning point of the naive plane* $\mathcal{P}(3,7,19,0)$ *with the configurations of* $(2,2)$*-cubes (or bicubes) and* $(3,3)$*-cubes (or tricubes).*

Definition 1.
- *(n,m)-cubes at points (x_1, y_1) and (x_2, y_2) are* **geometrically equal** *if and only if the difference $\mathcal{R}(x_1+k_1, y_1+k_2) - \mathcal{R}(x_2+k_1, y_2+k_2)$ is a constant for all $(k_1, k_2) \in [0, n-1] \times [0, m-1]$.*
- *(n,m)-cubes at points (x_1, y_1) and (x_2, y_2) are* **geometrically symetrics** *if and only if the sum $\mathcal{R}(x_1+k_1, y_1+k_2) + \mathcal{R}(x_2+n-1-k_1, y_2+m-1-k_2)$ is a constant for all $(k_1, k_2) \in [0, n-1] \times [0, m-1]$. The symetry is defined with respect to the center of the volume delimited by the (n,m)-cube.*

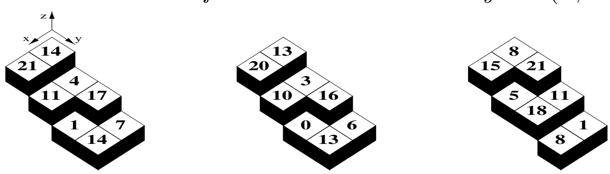


Fig. 3. *In the naive plane $\mathcal{P}(7, 13, 23, 0)$, the $(2,4)$-cube at point $(2,0)$ (a) is geometrically equal to the $(2,4)$-cube at point $(0,1)$ (b) and geometrically symetric to the $(2,4)$-cube at point $(3,-1)$ (c).*

Then we verify the new following proposition (see figure 4 for illustration):

Proposition 1. *Let x_l, y_l be the abscissa and ordinate of a lower leaning point and x_u, y_u those of an upper leaning point of a same naive plane $\mathcal{P}(a, b, c, r)$. Then, for all $(\alpha, \beta) \in \mathbb{Z}^2$ the configuration of the (n,m)-cube at point of abscissa $x_l + \alpha$ and ordinate $y_l + \beta$ is geometrically symetric, with respect to the center of the volume delimited by the (n,m)-cube, to the (n,m)-cube at point of abscissa $x_u - n + 1 - \alpha$ and ordinate $y_u - m + 1 - \beta$.*

Proof. For all $(\alpha, \beta) \in \mathbb{Z}^2$ we have the relation $\mathcal{R}(x_l+\alpha, y_l+\beta) + \mathcal{R}(x_u-\alpha, y_u-\beta) = c-1$. So according to definition 1, the configuration of (n,m)-cube at point $(x_l+\alpha, y_l+\beta)$ is geometrically symetric to the configuration of (n,m)-cube at point $(x_u-n+1-\alpha, y_u-m+1-\beta)$.

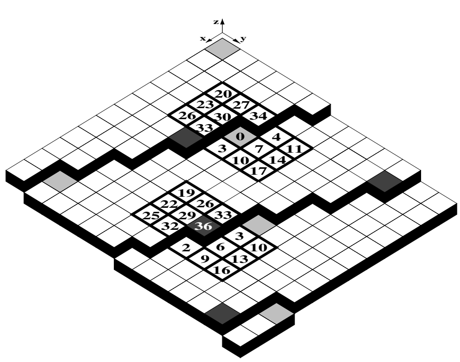


Fig. 4. *Representation with the remainders of the two symetric $(3,5)$-cubes which are attached to point $(2,2)$ and $(7,5)$ of the naive plane $\mathcal{P}(3, 7, 37, 0)$. The projection on the $(0xy)$ plane of the lower leaning point (voxel in light gray) is the point $(3,4)$. The projection on the $(0xy)$ plane of the upper leaning point (voxel in dark gray) is the point $(8,7)$.*

