Set Connections and Discrete Filtering

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Abstract. Connectivity is a topological notion for sets, often introduced by means of arcs. Classically, discrete geometry transposes to digital sets this arcwise appoach. An alternative, and non topological, axiomatics has been proposed by Serra. It lies on the idea that the union of connected components that intersect is still connected. Such an axiomatics enlarges the range of possible connections, and includes clusters of particles.
The main output of this approach concerns filters. Very powerful new ones have been designed (levelings), and more classical ones have been provided with new properties (openings, strong alternated filters)
The paper presents an overview of set connection and illustrates it by filterings on gray tone images. It is emphazised that all notions introduced here apply equally to both discrete and continuous spaces.

1 The connectivity concepts

1.1 Classical connectivity and image analysis

In mathematics, the concept of connectivity is formalized in the framework of topological spaces and is introduced in two different ways. First, a set is said to be connected when one cannot partition it into two non empty closed (or open) sets. This definition makes precise the intuitive idea that $[0,1] \cup [2,3]$ consists of two pieces, while $[0,1]$ consists of only one. But this first approach, extremely general, does not derive any advantage from the possible regularity of some spaces, such as the Euclidean ones. In such cases, the notion of *arcwise connectivity* turns out to be more convenient. According to it, a set A is connected when, for every $a, b \in A$, there exists a continuous mapping ψ from $[0,1]$ into A such that $\psi(0) = a$ and $\psi(1) = b$. Arcwise connectivity is more restrictive than the general one ; however, in R^d, any open set which is connected in the general sense is also arcwise connected.

A basic result governs the meaning of connectivity ; namely, the union of connected sets whose intersection is not empty is still connected :

$$\{A_i \text{ connected}\} \text{ and } \{\cap A_i \neq \emptyset\} \Rightarrow \{\cup A_i \text{ connected}\} \tag{1}$$

In discrete geometry, the digital connectivities transpose the arcwise corresponding notion of the Euclidean case, by introducing some elementary arcs

G. Bertrand, M. Couprie, L. Perroton (Eds.): DGCI'99, LNCS 1568, pp. 191–206, 1999.

between neighboring pixels. This results in the classical 4- and 8-square connectivities, as well as the hexagonal one, or the cuboctahedric one in 3-D space. Is such a metric approach to connectivity adapted to image analysis ? We can argue that

a/ Certain arcwise connections seem smowhat shaky, *e.g.* when they do not treat equally a set and its complement;

b/ In discrete motion analysis, the trajectories of fast moving objects often appear as dotted tubes, and arcwise connections are unable to handle such situations;

c/ more deeply, one can wonder what is actually needed in image processing. As a matter of fact, when we examine the requirements for connectivity, we observe that the basic operation they involve consists, given a set A and a point $x \in A$, in extracting the particle of A at point x. For such a goal, an arcwise approach is obviously sufficent. But is it necessary?

1.2 The notion of a connection

These criticisms led G. Matheron and J. Serra to propose a new approach, in 1988 [SER88] where they take not (1) as a consequence, but as a starting point. However, their definition is rather general and stated as follows.

Definition 1. *(G. Matheron and J. Serra) Let E be an arbitrary space. We call connected class or* connection $\mathcal{C}$ *any family in* $\mathcal{P}(E)$ *such that*

(i) $\emptyset \in \mathcal{C}$ *and for all* $x \in E$, $\{x\} \in \mathcal{C}$

(ii) for each family $\{C_i\}$ *in* $\mathcal{C}$, $\cap\, C_i \neq \emptyset$ *implies* $\cup C_i \in \mathcal{C}$.

As we can see, the topological background has been deliberately thrown out. The classical notions (e.g. connectivity based on digital or Euclidean arcs) are indeed particular cases, but the emphasis is put on another aspect, that answers the above criticism c/ in the following manner ([SER88], Chap. 2) :

Theorem 2. *The datum of a connection* $\mathcal{C}$ *on* $\mathcal{P}(E)$ *is equivalent to the family* $\{\gamma_x\,, x \in E\}$ *of openings such that*

(iii) for all $x \in E$, *we have* $\gamma_x(x) = \{x\}$

(iv) for all $A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ *and* $\gamma_y(A)$ *are equal or disjoint*

(v) for all $A \subseteq E$, *and all* $x \in E$, *we have* $x \notin A \Rightarrow \gamma_x(A) = \emptyset$.

An alternative, but equivalent, axiomatics has been proposed by Ch. Ronse [RON98] ; it contains, as a particular case, another one by R.M. Haralick and L.G. Shapiro [HAR92] ; however, both approaches are still set-oriented. The extension from sets to the general framework of complete lattices and in particular to numerical functions has been proposed by J. Serra [SER98a]

Historically speaking, the number of applications or of theoretical developments which was suggested (and permitted) by this theorem is considerable (see, among others[MAR94][MEY94][SAL96]). It has opened the way to an object-oriented approach for segmentation, compression and understanding of still and moving images.

