New Public-Key Schemes Based on Elliptic Curves over the Ring Z_n

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ABSTRACT Three new trapdoor one-way functions are proposed that are based on elliptic curves over the ring Z_n. The first class of functions is a naive construction, which can be used only in a digital signature scheme, and not in a public-key cryptosystem. The second, preferred class of function, does not suffer from this problem and can be used for the same applications as the RSA trapdoor one-way function, including zero-knowledge identification protocols. The third class of functions has similar properties to the Rabin trapdoor one-way functions. Although the security of these proposed schemes is based on the difficulty of factoring n, like the RSA and Rabin schemes, these schemes seem to be more secure than those schemes from the viewpoint of attacks without factoring such as low multiplier attacks. The new schemes are somewhat less efficient than the RSA and Rabin schemes.

1 Introduction

In their seminal 1976 paper [3], Diffie and Hellman introduced the concept of a trapdoor one-way function (TOF). A TOF is a function that is easy to evaluate but infeasible to invert, unless a secret trapdoor is known, in which case the inversion is also easy. Although no realisation of a TOF was proposed in [3], Diffie and Hellman observed that such a function would allow the construction of digital signature schemes and public-key cryptosystems, two concepts that they introduced.

The first implementation of a TOF was proposed by Rivest, Shamir and Adleman in 1978 [21]. Its security relies on the difficulty of factoring a composite number n. Some other implementations [20, 4] of TOFs have been proposed based on the difficulty of factoring and discrete logarithms. From another direction, one of the recent topics in the field of elliptic curves is their applicability to cryptography. The points of an elliptic curve E over a *finite field* form an abelian group, and hence the group E can be used to implement analogs of the Diffie-Hellman key exchange scheme and the ElGamal public key cryptosystem, as explained in [9]. The security of these analogous systems rests on the difficulty of the discrete logarithm problem on an elliptic curve.

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In this paper, we propose new TOFs (or public-key cryptographic schemes) based on elliptic curves over a *ring* $\mathbf{Z}_n$. The security of these TOFs depends on the difficulty of factoring n. Although these schemes are less efficient than the RSA and Rabin schemes, our schemes seem to be more secure from the viewpoint of some attacks that do not use factoring such as low multiplier attacks. In this case, even when the RSA system can be broken without factoring the modulus, our schemes seem to remain secure.

We begin with a brief review of the basic definitions and facts about elliptic curves over a finite field. In Section 3, we show some properties of elliptic curves over a ring, which are used in the succeeding sections. Section 4 proposes a naive construction of the TOF (Type 0 scheme) based on elliptic curves over a ring, but which can be used only in a digital signature scheme, and not in a public-key cryptosystem. In Sections 5 and 6, we propose the Type 1 and Type 2 schemes respectively based on the elliptic curve over a ring, and discuss their properties. Section 7 discusses the security of the proposed schemes, and Section 8 discusses their performance.

2 Elliptic Curves over a Finite Field

Let K be a field of characteristic $\neq 2, 3$, and let $a, b \in K$ be two parameters satisfying $4a^3 + 27b^2 \neq 0$. An elliptic curve over K with parameters a and b is defined as the set of points (x, y) with $x, y \in K$ satisfying the equation

$$y^2 = x^3 + ax + b,$$

together with a special element denoted $\mathcal{O}$ and called the point at infinity. We will mainly be interested in elliptic curves over the finite field $\mathbf{F}_p$ with p elements, for some prime p. Such a curve will be denoted $E_p(a, b)$. What makes elliptic curves interesting in cryptography is the fact that an addition operation on the points of an elliptic curve can be defined that makes it into an abelian group. This addition operation, which has but its name in common with the ordinary addition of integers, is described in the following.