With proposition 1 and property 2, we can limit our study to parts of plane that are centered on lower leaning points. We deduce by translation that the $(2n-1, 2m-1)$-cube $\mathcal{S}$ of the naive plane $\mathcal{P}(a,b,c,0)$ which is centered on the origin (a lower leaning point of this plane) contains all the configurations of (n,m)-cubes that are needed to generate the planes $\mathcal{P}(a,b,c,r)$. In fact we can remark that, for $r = 0, \cdots, c-1$, the (n,m)-cube $\mathcal{C}(0,0,n,m)$ of the naive plane $\mathcal{P}(a,b,c,r)$ is similar to one of the (n,m)-cubes contained in $\mathcal{S}$. Consequently, we will looking for the different configurations which appear around leaning points of naive planes passing through the origin. First we need to recall some notions about discretization of real planes.

3 Discretization of planes and hyper Farey nets

The discretization of an oriented curve by object boundary quantization (OBQ) consists to take the nearest points of the discrete grid which are on the curve or on the right of the curve.
The OBQ-discretization of a rational plane $z = -\frac{ax+by+r}{c}$ where $0 \leq a \leq c$, $0 \leq b \leq c$ and $c \neq 0$ gives the discrete naive plane $\mathcal{P}(a,b,c,r)$.

Let $\mathcal{P}_R$ be the real plane $z = -(\alpha x + \beta y + \gamma)$ where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \gamma < 1$. For $n \geq 2$ and $m \geq 2$, the OBQ-discretization of the plane $\mathcal{P}_R$ on the set $V(i,j) = \{(x,y) \in \mathbb{Z}^2 \,/\, i \leq x < i+n,\ j \leq y < j+m\}$ is the set $\mathcal{C}_R$ defined by:

$$\mathcal{C}_R(i,j) = \{(x,y,z) \in \mathbb{Z}^3 \,/\, (x,y) \in V(i,j),\ 0 \leq \alpha x + \beta y + z + \gamma < 1\}$$

As the set of rational numbers is dense in the set of real numbers, we always can find four integers a, b, c and r satisfying the following conditions:

1) $0 \leq a \leq c$, $0 \leq b \leq c$, $c \neq 0$,
2) $0 \leq r < c$,
3) $\gcd(a,b,c) = 1$,
4) the discretization on $V(i,j)$ of the plane $z = -(\alpha x + \beta y + \gamma)$ is similar to the discretization of the rational plane $z = -\frac{ax+by+r}{c}$. Note that by definition the set of discrete points $\mathcal{C}_R(i,j)$ is exactly the (n,m)-cube $\mathcal{C}(i,j,n,m)$ of the discrete naive plane $\mathcal{P}(a,b,c,r)$.

Consequently, all rational numbers $\frac{a}{c}$, $\frac{b}{c}$ and $\frac{r}{c}$ satisfying these conditions are "good" rational approximations of the real values α, β and γ.
By taking the discretization of the real plane $z = -(\alpha x + \beta y)$ on the rectangle of size $(2n-1) \times (2m-1)$ centered on the point $(0,0)$, we obtain the discrete set $\mathcal{S}$ defined by:

$$\mathcal{S} = \{(x,y,z) \in \mathbb{Z}^3 \,/\, -n < x < n,\ -m < y < m,\ z-1 < -(\alpha x + \beta y) \leq z\}$$

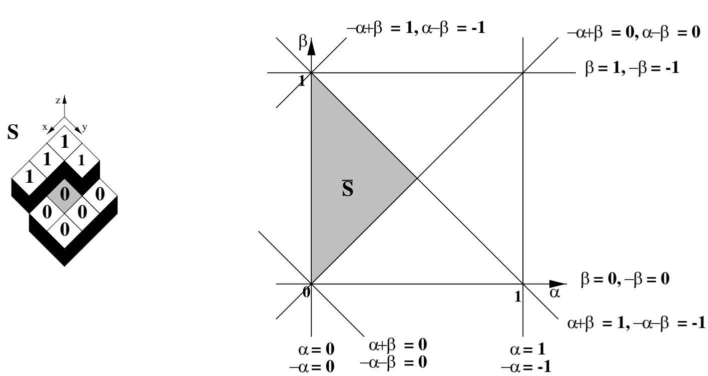
Using property 2, $\mathcal{S}$ contains all the different configurations of (n,m)-cubes appearing in the discretization of rational planes for which parameters are approximations of α and β.