2 Examples of connections on $\mathcal{P}(E)$

Several instructive examples of connections on $\mathcal{P}(E)$ can be found in [HEI97], in [RON98] and in [SER98a]. Here we just recall a few of them, which are of interest for the present study.

i/ All arcwise connectivities on digital spaces are connections in the sense of definition 1;

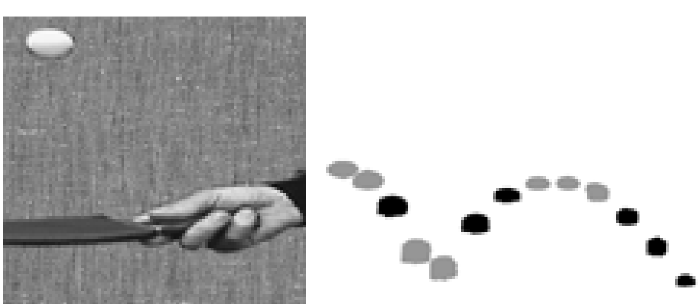
ii/ In [SER88] ch.2, J. Serra provides E with a first connection $\mathcal{C}$ and considers an extensive dilation $\delta : \mathcal{P}(E) \to \mathcal{P}(E)$ that preserves $\mathcal{C}$ (*i.e.* $\delta(\mathcal{C}) \subseteq \mathcal{C}$). Then the inverse image $\mathcal{C}' = \delta^{-1}(\mathcal{C})$ of $\mathcal{C}$ under δ defines a new connection on $\mathcal{P}(E)$, which is richer. The $\mathcal{C}$ -components of $\delta(A)$, $A \in \mathcal{P}(E)$, are exactly the images $\delta(Y_i')$ of the $\mathcal{C}'$-components of A. If γ_x stands for the connected opening associated with connection $\mathcal{C}$ and ν_x for that associated with $\mathcal{C}'$, we have

$$\nu_x(A) = \gamma_x \delta(A) \cap A \quad \text{when} \quad x \in A \quad ; \quad \nu_x(A) = \emptyset \quad \text{when not} \qquad (2)$$

(similar technique applies also when stands for an opening, but wihout the statement on the connected components, and without Eq.2 [SER98b])

In practice, the openings ν_x characterize the *clusters* of objects from a given distance d apart. Figure 1 illustrates this point by "reconnecting" dotted lines trajectories. But *a contrario*, such an approach can also provide a means to extract the objects which are isolated.They will be defined by the fact that for them $\nu_x(A) = \gamma_x(A)$, an equality which yields easy implementation [SER98b].

iii/ In [RON98], Ch. Ronse starts also from a first connection on $\mathcal{P}(E)$, and proposes, as a new connection, the class generated by the points and the connected sets opened by a given structuring element B. If $x \in X \circ B$, then $\gamma_x(X)$ is the initial connected component of $X \circ B$ containing x, and when point $x \in X \backslash X \circ B$, then $\gamma_x(X) = \{x\}$. For example, for such an "open"connection by a 3x3x square, the set of fig.2a has 16 particles: the two surrounded squares, plus 14 isolated points. Also, the six points of the vertical gulf are isolated pores.



a) First image of a sequence *b) Space Time display of the ball*

Fig. 1. In a dilation based connection, the three clusters in grey are considered as particles. They correspond to the slow motions.

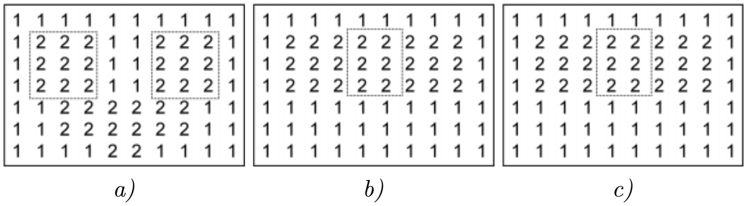


Fig. 2. a) initial set A. For the "open" connection by the 3x3 square, A is made of 16 grains, namely the two $3x3$ squares plus 14 isolated points ; b) $\varphi_{M^c}\gamma_M(A)$ for M equal to the six pores of the central gulf (surrounded); c) $\gamma_M\ \varphi_{M^c}(A)$ for the same marker.The difference comes from that the "open" connection is not adjacency preventing.

2.1 Connected Filters

For now on E is an arbitrary set, and $\mathcal{P}(E)$ is supposed to be equipped with connection $\mathcal{C}$. For every set $A \in \mathcal{P}(E)$, the two families of the connected components of A (the "grains") and of A^c (the "pores") partition space E. Then, an operation $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is said to be *connected* when the partition associated with $\psi(A)$ is coarser that that of A [SER93]. Clearly, taking the complement of a set, or removing some grains, or filling pores generate connected operators. The major class of mappings we have in view is that of the (connected or not) morphological filters. Let us briefly recall it

- A mapping ψ is said to be a *morphological filter* on $\mathcal{P}(E)$ when it is increasing and idempotent:

$$A, B \subseteq E,\ \ A \subseteq B \Rightarrow \psi(A) \subseteq \psi(B) \qquad \textit{increasingness}$$
$$\psi(\psi(A)) = \psi(A) \qquad \textit{idempotence}$$

- In particular, a filter that is extensive (resp. anti-extensive) is called a *closing* (resp. an *opening*) :

$$\gamma \quad \text{an opening} \quad : \quad \gamma = \text{a filter and} \quad \gamma(A) \subseteq A \quad , \quad A \subseteq E$$
$$\varphi \quad \text{a closing} \quad : \quad \varphi = \text{a filter and} \quad \varphi(A) \supseteq A \quad , \quad A \subseteq E$$