Let E be an elliptic curve, and let P and Q be two points on E. The point $P + Q$ is defined according to the following rules. If $P = \mathcal{O}$, then $-P = \mathcal{O}$, and $P + Q = Q$ (i.e., $\mathcal{O}$ is the neutral element of E). Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. If $x_1 = x_2$ and $y_1 = -y_2$, then $P + Q = \mathcal{O}$ (i.e., the negative of the point (x, y) is the point $(x, -y)$). In all other cases the coordinates of $P + Q = (x_3, y_3)$ are computed as follows. Let λ be defined as

$$\lambda = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} & \text{if } x_1 \neq x_2 \\ \dfrac{3x_1^2 + a}{2y_1} & \text{if } x_1 = x_2. \end{cases}$$

(When $P + Q \neq \mathcal{O}$, then the denominator is always non-zero and thus the quotient is defined.) The resulting point $P + Q = (x_3, y_3)$ is defined by

$$\begin{aligned} x_3 &= \lambda^2 - x_1 - x_2 \\ y_3 &= \lambda(x_1 - x_3) - y_1. \end{aligned}$$

Clearly, the first equation is equivalent to $x_3 = \lambda^2 - 2x_1$ when $P = Q$. All computations are in the field over which E is defined. In particular, when the field is $\mathbf{F}_p$, all computations are modulo p.

Let $\#E_p(a,b)$ denote the order (i.e., the number of points) of the elliptic curve $E_p(a,b)$. It is well-known that $\#E_p(a,b) = p+1+t$ where $|t| \leq 2\sqrt{p}$ for every elliptic curve over $\mathbf{F}_p$. Every value of t within the given bounds is taken for some pair (a,b), but this fact will not be used in this paper. There exists a polynomial-time algorithm due to Schoof [22] for computing the order of an elliptic curve, but this algorithm is quite impractical for large p. It is known that $E_p(a,b)$ is either cyclic or the product of two cyclic groups. In the latter case, $E_p(a,b) \cong \mathbf{Z}_{N_1} \times \mathbf{Z}_{N_2}$ where $N_1 \cdot N_2 = \#E_p(a,b)$, where N_2 divides N_1 and where N_2 also divides $p-1$. We refer to [9] for a more detailed introduction to elliptic curves, and to [8] for some further cryptographically useful properties of elliptic curves.

For some special classes of elliptic curves the order and group structure is easily determined. The following well known two lemmas illustrate this point.

Lemma 1. *Let p be an odd prime satisfying $p \equiv 2 \pmod 3$. Then, for $0 < b < p$, $E_p(0,b)$ is a cyclic group of order*

$$\#E_p(0,b) = p+1.$$

Proof. We first prove that $\#E_p(0,b) = p+1$. When $p \equiv 2 \pmod 3$ then the mapping $x \mapsto x^3$ is a permutation on $\mathbf{F}_p$. Hence for every b there are exactly $(p-1)/2$ numbers $x \in \mathbf{F}_p$ for which x^3+b is a quadratic residue, and for each such x there are two points on $E_p(0,b)$, viz., the points $(x, \pm\sqrt{x^3+b})$. Together with the points $(\sqrt[3]{-b}, 0)$ and $\mathcal{O}$ there are $p+1$ points on $E_p(0,b)$. To prove that $E_p(0,b)$ is cyclic (see also [8]), suppose it is not. Then $E_p(0,b) \cong \mathbf{Z}_{N_1} \times \mathbf{Z}_{N_2}$ where $N_1 N_2 = p+1$ and N_2 divides $p-1$. Hence $N_2 = 2$ and N_1 is even. Then the group $\mathbf{Z}_{N_1} \times \mathbf{Z}_2$ must have four elements P for which $-P = P$. However, there are exactly two points, $P = \mathcal{O}$ and $(\sqrt[3]{(-b)}, 0)$ for which $-P = P$, since the only points P on $E_p(0,b)$ for which $-P = P$ are the points (x,y) with $y = 0$. This contradiction implies $E_p(0,b)$ is cyclic. □

Lemma 2. *Let p be a prime satisfying $p \equiv 3 \pmod 4$. Then, for $0 < a < p$, we have*

$$\#E_p(a,0) = p+1.$$

Moreover, $E_p(a,0)$ is cyclic if a is a quadratic residue modulo p and $E_p(a,0) \cong \mathbf{Z}_{(p+1)/2} \times \mathbf{Z}_2$ otherwise.