One can notice that there exists an infinite number of real planes which have the same discretization. So we define the equivalence class of $\mathcal{S}$ by:

$$\overline{\mathcal{S}} = \left\{(\alpha,\beta) \in [0,1]^2 \,/\, \forall(x,y,z) \in \mathcal{S} \quad z-1 < -(\alpha x + \beta y) \leq z\right\}.$$

There is a duality between the cartesian representation $\alpha x+\beta y+z=0$ of the real plane and its parametric representation by the point (α,β) . A line $x\alpha+y\beta+z=0$ of the parametric space is the set of points (α,β) which represent the parameters of real planes passing through the point (x,y,z). The equivalence classes of the different configurations appearing around a leaning point are defined by sets of inequalities that decompose the unit square $\{(\alpha,\beta)/0 \leq \alpha \leq 1, 0 \leq \beta \leq 1\}$ of the parametric space into a set of polygons. This combination of lines is called an **hyper Farey net associated with** (n,m)**-cubes**.An illustration will be provided on figure 7.
Since the cartesian representation of a discrete plane is given by a double inequality, each discrete point (x,y,z), $x=-(n-1),\cdots,n-1$, $y=-(m-1),\cdots,m-1$ and $z=\min(0,-(x+y),-x,-y),\cdots,\max(0,-(x+y),-x,-y)$, is associated with an half-open band $B(x,y,z)$ in the parametric space delimited by two parallel lines $\mathcal{D}(x,y,z)$ and $\mathcal{D}(x,y,z-1)$, where $\mathcal{D}(x,y,z) = \left\{(\alpha,\beta) \in [0,1]^2 \,/\, x\alpha+y\beta+z=0\right\}$. Each band $B(x,y,z)$ represents the set of parameters (α,β) of real planes for which the discretization includes the point (x,y,z).

Example 1. Let us analyze the (3×3) bloc $\mathcal{S}$ centered on the origin of a naive plane (illustration on figure 5(a)). Its equivalence class $\overline{\mathcal{S}}$ is defined by the inequalities $-1 \leq -\alpha-\beta < 0$, $-1 \leq -\alpha < 0$, $0 \leq -\alpha+\beta < 1$, $-1 \leq -\beta < 0$, $0 \leq \beta < 1$, $-1 \leq \alpha-\beta < 0$, $0 \leq \alpha < 1$, $0 \leq \alpha+\beta < 1$.
After reductions, the domain $\overline{\mathcal{S}}$ illustrated on figure 5(b) is defined by the 4 inequalities $0 < \alpha+\beta < 1$, $0 < \alpha < 1$, $0 < \beta < 1$, $0 < -\alpha+\beta < 1$.



(a) (b)

Fig. 5. *(a)* $(3,3)$*-cube* $\mathcal{S}$ *centered on the origin; (b) domain* $\overline{\mathcal{S}}$ *of parameters* (α,β) *related to the real plane* $\alpha x+\beta y+z=0$ *having* $\mathcal{S}$ *as discretization around the origin.*

Figure 6 represents all the configurations of $(3,3)$-cubes which can appear around the origin and the corresponding hyper Farey net.