- A *granulometry* is a family $\{\gamma_d\, ,\, d > 0\}$ of decreasing openings (i.e. $d \geq d' \Rightarrow \gamma_d\ \subseteq \gamma_{d'}$) and an anti-granulometry is a family $\{\varphi_d\, ,\, d > 0\}$ of increasing closings, both depending on a positive parameter. Every granulometry and every anti-granulometry satisfy the following law of a semi-group, where the more severe operation imposes its conditions

$$p \geq n \Rightarrow \psi_n\psi_p = \psi_p\psi_n = \psi_p. \tag{3}$$

- The products $\varphi\gamma$ and $\gamma\varphi$ of two arbitrary opening γ and closing φ turn out to be idempotent. They are called *alternated filters.* Note that the product $\gamma\gamma'$ of two openings (or $\varphi\varphi'$ of two closings) is *a priori* not idempotent. Denoting by Id the identity operator, we have the following equivalences[MAT88]

$$\psi = \varphi\gamma \quad \Longleftrightarrow \psi(Id \cap \psi) \ \text{ and } \ \psi = \gamma\varphi \quad \Longleftrightarrow \psi(Id \cup \psi) \tag{4}$$

- When a morphological filter ψ satisfies both conditions (4), which provide it with robustness, *i.e.* when

$$A, B \subseteq E, \ \ \psi(A) \cap A \subseteq \psi(B) \subseteq \psi(A) \cup A \quad \Rightarrow \quad \psi(B) = \psi(A)$$

one says that ψ is *strong*. In particular, openings and closings are strong .

- The notion of Alternated Sequential Filters plays, with the alternated filters, the same role as the granulometries do with γ and φ . Let $\{\gamma_i\}$ and $\{\varphi_i\}$, $i \in Z^+$, be a granulometry and an antigranulometry. The composition products

$$P_n = \varphi_n \gamma_n ... \varphi_i \gamma_i ... \varphi_1 \gamma_1 \quad and \quad \Sigma_n = \gamma_n \, \varphi_n ... \, \gamma_i \varphi_i \, ... \, \gamma_1 \varphi_1.$$

are idempotent and called Alternated Sequential Filters (A.S.F).They do not satisfy implication (3) , but only the following absorption law

$$p \geq n \Rightarrow \psi_p \psi_n = \psi_p$$

2.2 Set opening by reconstruction and some derivatives

A comprehensive class of connected filters derives from the classical *opening by reconstruction.* Its definition appears in [SER88], ch.7.8. Significant studies which use this notion may be found in literature, such as [SER93] (connected operators),[CRE97],(stable operators) [MEY94],(spanning trees), [HEI97],(grain oprators).

An opening by reconstruction is obtained by starting from an increasing binary criterion τ (e.g. "*the area of A is $\geq$ 10*"), to which one associates the trivial opening

$$\begin{array}{ll} \gamma^\tau(A) = A \text{ when } A \text{ satisfies the criterion} \\ \gamma^\tau(A) = \emptyset \qquad\qquad \text{when not} \end{array}$$

The corresponding opening by reconstruction γ is then generated by applying the criterion to all grains of A, independently of one another, and by taking the union of the results :

$$\gamma(A) = \cup \{ \gamma^\tau \gamma_x(A), \quad x \in E \}$$

The closing by reconstruction φ (for the same criterion) is the dual of γ for the complement, *i.e.* if $-$ stands for the complement operator, then

$$\varphi = -\gamma- .$$

For example, in R^2, if we take for criterion τ, " *have an area* $\geq$ 10", then $\gamma(A)$ is given by the union of grains of A whose areas are $\geq$ 10, and $\varphi(A)$ is the union of A and all its pores whose areas are $\leq$ 10. Similarly, if criterion τ is expressed by "hit a fixed marker M", then $\gamma(A)$ is the union of the grains that hit A, wheras $\varphi(A)$ is composed of A and of all pores that miss M .

The operators by reconstruction satisfy number of nice properties. The three following ones are typical examples of them.

Fig. 3. An example of a pyramid of connected alternated sequential filters. Each contour is preserved or suppressed, but never deformed : the initial partition increases under the successive filters, which are strong and form a semi-group.

Proposition 3. *[SER93]Let γ be an opening by reconstruction, and φ be a closing that does not create connected components, i.e. such that*

$$x \in \varphi(A) \Longrightarrow A \cap \gamma_x \varphi(A) \neq \emptyset \qquad (5)$$

Then the associted alternated filters are ordered, and we have $\gamma\varphi \geq \varphi\gamma$

Proof. Consider $\varphi\gamma(A)$, for $A \subseteq E$. Since the (extensive) closing φ does not create new connected components, it can only enlarge those of $\gamma(A)$; now γ acts grain by grain, hence $\gamma\varphi\gamma = \varphi\gamma$. According to criterion 6.6 in [MAT88c]this equality is equivalent to $\gamma\varphi \geq \varphi\gamma$.

The most common closings may not satisfy condition (5). It is the case for intersections of closings by segments, for example. However, if starting from an arbitrary closing φ, we restrict $\varphi(A)$ to its grains that contain at least one point $x \in A$, the resulting operation is still a closing. It is the reason for which condition (5) is always assumed implicity in practice.