Proof. Let $f(x) = x^3 + ax$. $f(x)$ is an odd function, i.e., $f(-x) = -f(x)$. The condition $p \equiv 3 \pmod 4$ implies that for every $s \in \mathbf{Z}_p^*$, exactly one of the two numbers s or $-s$ is a quadratic residue modulo p. This follows from the fact that $(p-1)/2$ is odd and thus -1 is a quadratic non-residue modulo p. Consider the $(p-1)/2$ pairs $[x,-x]$ for $0 < x \leq (p-1)/2$. For every such pair, either $f(x) = f(-x) = 0$ or $f(x)$ is a quadratic residue or $f(-x)$ is a quadratic residue. In either of the three cases, there exist 2 point on $E_p(a,b)$ associated with the pair $[x,-x]$, viz., $(\pm x, 0)$, $(x, \pm\sqrt{f(x)})$ or $(-x, \pm\sqrt{-f(x)})$, respectively. Together with $(0,0)$ and $\mathcal{O}$ the total number of points on $E_p(a,b)$ is $p+1$. The proof of the last claim is similar to the proof given for Lemma 1. □

3 Elliptic Curves over a Ring

We now consider elliptic curves over the ring $\mathbf{Z}_n$, where n is an odd composite squarefree integer. (An alternative notation for $\mathbf{Z}_n$ used in the literature is $\mathbf{Z}/n\mathbf{Z}$.) Similar to the

definition of $E_p(a,b)$, an elliptic curve $E_n(a,b)$ can be defined as the set of pairs $(x,y) \in \mathbf{Z}_n^2$ satisfying $y^2 \equiv x^3 + ax + b \pmod{n}$, together with a point $\mathcal{O}$ at infinity. An addition operation on $E_n(a,b)$ can be defined in the same way as the addition operation on $E_p(a,b)$, simply by replacing computations in $\mathbf{F}_p$ by computations in $\mathbf{Z}_n$. However, two problems occur. The first problem is that because the computation of λ requires a division which in a ring is defined only when the divisor is a unit, the addition operation on $E_n(a,b)$ is not always defined. The second problem, which is related to the first is that $E_n(a,b)$ is not a group. It would therefore seem impossible to base a cryptographic system on $E_n(a,b)$. In the following we present a natural solution to these problems.

For the sake of simplicity, let $n = pq$ in the sequel be the product of only two primes as in the RSA system. Moreover, the addition operation on $E_n(a,b)$ described above, whenever it is defined, is equivalent to the (componentwise defined) group operation on $E_p(a,b) \times E_q(a,b)$. By the Chinese Remainder Theorem, every element c of $\mathbf{Z}_n$ can be represented uniquely as a pair $[c_p, c_q]$ where $c_p \in \mathbf{Z}_p$ and $c_q \in \mathbf{Z}_q$. Thus every point $P = (x,y)$ on $E_n(a,b)$ can be represented uniquely as a pair $[P_p, P_q] = [(x_p, y_p), (x_q, y_q)]$ where $P_p \in E_p(a,b)$ and $P_q \in E_q(a,b)$, with the convention that $\mathcal{O}$ is represented by $[\mathcal{O}_p, \mathcal{O}_q]$, where $\mathcal{O}_p$ and $\mathcal{O}_q$ are the points at infinity on $E_p(a,b)$ and $E_q(a,b)$, respectively. By this mapping, all elements of $E_p(a,b) \times E_q(a,b)$ are exhausted except the pairs of points $[P_p, P_q]$ for which exactly one of the points P_p and P_q is the point at infinity. Note that the addition operation on $E_n(a,b)$ described above is undefined if and only if the resulting point, when interpreted as an element of $E_p(a,b) \times E_q(a,b)$, is one of these special points.

It is important to note that when all prime factors of n are large, it is extremely unlikely that the sum of two points on $E_n(a,b)$ is undefined. In fact, if the probability of the addition operation being undefined were non-negligible, then the very execution of a computation on $E_n(a,b)$ would be a feasible factoring algorithm, which is assumed not to exist. Therefore, the first problem will cause no difficulties in practice.

The second problem, that $E_n(a,b)$ is not a group, can be solved by the following lemma. That is, although we cannot use the properties of a finite group directly, we can use a property of $E_n(a,b)$ which is similar to that of a finite group. The following lemma can be easily obtained from the Chinese Remainder Theorem.