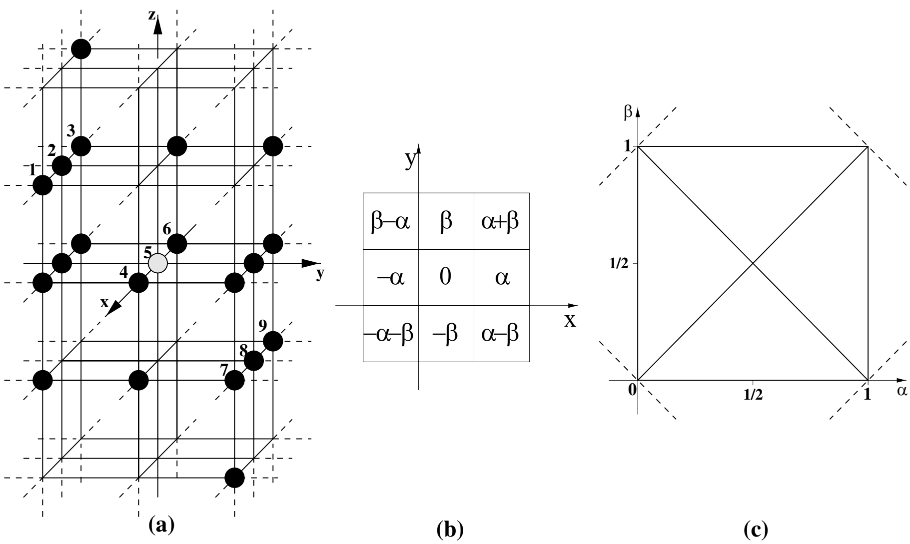


Fig. 6. *(a) Black points are points appearing in naive planes $\mathcal{P}(a,b,c,0)$ in the 24^{th} part of space. For instance, the points numbered from 1 to 9 belong to the naive plane $0 \leq y+z < 1$. (b) Representation with the values $x\alpha + y\beta$ of the $(3,3)$-cubes centered on the origin. (c) Hyper Farey net attached to $(2,2)$-cubes. Each band $B(x,y,z) = \cup_{k\in \mathbb{R}, 0\leq k<1} \mathcal{D}(x,y,z-k)$ of the net is the parameter set of real planes $\mathcal{P}_R$ such that the point $M(x,y,z)$ (black point of (a)) is in the discretization of $\mathcal{P}_R$.*

Definition 2. *A two-dimensional Farey series of order $q \in \mathbb{N}^*$ is the set F_q of rational points defined by:*

$$F_q = \bigcup_{c=1}^{q} \left\{ \left(\frac{a}{c}, \frac{b}{c} \right) \in \mathbb{Q}^2 \,/\, 0 \leq a \leq c,\ 0 \leq b \leq c \right\}.$$

The construction of Farey series of order $q \geq 2$ is recalled here.
Let $A_1\left(\frac{a_1}{c_1}, \frac{b_1}{c_1}\right)$ and $A_2\left(\frac{a_2}{c_2}, \frac{b_2}{c_2}\right)$ be two points of the Farey series of order $(q-1)$ satisfying $c_1 + c_2 = q$. The median point between A_1 and A_2, noted by $A_1 + A_2$, is the point with coordinates $\left(\frac{a_1+a_2}{c_1+c_2}, \frac{b_1+b_2}{c_1+c_2}\right)$. The Farey series of order q is obtained by adding to the Farey series of order $(q-1)$ all median points for which “denominator” are equal to q.

Definition 3. *Let $\mathcal{A}$ be a subset of the Farey serie $\mathcal{F}_q$ of order $q \geq 1$.*
An **hyper Farey net attached to** *$\mathcal{A}$ is a partition of the convex hull of $\mathcal{A}$ into triangles of vertices*

$$M_1\left(\frac{a_{1,1}}{c_{1,3}}, \frac{a_{1,2}}{c_{1,3}}\right), \quad M_2\left(\frac{a_{2,1}}{a_{2,3}}, \frac{a_{2,2}}{a_{2,3}}\right), \quad M_3\left(\frac{a_{3,1}}{a_{3,3}}, \frac{a_{3,2}}{a_{3,3}}\right)$$

in $\mathcal{A}$ such as the determinant of the matrix (a_{ij}) is ± 1 and containing no points of $\mathcal{A}$ except for its vertices.