Corollary 4. *Let $\{\gamma_i\}$ be a granulometry by reconstruction, and $\{\varphi_i\}$ be a anti-granulometry that does not create connected components, then the A.S.F. $\Sigma_n = \gamma_n \varphi_n ... \gamma_i\varphi_i ... \gamma_1\varphi_1$ satisfy the semi-group relation (3)*

(*easy proof*). This corollary explains, partly at least, why the A.S.F. by reconstruction are so often involved in pyramids, for coding, segmentation, or indexation purposes. In such pyramids, the additional information to get finer levels is concentrated in subdivisions the flat zones [MEY94]. An example of such a behaviour is presented in Fig.3. Each cross section of the gray tone image has processed by an alternating sequential filter by reconstruction. The underlying binary criterion was here associated with the size of the disc inscribable in each grain.

Another point of interest is the following. The infimum of openings is generally not idempotent. But consider a family $\{\gamma_i, i \in I\}$ of openings by reconstruction associated with criteria $\{\tau_i\}$. Clearly, their infimum $\gamma = \cap\gamma_i$ is still an opening, where each grain of A must fullfill all criteria τ_i to be retained. On the other hand, $\cup\gamma_i$ is the opening by reconstruction where each grain must satisfy one criterion τ_i at least. However, the largest opening is here the identity mapping, and not the largest increasing operator (*i.e.* $A \to E, \forall A \in \mathcal{P}(E)$). Hence we may state:

Proposition 5. *In the lattice of the increasing operators from $\mathcal{P}(E)$ into itself, the openings and the closings by reconstruction constitute two complete quasi sub-lattices.*

2.3 Adjacency

The central notion of adjacency[SER98b], which governs the structure of the levelings below, is defined as follows

Definition 6. *Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$, and let $X, Y \in \mathcal{C}$. Sets X and Y are said to be adjacent when $X \cup Y$ is connected, whereas X and Y are disjoint.*

Definition 7. *Given a connected component $A \in \mathcal{C}$ and a set $M \in \mathcal{P}(E)$, one says that A* touches *M, and one writes $A \parallel M$ when either $A \cap M \neq \emptyset$, or the grain of $A \cup M$ which contains A is strictly larger than A. By duality, one says that A* lies in *M when A does not touch M^c; one writes A j M.*

The duality under complement provides the two following equivalences

$$A \parallel M \iff A \text{ " } M^c \text{ and } A \text{ , } M \iff A \text{ j } M^c \tag{6}$$

Note that relation $A \parallel M$ (A touches M) is less demanding than $A \cap M \neq \emptyset$ (A hits M), since it accepts in addition that A and M be adjacent. Similarly, A j M (A lies in M), is more severe than $A \subseteq M$, since none of the grains of A and of M must be adjacent to each other.

When $\gamma_x(A) \neq \gamma_y(A)$ for an arbitrary $A \in \mathcal{P}(E)$ one cannot have $\gamma_x(A) \parallel \gamma_y(A)$ since $\gamma_x(A)$ is the largest element of $\mathcal{C}$ included in A. But $\gamma_x(A)$ may not touch some pores Y_i of A and, nevertheless, touch their union $\cup Y_i$ For example, for the "open" connection *iii/* of section 2, none of the six point pores of the central gulf, in fig.2a, is adjacent to the set, whereas their union touches it. The most powerfull coonections are those which prevent this perverse effect, i.e. which fullfill the following condition

Condition 8 *A connection $\mathcal{C}$ on $\mathcal{P}(E)$ is* adjacency preventing *when for all $x \in E$ and all sets $A, M \in \mathcal{P}(E) y \in M$ and $\gamma_x(A)$, $[\gamma_y(A) \cup \gamma_y(A^c)] \Longrightarrow \gamma_x(A)$, $\cup\{[\gamma_y(A) \cup \gamma_y(A^c)] y \in M\}$*

In particular, adjacency prevention governs the strenght of the filters by reconstruction, as proved in proposition 9 and in theorem 12 .

Proposition 9. *Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$ such that if for all $i \in I$ we have A , B_i, then A , $\cup B_i$. If γ and φ stand for an opening and a closing by reconstruction based on connection $\mathcal{C}$, then both alternated filters $\gamma\varphi$ and $\varphi\gamma$ are strong.*

Proof. We shall prove the proposition for $\varphi\gamma$. Because of Eq.(4), we have only to show that for all $A \in P(E)$, if $x \in E$ is an arbitrary point, then $x \notin \varphi\gamma(A)$ implies $x \notin B = \varphi\gamma[A \cup \varphi\gamma(A)]$. Suppose first that $x \notin A$. Opening γ can only enlarge pore $\gamma_x(A^c)$, and closing φ keep it unchanged (if not, we would not have $x \notin \varphi\gamma(A)$). Hence $\gamma_x(A^c)$ is equal to $\gamma_x[A \cup \varphi\gamma(A)]^c$, and finally $x \notin B$. Suppose now that $x \in A$. The grain $\gamma_x(A)$ touches none of the grains and the pores of A that compose $\varphi\gamma(A)$ (if not, $\gamma_x(A)$ would belong to $\varphi\gamma(A)$, now $x \notin \varphi\gamma(A)$). Then, according to the assumption of the proposition, $\gamma_x(A)$ does not touch $\varphi\gamma(A)$, neither $[A \cup \varphi\gamma(A)] \setminus \gamma_x(A)$, hence $\gamma_x(A) = \gamma_x[A \cup \varphi\gamma(A)]$ and finally $x \notin B$, which achieves the proof.

According to Equivalences (4), the proposition implies that $\varphi\gamma$ admits a decomposition as $\gamma'\varphi'$, but for a $\gamma'$and a φ' priori *different* from γ and φ. We will now see under which condition these primitives can be the same.