Lemma 3. *Let $E_n(a,b)$ be an elliptic curve such that $\gcd(4a^3 + 27b^2, n) = 1$ and $n = pq$ (p, q: prime). Let N_n be $\mathrm{lcm}(\#E_p(a,b), \#E_q(a,b))$. Then, for any $P \in E_n(a,b)$, and any integer k,*

$$(k \cdot N_n + 1) \cdot P = P \text{ over } E_n(a,b).$$

We should note that it is possible to define an elliptic curve over a ring so that the resulting structure is a group. For our purposes, this is unnecessary.

4 Naive Construction of TOF Based on Elliptic Curves over a Ring

In this section, we show a naive construction of TOFs (Type 0 scheme) which are based on elliptic curves over a ring. These TOFs can be used only in a digital signature scheme, and not in a public-key cryptosystem. The shortcomings of the TOFs of this section are elliminated in the Type 1 and 2 schemes shown in following sections.

A digital signature scheme based on $E_n(a,b)$ can be set up as follows. The signer Alice chooses two primes p and q (or, more generally, a set of two or more distinct primes) and two parameters a and b satisfying $\gcd(4a^3 + 27b^2, n) = 1$, where $n = pq$. She

then computes the orders of the elliptic curves $E_p(a,b)$ and $E_q(a,b)$ (for example using Schoof's algorithm [22]), chooses a public encryption multiple e relatively prime to both $\#E_p(a,b)$ and $\#E_q(a,b)$, and computes the secret decryption multiple d according to

$$d \equiv e^{-1} \pmod{\mathrm{lcm}(\#E_p(a,b), \#E_q(a,b))}.$$

Alice releases as public parameters n, a, b and e. When she later wants to sign a message M she associates a point $P = (x,y) \in E_n(a,b)$ with M in a publicly-known way (see below) and computes the point $Q = (s,t)$ on $E_n(a,b)$ according to

$$Q = (s,t) = d \cdot P.$$

The signature for the message M is the pair (s,t), which can be checked by computing

$$P = (x,y) = e \cdot Q$$

on $E_n(a,b)$ and extracting the message M from (x,y) (because $(ed) \cdot P = P$ from Lemma 3).

Here, given a message M, a point (x,y) on $E_n(a,b)$ can efficiently be associated with M. M is first padded with sufficient redundancy, for instance by appending zero's to M, resulting in M'. x is defined as the smallest integer greater or equal to M' such that $x^3 + ax + b$ is a quadratic residue modulo n, and y is defined as one of the square roots modulo n of this number.

The shortcomings of this scheme are as follows:
(1) Schoof's algorithm [22] to compute $\#E_p(a,b)$ and $\#E_q(a,b)$ is infeasible for large p.
(2) The signature is roughly twice as long as the original message M.
(3) This scheme cannot be used for a public-key cryptosystem, since knowledge of the trapdoor is required to create a point on $E_n(a,b)$, which corresponds to a plaintext.

This scheme may be advantageous in some circumstances. It does allow digital signature without the possibility of encryption.

5 Basic TOF Based on Elliptic Curves over a Ring

In this section, we propose a new TOF (Type 1 scheme) that is based on elliptic curves over a ring. It overcomes the three shortcomings of the Type 0 scheme. For simplicity, we show a protocol for a public-key cryptosystem based on elliptic curves as described in Lemma 1. We can easily construct a public-key cryptosystem in the case of Lemma 2, and digital signature schemes, although we omit a description.

Step 0 (Key Generation) User U chooses large primes p and q such that

$$p \equiv q \equiv 2 \pmod 3.$$

U computes the product $n = pq$, and $N_n = \mathrm{lcm}(\#E_p(0,b), \#E_q(0,b)) = \mathrm{lcm}(p+1, q+1)$.

U chooses an integer e which is coprime to N_n, and computes an integer d such that

$$ed \equiv 1 \pmod{N_n}.$$

Summarizing, U's secret key is d, (p, q, $\#E_p(0,b)$, $\#E_q(0,b)$, N_n), and U's public key is n, e.