Theorem 1. *The intersection between a line of the hyper Farey net associated to (n,m)-cubes with another lines of the same Farey net is a subset of the two-dimensional Farey series of order $q \leq 2(n-1)(m-1)$. The two vertices of an edge of the Farey net are consecutive fractions of the one dimensional Farey series in abscissa and ordinate.*

Proof. The intersection of the line $\mathcal{D}(x,y,z)$ with the line $\mathcal{D}(x',y',z')$, for $xy' - x'y \neq 0$, is the point

$$\left(\frac{|yz' - y'z|}{|xy' - x'y|}, \frac{|x'z - xz'|}{|xy' - x'y|} \right)$$

So the intersection of the line $\mathcal{D}(x,y,z)$ with the lines $\mathcal{D}(x',y',z')$, $x' = -(n-1), \cdots, (n-1)$ and $y' = -(m-1), \cdots, (m-1)$ is the set of rational points $\left(\frac{a}{c}, \frac{b}{c}\right)$ satisfying $0 \leq a \leq c$, $0 \leq b \leq c$ and $1 \leq c \leq 2(n-1)(m-1)$.

Example 2. In the hyper Farey net associated to $(2,3)$-cubes (fig. 7), the intersection of the line $\alpha - \beta = 0$ with the lines $\alpha + 2\beta = 0$, $\alpha + 2\beta = 1$, $\alpha + \beta = 1$, $\alpha + 2\beta = 2$ and $\alpha + 2\beta = 3$ generates the points $(0,0)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{2}{3}, \frac{2}{3}\right)$ and $(1,1)$. The set of abscissa and the set of ordinates taken in ascending order are associated with the one dimensional Farey series of order 3: $\left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$

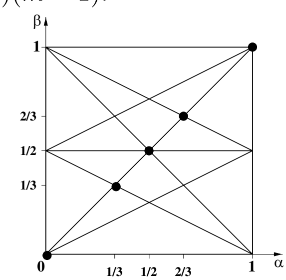


Fig. 7. *Hyper Farey net associated to* $(2,3)$*-cubes.*

We study now the relation between the position of the parameters (α,β) in the Farey net and the different configurations of (n,m)-cubes of the naive plane $\mathcal{P}(a,b,c,0)$ where $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a "good" approximation of (α,β) (as it was introduced at the begining of this section). Let have a look at the different positions for (α,β) with respect to the k-faces, k=0,1,2, of the lines arrangement of the hyper Farey net.

Case 1: If the point (α,β) is a vertex of the arrangement then it is a rational point expressed by $\left(\frac{a}{c}, \frac{b}{c}\right)$ with $c \leq 2(n-1)(m-1)$. The discretization of the rational plane of parameters (α,β) which includes the origin is the naive plane $\mathcal{P}(a,b,c,0)$ containing strictly p ($p < nm$) configurations of (n,m)-cubes. Only the planes of normal vector (a,b,c) are generated by these p (n,m)-cubes. The point (α,β) is also a vertex of the arrangement of the Farey net associated with (n',m')-cubes where $n' \geq n$ and $m' \geq m$. The associated naive plane is generated by exactly p (n',m')-cubes.

Case 2: If the point (α,β) belongs to an edge of the arrangement then the median point $\left(\frac{a_1+a_2}{c_1+c_2}, \frac{b_1+b_2}{c_1+c_2}\right)$ between the two vertices $\left(\frac{a_1}{c_1}, \frac{b_1}{c_1}\right)$ and $\left(\frac{a_2}{c_2}, \frac{b_2}{c_2}\right)$, is a "good" rational point of minimal denominator approaching (α, β). The real plane of parameters (α, β) and the rational plane of parameters $\left(\frac{a_1+a_2}{c_1+c_2}, \frac{b_1+b_2}{c_1+c_2}\right)$ have the same discretization on the set $\{(x, y) \in \mathbb{Z}^2 \;/\; -n < x < n, \; -m < y < m\}$. So they have the same configurations of (n, m)-cubes. If L_1 and L_2 are the lists of (n, m)-cubes appearing in the planes of normal (a_1, b_1, c_1) and (a_2, b_2, c_2) then $L_1 \cup L_2$ is the list of (n, m)-cubes appearing in the naive plane of normal $(a_1 + a_2, b_1 + b_2, c_1 + c_2)$ with $\mathrm{Card}(L_1 \cup L_2) < nm$.