2.4 Set Levelings

Levelings have been introduced by F. Meyer, in [MEY98b], as gray tone connected operators on digital spaces, for the usual digital arcwise connections based on neighbor pixels in square or hexagonal grids. In [MAT97], G. Matheron proposes a generalization to an arbitrary space (hence, without *a priori* connection). Here, connection arrives as a final result, and is generated by an extensive dilation. Now in both cases, levelings turn out to be *flat* operators, *i.e.* that treat each grey level independently of the others. This circumstance suggests to try and generalize F. Meyer's approach by focusing on *set* levelings, but re-interpreted in the framework of an arbitrary connection $\mathcal{C}$. J. Serra entered this way of thinking [SER98b], which allowed him to obtain theorem 12

Independently of these approaches, H.Heijmans has introduced and studied the class of "grain operators" in[HEI97]. Levelings, in the sense of definition 10 below, are particular grain operators. However, the "good" properties of these grain operators appear when they derive from markers based openings and closings. So we will restrict ourselves to such criteria (for example, we will not accept or reject a particle according to its area).

From now on, we denote by $\gamma_M(A)$ the union of all grains of set A that touch an arbitrary set M, called *marker:*

$$\gamma_M(A) = \cup\{\gamma_x(A)\, ,\, x \in E\, ,\, \gamma_x(A) \parallel M\}$$

Similarly, the complement of closing $\varphi_{N^c}(A)$ is the union of those pores of A that hit marker N^c,

$$[\varphi_{N^c}(A)]^c = \cup\{\gamma_x(A^c)\, ,\, x \in E\, ,\, \gamma_x(A^c) \parallel N^c\}\, ; \quad (7)$$

hence

$$A^c \cap \varphi_{N^c}(A) = \cup\{\gamma_x(A^c)\,,\, x \in E\,,\, \gamma_x(A^c) \;\mathrm{j}\; N\} \tag{8}$$

is the union of those pores of A lying in marker N.

Definition 10. *Let E be an arbitrary set, and $\mathcal{C}$ be a connection on $\mathcal{P}(E)$. Let γ_M and φ_{N^c} an opening and a closing, both by marker reconstruction, from $\mathcal{P}(E)$ into itself. The leveling $\lambda : \mathcal{P}(E) \to \mathcal{P}(E)$, of primitives γ_M and φ_{N^c} is then defined by the relation*

$$\lambda = \gamma_M \cup (- \cap \varphi_{N^c}) = \varphi_{N^c} \cap (- \cup \gamma_M) \tag{9}$$

where $-$ stands for the complement operation on $\mathcal{P}(E)$.

When applied to set A, leveling λ yields the two equalities

$$\begin{aligned} A \cap \lambda(A) &= A \cap \gamma_M(A) \\ A^c \cap \lambda(A) &= A^c \cap \varphi_{N^c}(A) \; (\Longleftrightarrow A \cup \lambda(A) = A \cup \varphi_{N^c}(A)) \end{aligned} \tag{10}$$

so that $\lambda(A)$ acts inside A as opening γ_M, and inside A^c as closing φ_{N^c}. System (10) also relates to the *activity lattice*, where a mapping ψ on $\mathcal{P}(E)$ is said to be less active than another, ψ', when $\psi'(A)$ modifies more points of A than $\psi(A)$ does, $\forall A \in \mathcal{P}(E)$, (ch.8 in [SER88]). If Id stands for the identity operator, the activity ordering is as follows

$$\begin{aligned} Id \cap \psi &\supseteq Id \cap \psi' \\ Id \cup \psi &\subseteq Id \cup \psi' \end{aligned}$$

and one notes $\psi \preceq \psi'$. A complete lattice is associated with this ordering, where the supremum and the infimum of a family $\{\psi_i\,,\, i \in I\}$ are given by

$$\begin{aligned} \mathrm{g}\psi_i &= [- \cap (\cup\psi_i)] \cup [\cap\psi_i] \\ \mathrm{f}\psi_i &= [Id \cap (\cup\psi_i)] \cup [\cap\psi_i] \;. \end{aligned}$$

When applying this system to the family $\{\gamma_M, \varphi_{N^c}\}$ of the two leveling primitives, we draw from (9) that

$$\gamma \,\mathrm{g}\, \varphi = \lambda \qquad \gamma \,\mathrm{f}\, \varphi = Id\,.$$

Conversly, the relation $\gamma \mathrm{g} \varphi = \lambda$ yields equation (9), hence may be considered as an alternative definition for leveling.

An operation whose definition involves the complement $-$ risks not to be increasing. But in the present case, we will now see that the condition under which λ is increasing makes it also a strong filter, which means much more.

Lemma 11. *Let $A, N \in \mathcal{P}(E)$, and let Y be a pore of A. If Y lies in N, then all grains of A which are adjacent to Y hit N. By duality, if a grain X of A does not touch set N, then none of the pores of A adjacent to X is included in N .*

Proof. Consider a pore Y of A, with Y j N, and a grain X of A adjacent to Y. Since there exists a point $x \in X$ that is adjacent to Y, and since Y , N^c (Eq (6)), x belongs necessarily to N; hence $X \cap N \neq \emptyset$.

Take now a grain X of A that does not touch N, i.e. such that X j N^c. We draw from the first part of the proof that every pore Y of A that is adjacent to X meets N^c, hence is not included in N.