Step 1 (Encryption) A plaintext $M = (m_x, m_y)$ is an integer pair, where $m_x \in \mathbf{Z}_n$, $m_y \in \mathbf{Z}_n$. Let $M = (m_x, m_y)$ be a point on the elliptic curve $E_n(0, b)$, where b is determined by m_x and m_y.

Sender A encrypts the point M by encryption function $\mathbf{E}(\cdot)$ with the receiver's public key e and n as

$$C = \mathbf{E}(M) = e \cdot M \text{ over } E_n(0, b),$$

and sends a ciphertext pair $C = (c_x, c_y)$ to a receiver B.

Step 2 (Decryption) Receiver B decrypts a point C by decryption function $\mathbf{D}(\cdot)$ with his secret key d and public key n as

$$M = \mathbf{D}(C) = d \cdot C \text{ over } E_n(0, b).$$

[Notes]

1. In the case of Lemma 1, the minimum possible value of e is 5 because $2|N_n$ and $3|N_n$. In the case of Lemma 2, the minimum possible value of e is 3 because $2|N_n$.
2. For elliptic curves, the addition formula is independent of a and b, and the doubling formula is independent of b. Thus, the above protocol does not require computation of the value $b = y^2 - x^3 \bmod n$. If Lemma 2 is adopted, for the addition formula the sender S must compute a such that $a = (m_y^2 - m_x^3)/m_x \bmod n$, and the receiver R must compute a such that $a = (c_y^2 - c_x^3)/c_x \bmod n$.
3. This scheme has the interesting property that it is not defined on a single group but on a large class of groups, all with the same order. The curve to be used is determined by the plaintext to be transmitted.

6 Rabin-type Generalization

6.1 Protocol

We propose another TOF (Type 2 Scheme) also based on elliptic curves over a ring, which is the Rabin-type generalization of the basic TOF (Type 1 scheme). The Type 2 scheme also overcomes the three deficiencies of the Type 0 scheme. For simplicity, we described the protocol for a public-key cryptosystem based on elliptic curves described in Lemma 1.

Step 0 (Key Generation) User U chooses large primes p and q such that

$$p \equiv q \equiv 2 \pmod 3.$$

U computes the product $n = pq$, and the orders $N_p = \#E_p(0, b) = p + 1$ and $N_q = \#E_q(0, b) = q + 1$.

Summarizing, U's secret key is p, q, N_p, N_q, and U's public key is n.

Step 1 (Encryption) A plaintext $M = (m_x, m_y)$ is an integer pair, where $m_x \in \mathbf{Z}_n$, $m_y \in \mathbf{Z}_n$. Let $M = (m_x, m_y)$ be a point on the elliptic curve $E_n(0, b)$, where b is determined by m_x and m_y.

Sender A encrypts the point M by doubling on the elliptic curve E_n with the receiver's public key n as

$$C = 2 \cdot M \text{ over } E_n(0, b),$$

and sends a ciphertext pair $C = (c_x, c_y)$ to a receiver B.

Step 2 (Decryption) Receiver B computes $M_p \in E_p(0, b)$ and $M_q \in E_q(0, b)$ from $C_p = (c_x \bmod p,\ c_y \bmod p) \in E_p(0, b)$ and $C_q = (c_x \bmod q,\ c_y \bmod q) \in E_q(0, b)$ such that

$$C_p = 2 \cdot M_p \text{ over } E_p(0, b), \quad C_q = 2 \cdot M_q \text{ over } E_q(0, b),$$

by using a halving algorithm, which is described in Section 6.2.

B computes $M = (m_x, m_y) \in E_n$ from $M_p = (m_{px}, m_{py}) \in E_p(0, b)$ and $M_q = (m_{qx}, m_{qy}) \in E_q(0, b)$ using the Chinese Remainder Theorem.

[Notes]

1. Since both N_p and N_q are even, 2 is not coprime to N_p, N_q and N_n.
2. The Type 2 scheme has the drawback that there is 4:1 ambiguity in the decrypted messages, as is true for the original Rabin scheme.
3. In decryption based on a halving formula, the algorithm for finding a non-double point requires an exact expression of the elliptic curve. Thus, the receiver B must compute b such that $b = c_y^2 - c_x^3 \bmod n$.