Case 3: If the point (α, β) is on a face of the arrangement, we have 3 or 4 vertices bounding the face. The median point in case of 4 vertices corresponds to the median point issued from two opposite vertices (see case 2). In case of 3 vertices, the median point of the 3 vertices is a "good" rational point $\left(\frac{a}{c}, \frac{b}{c}\right)$ with minimal denominator approaching (α, β). The naive plane with parameters corresponding to the median point is composed by union of (n, m)-cubes issued from vertices. The plane is composed of exactly nm different configurations of (n, m)-cubes. All rational plane having their parameters in the face of the arrangement are discretized in naive planes generated by exactly nm (n, m)-cubes.

4 Relation between Farey series and representation by (n, m)-cubes

The different configurations of (n, m)-cubes appearing around leaning points are in one to one correspondance with vertices, edges and faces of the lines arrangement of the hyper Farey net. We are going to see through some examples how (n, m)-cubes can be constructed similarly with the computation of Farey series.

Farey serie of order 1 and basic elements of planes

The two-dimensional Farey serie of order 1 is the set of points $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ corresponding to the dual representation of the basic naive planes $0 \leq z < 1$, $0 \leq y + z < 1$, $0 \leq x + z < 1$, $0 \leq x + y + z < 1$ of the same 24^{th} part of space (see figure 8). We define a **basic element** as to be a set of three neighbour voxels not aligned. The basic planes are obtained by repetition of a basic element (figure 9). The configurations of basic elements are sufficient to generate naive planes.

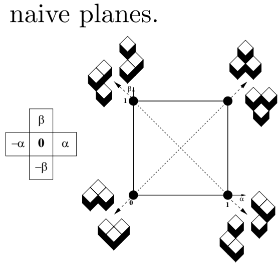


Fig. 8. *Hyper Farey net of order 1 and basic elements generators of the basic planes.*

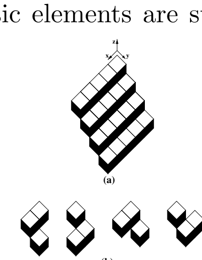


Fig. 9. *(a) Basic naive plane of normal vector (0,1,1); (b) list of the 4 basic elements involved in this plane.*

Hyper Farey net associated with bicubes
On figure 10 we can see the basic planes generated by $(2, 2)$-cubes. On this example the new point of coordinates $\left(\frac{1}{2}, \frac{1}{2}\right)$ is introduced by using the transition rules from the Farey serie of order 1 to the Farey serie of order 2. We introduce the median point between $\left(\frac{0}{1}, \frac{0}{1}\right)$ and $\left(\frac{1}{1}, \frac{1}{1}\right)$ or $\left(\frac{0}{1}, \frac{1}{1}\right)$ and $\left(\frac{1}{1}, \frac{0}{1}\right)$. The naive plane of normal vector $(1, 1, 2)$ is generated by two $(2, 2)$-cubes. The representation in $(2, 2)$-cubes of each vertex of the hyper Farey net is obtained by concatenation of the basic elements. Concatenation is defined as following.