Theorem 12. *Let $\mathcal{C}$ be an adjacency preventing connection on $\mathcal{P}(E)$. Given $M, N \subseteq E$ with $N \subseteq M$, the leveling $\lambda_{M,N} : \mathcal{P}(E) \to \mathcal{P}(E)$ of primitives γ_M and φ_{N^c} is a* strong *connected filter, and admits the double decomposition*

$$\lambda = \gamma_M \ \varphi_{N^c} = \varphi_{N^c}\gamma_M \ .$$

Proof. We have to prove that the three following oprations are identical:

i/ to take the union of the pores of A lying in N and of the grains of A touching M;

ii/ to take the union A'of the grains of A touching M, and to add it to the pores of A' that lie in N;

iii/ to add to A all its pores lying in N, and to extract from the result the union of all grains touching M.

Indeed, when $N \subseteq M$, the lemma states that all grains of A adjacent to a pore Y j N hit N, hence hit also M. On the other hand, a grain $\gamma_x(A)$ of A which is not adjacent to various grains X and pores Y of A, with Y j N , are neither adjacent to the union of these X and Y (assumption of adjacency prevention), so that the two processings *i/* and *iii/* are identical. The proof is achieved by observing that *i/* is a self-dual procedure, and that *ii/* and *iii/* are dual of each other.

Remark that, when $N \subseteq M$, the supremum of the two logical conditions A , M and A j N^c is the certainty. Then, according to proposition 8.5 in [HEI97], we find again the increasingness of λ.For extending levelings from sets to numerical functions, we need to consider them as functions of their three arguments A, M and N. Now, is the mapping $\lambda(A, M, N)$ from $[\mathcal{P}(E)]^3$ into $\mathcal{P}(E)$ increasing ?

Corollary 13. *The leveling $\lambda : [\mathcal{P}(E)]^3 \to \mathcal{P}(E)$ is increasing if and only if the two operands M and N are ordered by $N \subseteq M$*

Proof. We draw from the theorem that, given M and N, with $N \subseteq M$,

$$A \subseteq A' \Longrightarrow \lambda(A, M, N) \subseteq \lambda(A', M, N).$$

On the other hand, given A', when $M \subseteq M'$ and $N \subseteq N'$ more grains of A' are touched and more pores of A' are lying, hence

$$\lambda(A', M, N) \subseteq \lambda(A', M', N')$$

which achieves the proof (the only if part is given by the counter example of fig.4a

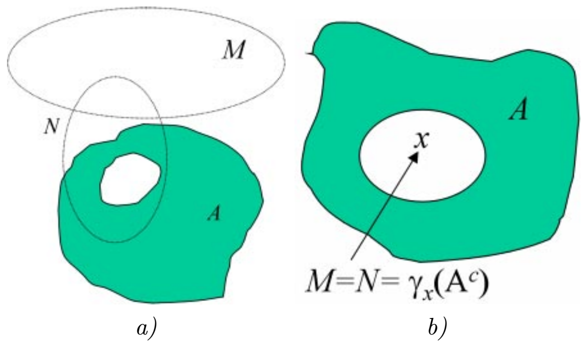


Fig. 4. a)Non increasingness of λ when $N * M$. Take for A' grain A plus its pore; then A$\subseteq A'$ whereas $\lambda(A') \subseteq \lambda(A)$. b) Take the internal pore of grain A as M and N, then $\lambda(A)$equals the pore without the grain (flip-flop effect)

An interesting feature of levelings concerns their possible self-duality. Firstly, we may consider the behaviour, under complement, of the triple mapping $(A, M, N) \to \lambda(A, M, N)$. We have

$$[\lambda(A^c, M^c, N^c)]^c = [\gamma_{M^c}(A^c)]^c \cap [A \cap [\gamma_N(A)]^c]^c = \varphi_{M^c}(A) \cap [A^c \cup \gamma_N(A)],$$

$$\text{hence } [\lambda(A^c, M^c, N^c)]^c = \gamma_N(A) \cup [A^c \cap \varphi_{M^c}(A)] = \lambda(A, N, M)$$

Therefore self-duality of $\lambda(A, M, N)$ is reached when and only when the two markers N and M are identical (a result that can also be drawn from proposition 8.3 in [HEI97]). Since, in addition, condition $M \equiv N$ implies the inreasingness of λ, we may state

Proposition 14. *The leveling $(A, M, M) \to \lambda(A, M, M)$ is an increasing self-dual mapping from $\mathcal{P}(E) \times \mathcal{P}(E)$ into $\mathcal{P}(E)$.*

In this approach, we implicitely supposed that the data of A and of M are independent. In practice, it often occurs that marker M derives from a previous tranformation of A itself, $M = \mu(A)$, say. Then the proposition shows that the leveling $\lambda : \mathcal{P}(E) \to \mathcal{P}(E)$, with $\lambda = \lambda(A, \mu(A), \mu(A))$ is self-dual if and only if *mapping μ itself* is already self-dual.