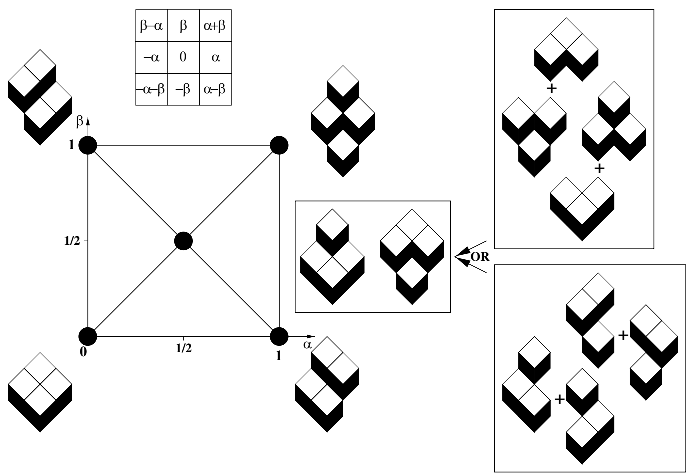


Fig. 10. *Hyper Farey net associated with* $(2, 2)$*-cubes.*

The rules of construction of the (n, m)-cubes, for $n \geq 2$ and $m \geq 2$, are defined as follows:
$[S]$ and $[S']$ will design two (n_1, m_1)-cubes. Their symetric will be noted $[-S]$ and $[-S']$ (symetry with respect to the center of the volume including the (n, m)-cube).
$[S'']$ and $[-S'']$ will design two symetric (n_2, m_2)-cubes.
$[N]$ and $[N']$ will design two neutral (n_1, m_1)-cubes (geometrically equal to their symetric).
$[N'']$ will design a neutral (n_2, m_2)-cube.
The operator of concatenation (noticed by "+") between two (n_1, m_1)-cubes is defined as the union of them by a common band of voxels along lines or columns.
The concatenation of two (n_1, m_1)-cubes generates a (n_2, m_2)-cube with the following constraint:

$$n_1 = n_2 \text{ and } m_1 < m_2 \quad \text{or} \quad n_1 < n_2 \text{ and } m_1 = m_2$$

With respect to the classification of (n, m)-cubes in terms of symetric or neutral, we have:

- $[S] + [N] = [S'']$
- $[S] + [S'] = [S'']$
- $[S] + [-S] = [N'']$
- $[N] + [N'] = [S'']$

The different examples will illustrate the different rules to construct the (n, m)-cubes appearing in a plane. To do that we will increase n and m values.

Representation by $(2, 3)$-cubes
In figure 11, the point $C\left(\frac{1}{3}, \frac{1}{3}\right)$ is the median point (in the sense of Farey series) between $B\left(\frac{1}{2}, \frac{1}{2}\right)$ and $A\left(\frac{0}{1}, \frac{0}{1}\right)$. The plane corresponding to B is generated by two bicubes and the plane corresponding to A is generated by one bicube. The concatenation of the two bicubes of B generates two neutral $(2, 3)$-cubes but only one can coexist in a plane with the bicube of A. The association of the bicube of A with the two bicubes of B generates two symetric $(2, 3)$-cubes. Finaly, C is generated by three $(2, 3)$-cubes. The point $D\left(\frac{2}{4}, \frac{1}{4}\right)$ is generated by four $(2, 3)$-cubes issued from the concatenation of the three bicubes appearing in C (bicubes of A and B) with the bicube appearing in E (there is not neutrals in E).

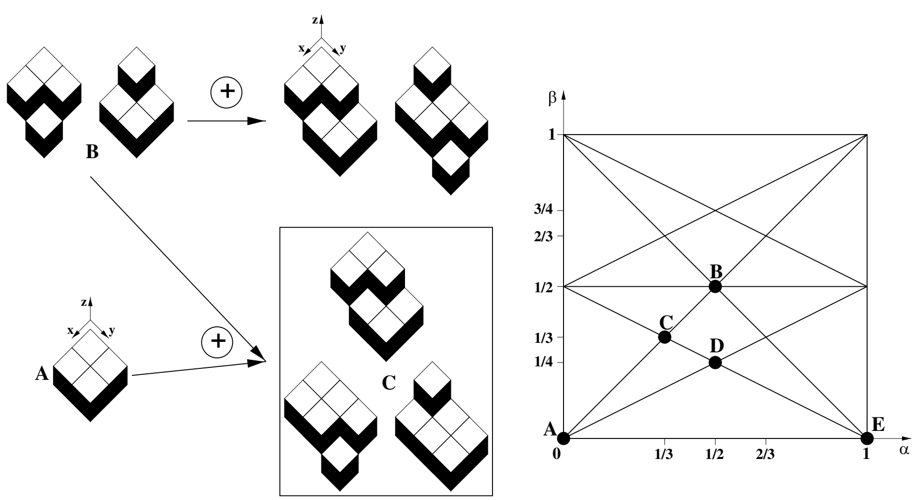


Fig. 11. *Hyper Farey net associated to* $(2, 3)$*-cubes.*

Representation by $(3, 3)$-cubes
In figure 12, the point $F\left(\frac{1}{4}, \frac{1}{4}\right)$ is the median point between $C\left(\frac{1}{3}, \frac{1}{3}\right)$ and $A\left(\frac{0}{1}, \frac{0}{1}\right)$. It is composed of four $(3, 3)$-cubes issued from the concatenation of the neutral $(2, 3)$-cube attached to A with the neutral and the two symetric $(2, 3)$-cubes attached to C.