We conclude this section by exhibiting two examples showing how necessary are some assumptions above. Take for A a single grain with an internal pore, and for $M \equiv N$ the set made by the pore of A in fig.4b. Suppose we replace, in definition 10, the condition $\gamma_x(A) \parallel M$ by $\gamma_x(A) \cap M = \emptyset$, and $\gamma_x(A^c)$ j N by $\gamma_x(A^c) \subseteq N$ Clearly, we have

$$\begin{array}{lll} & \varphi_{M^c}(A) = A \cup M \implies & \gamma_M \varphi_{M^c}(A) = A \cup M, \\ \text{but} & \gamma_M(A) = \emptyset \implies & \varphi_{M^c} \gamma_M(A) = \emptyset \end{array}$$

whereas $\lambda(A) = M$ is neither $\gamma_M \varphi_{M^c}(A)$ nor $\varphi_{M^c}\gamma_M(A)$. Moreover, the example shows that $\lambda(A \cap \lambda(A)) = \emptyset$ and that $\lambda(A \cup \lambda(A)) = A \cup M$; this implies that λ cannot be decomposed into the product of an opening by a closing or *vice versa* (theorem 6-11, corollary 2 in[MAT88c]). Notice also, finally, that in the example of fig.4b the border between the grain and its internal pore is preserved, but not the sense of variation. As a matter of fact, such a "flip-flop" effect is due to the case when M contains a pore of A, but misses the surrounding grain(s). It *cannot appear* in the actual levelings of definition 10.

The second counter-example concerns adjacency prevention. Let us adopt the "open " connection, and take for A the set fig.2a, and for marker $M = N$ the six point pores of the central gulf. Fig.2b and fig.2c show the two transforms $\varphi_{M^c}\gamma_M(A)$ and $\gamma_M\ \varphi_{M^c}(A)$ which are obviously different : one cannot drop the adjacency prevention, in theorem 12 !

2.5 Levelings as function of their markers

For the sake of simplicity, we shall take $M = N$ through this section, although self-duality is not really required here, and write $\lambda_A(M)$ for $\lambda(A, M, M)$.

Theorem 15. *Let $\mathcal{C}$ be an adjacency preventing connection on $\mathcal{P}(E)$. Given $A \subseteq E$ with $N \subseteq M$, the mapping $\lambda_A : \mathcal{P}(E) \to \mathcal{P}(E)$ is a morphological filter from $\mathcal{P}(E)$ into itself.*

Proof. For $A, M \in \mathcal{P}(E)$, A given and M variable, $\lambda_A(M)$ is the union of some grains and some pores of A, in such a way that each accepted pore arrived in $\lambda_A(M)$ accompanied by the whole collection of its adjacent grains. So a grains of $\gamma_x(A)$ that does not participate to $\lambda_A(M)$ does not touch any of the A-connected elements (grains or pores) involved in $\lambda_A(M)$; hence, by adjacency prevention, $\gamma_x(A)$, $\lambda_A(M)$. By duality, $\gamma_x(A^c)$ " M implies $\gamma_x(A^c)$ " $\lambda_A(M)$, so that $\lambda_A[\lambda_A(M)] = \lambda_A(M)$.

The relevant formalism to go further is that of the activity ordering for sets (and no longer for set mappings)[MAT97]. As a matter of fact, any fixed set A generates an ordering denoted by $\preceq_A$, from the two relationships

$$M_1, M_2 \subseteq E \quad \begin{array}{c} M_1 \cap A \supseteq M_2 \cap A \\ \\ M_1 \cap A^c \subseteq M_2 \cap A^c \end{array} \quad \Leftrightarrow \quad M_1 \preceq_A M_2$$

From this ordering derives the so called *A-activity lattice*, where the supremum and the infimum of a family $\{M_i, i \in I\}$ of sets are given by

$$\begin{aligned} \mathrm{g}_A M_i &= [A^c \cap (\cup M_i)] \cup [\cap M_i] \\ \mathrm{f}_A M_i &= [A \cap (\cup M_i)] \cup [\cap M_i] \end{aligned}$$

with A itself as the minimum element, and A^c as the maximum one (a system very similar to that presented above about the activity lattice for operators). In this framework, the following theorem holds[MAT97][SER98b]

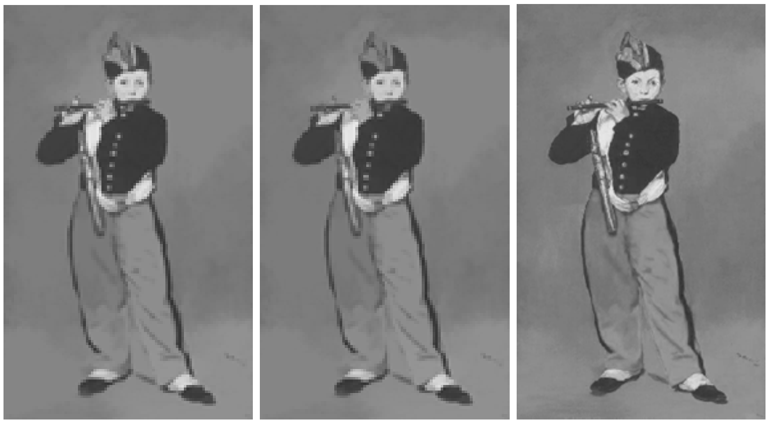


Fig. 5. a) Manet' Joueur de fifre b)and c) levelings of a) by extended extrema of dynamics 30 (b) and 60 (c).

Theorem 16. *Given set A, the leveling $M \rightarrow \lambda_A(M)$ from the A-activity lattice of $\mathcal{P}(E)$ into itself are openings. Moreover, when the A-activity of A increases, we have*

$$M_1 \preceq_A M_2 \quad \Rightarrow \quad \begin{array}{l} \lambda_{\lambda_A(M_1)}(M_2) = \lambda_{\lambda_A(M_2)}(M_1) = \lambda_A(M_2) \\ \lambda_{\lambda_A(M_1)}(M_2) = \lambda_{\lambda_A(M_2)}(M_1) = \lambda_A(M_2)\ . \end{array}$$

This last granulometric type pyramid is specially usefull in practice, for it allows to grade the activity effects of markers: it means that we can directly implement a highly active marker, or, equivalently, reach it by intermediary steps. An example is given in fig.5.

2.6 Function levelings

Let T be a discrete axis; denote by T^E the lattice of all numerical functions $f : E \longrightarrow T$. An increasing operator Ψ on T^E is said to be *flat* if there exists an increasing set operator ψ such that

$$X\left[\Psi(f), t\right] = \psi\left[X(f), t\right] \tag{11}$$

where stands for the thresholding of function f at level t, i.e. :

$$X(f,t) = \{x : x \in E, \quad f(x) \geq t\} \tag{12}$$

In the discrete cases of digital imagery, relation (11) is sufficient to characterize the function operator Ψ associated with an increasing set operator ψ.

Definition 17. *Let f, g, h, be three functions from E into T,with $g \leq h$.Then the relation*

$$X\left[\Lambda(f), t\right] = \lambda\left[X(f,t)\right], X(g,t), X(h,t)]$$

defines one and only one leveling $\Lambda(f)$ on T^E.

When connection $\mathcal{C}$ is obtained from the iterations of an elementary dilation δ, of adjoint erosion ε, then a digital algorithm for $\Lambda(f)$ from the data of f, g and h derives from the decomposition theorem 12, by computing successively the opening by reconstruction $g_\infty(f)$ and then $\Lambda(f) = h_\infty[g_\infty(f)]$. The first operation is thus given by the limit of the sequence

$$g_n = (f \wedge \delta g_{n-1})$$
$$\text{with } g_1 = (f \wedge \delta g)$$

and the second one by

$$h_n = [g_\infty(f) \vee \varepsilon h_{n-1}]$$
$$\text{with } h_1 = [g_\infty(f) \vee \varepsilon h]$$

All theorems and propositions 11 to16 of the binary case extend directly to numerical one. Concerning self-duality for example, if 0 and m stand for the two extreme bounds of the gray axis T, we have

$$m - \Lambda(m - f, m - g, m - g) = \Lambda(f, g, g)$$

which means that the leveling $f, g \to \Lambda(f, g)$ is always a self-dual mapping. In addition, when one takes for marker g a self-dual mapping (*e.g.* convolution, median operator, etc..), then the leveling Λ, considered as a function of f only, becomes in turn self-dual, and we have

$$m - g(m - f) = g(f) \Rightarrow m - \Lambda[m - f, g(m - f), g(m - f)] = \Lambda[f, g(f), g(f))]$$

In practice, the role of the marker is crucial. In fig.5, the marker is obtained by replacing f by zero on the extended maxima and minima of f, and by leaving f unchanged elsewhere (*extended maxima of* f : do the opening by reconstruction $\gamma_{\text{rec}}(f)$ of f from $f - k$, where k is a positive constant. Then the maxima of $\gamma_{\text{rec}}(f)$ define the so called *extended maxima* of f, and those points x where $f(x) - \gamma_{\text{rec}}(f)(x) = k$ define the (non extended) maxima of f of dynamics ¿ k ; the extended minima are obtained by duality).The corresponding levelings are shown in fig.5a and 5b, for markers g_{30} and g_{60}, of dynamics 30 and 60 respectively (over 256 gray levels).

These two markers are self-dual by construction, and satisfy the condition of activity increasingness of theorem 16. Their progressive leveling action appears clearly when confronting fig.5a and 5b. Notice the relatively correct preservation of some fine details such as buttons, eyes, eyebrows, fingers, etc.. These details are preserved because of their high dynamics.

In figure 6, the leveling is used for noise reduction, from a marker obtained by Gaussian moving average of size 5, namely fig.6b, of the initial noisy image fig.6a. It results in fig.6c where the noise reduction of fig.6b is preserved, but where the initial sharpness of the edges is recovered.

A last word. There are two ways for developing a theory in discrete geometry. One can start from some Euclidean notions and adapt them to discrete spaces,

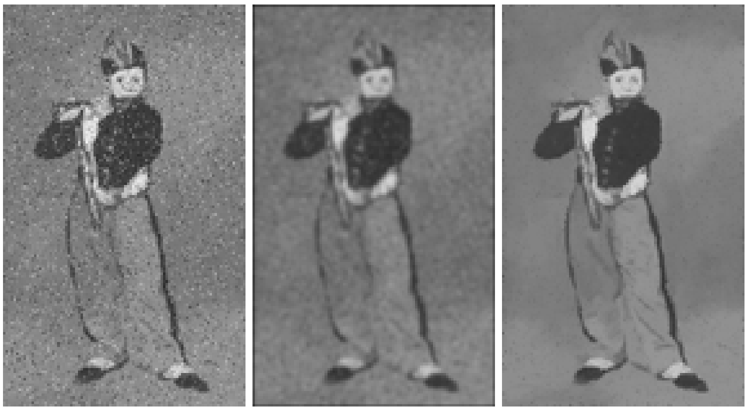


Fig. 6. a) noisy version, b) gaussian convolution of a, c) leveling of a) by marker b)

or elaborate the whole approach independently of the fact that it may apply to a continous, or a discrete, or a finite, space E. It is this second that was chosen here.

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