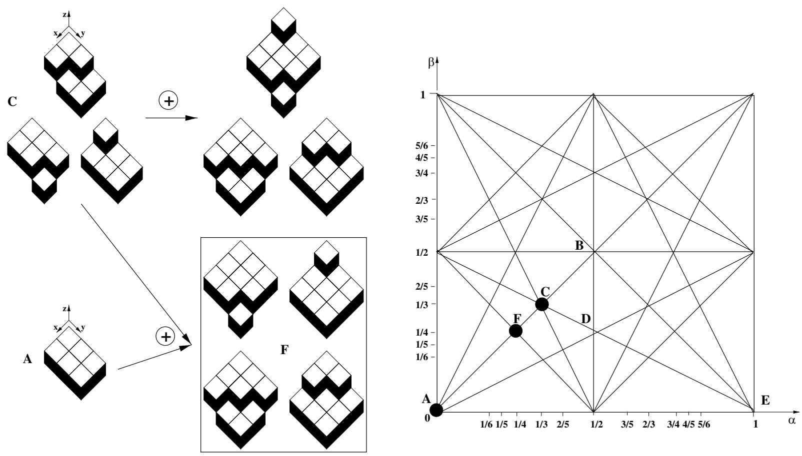


Fig. 12. *Hyper Farey net associated to* $(3,3)$*-cubes.*

Representation by $(3,4)$-cubes
In figure 13, the point $G\left(\frac{1}{5}, \frac{1}{5}\right)$ is the median point between $F\left(\frac{1}{4}, \frac{1}{4}\right)$ and $A\left(\frac{0}{1}, \frac{0}{1}\right)$. The plane associated with point F is generated by four $(3,3)$-cubes two by two symetrics. The concatenation of these $(3,3)$-cubes generate four $(3,4)$-cubes, two symetrics and two neutrals with only one compatible with the $(3,3)$-cubes of A. So the plane associated with point G is composed of three among the four $(3,4)$-cubes of F and of the $(3,4)$-cubes generated by concatenation between the $(3,3)$-cubes of F and the $(3,3)$-cube de A.

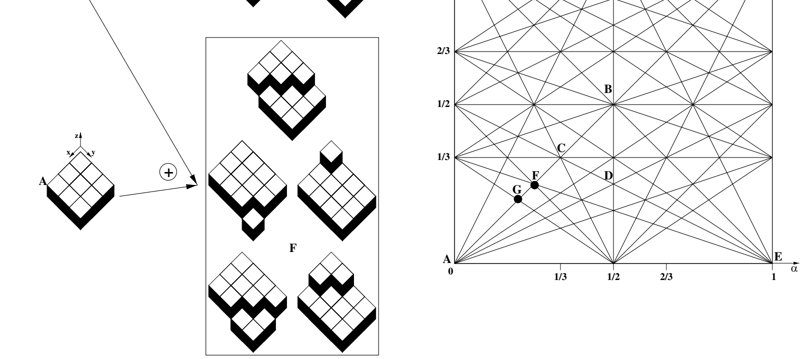


Fig. 13. *Hyper Farey net associated with* $(3,4)$*-cubes.*

5 Conclusion

A new approach has been discussed for digital naive plane understanding. Starting from the arithmetic formulation of a digital plane given by $0 \leq ax + by + cz + r < c$, we propose an identification process to list the set of bloc elements called (n, m)-cubes encountered on the plane. We proved that the incremental characterization of a Farey net (issued from the parametric representation $\left(\frac{a}{c}, \frac{b}{c}\right)$ of the plane) can be used to identify the list of different (n, m)-cubes.
Future works will focuse on the definition of an algorithmic way to recognize digital planes. The method will be incremental on n and m values. It will pass alternatively from (n, m)-cubes structure to the corresponding Farey net.

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