6.2 Halving Algorithm

In general, points on $E_p(a, b) : y^2 = x^3 + ax + b \bmod p$ can be separated into 2 classes, as integers in $\mathbf{Z}_p$ are classified into quadratic residue and quadratic non-residue modulo p.

Definition If $P = 2 \cdot X$ over $E_p(a, b)$ for some point X on the curve $E_p(a, b)$, then we call point P a *double point*, where we denote the set of all double points by DP_p. If $P \neq 2 \cdot X$ over $E_p(a, b)$ for any point X, then we call point P a *non-double point*, where we denote the set of all non-double points by NDP_p. □

Double points and non-double points are distinguishable by using the following three lemmas, when the group structure of $E_p(a, b)$ is known.

Lemma 4. Assume that E' is a cyclic subgroup of $E_p(a, b)$ having the maximum order N'. Let P be in E', and N' be even. Then

$$P \in DP_p \text{ if and only if } N'/2 \cdot P = \mathcal{O} \text{ over } E_p(a, b),$$

Lemma 5. Assume that E' is a cyclic subgroup of $E_p(a, b)$ with the maximum order N'. Let α be the cardinality of DP_p in E'. Then

$$\alpha = \begin{cases} N'/2, & \text{if } N' \text{ is even;} \\ N', & \text{if } N' \text{ is odd.} \end{cases}$$

Lemma 6. Assume that $E_p(a,b)$ has the group structure $\mathbf{Z}_{(p+1)/2} \times \mathbf{Z}_2$. Let E' be a cyclic subgroup of $E_p(a,b)$ with the maximum order $(p+1)/2$ and let Q be a point in this subgroup. Then

$P \in DP_p$ and $P \in E'$ if and only if $e_{(p+1)/2}(P,Q) = 1$ and $(p+1)/4 \cdot P = \mathcal{O}$ over $E_p(a,b)$,

where $e_{(p+1)/2}$ is the Weil pairing function [8, 19]. Note that $(p+1)/2$ is always even.

Next, consider a halving algorithm on elliptic curve $E_p(a,b)$ which outputs a half point of a given point over $E_p(a,b)$.

The algorithm of Adleman, Manders and Miller [1, 11] for computing a square root mod p is easily adapted to a halving algorithm in $E_p(a,b)$. For completeness we describe the result.

Theorem 7. There exists an expected polynomial time algorithm which, given an odd prime p, an elliptic curve $E_p(a,b)$ in the case of Lemma 1 or 2, N_p, and a point $Q \in DP_p$ as inputs, will output a half point of Q over $E_p(a,b)$.

The proof of Theorem 7 is a direct consequence of the following algorithm.

Halving Algorithm on Elliptic Curve for Type 2 scheme

Input: p (prime), $E_p(a,b)$, N_p, $Q\ (= 2 \cdot H) \in E_p(a,b)$.

Step 1. Compute an odd c, and h such that $N_p = 2^h c$.

Step 2. Choose random point T such that $T \in NDP_p$ and T is in the maximum cyclic subgroup including Q.

Step 3. Set $Y = Q$, $H = (c+1)/2 \cdot Q$ over $E_p(a,b)$.

Step 4. Find the least k such that $(2^k c) \cdot Y = \mathcal{O}$ over $E_p(a,b)$.

Step 5. If $k = 0$ then output H; else set

$$Y = Y - 2^{h-k} \cdot T \text{ over } E_p(a,b), \quad H = H - 2^{h-k-1} \cdot c \cdot T \text{ over } E_p(a,b)$$

and go to step 4.

Output: H.

An algorithm for finding a non-double point T is derived from Lemmas 4 and 6 as follows:

Algorithm 1 for Finding a Non-Double Point ($E_p(a,b)$: cyclic)

Input: p (prime), $E_p(a,b)$, N_p.

Step 1. Choose a random point $T = (t_x, t_y)$ on the curve.

Step 2. If T is a non-double point, that is, $N_p/2 \cdot T \neq \mathcal{O}$ over $E_p(a,b)$, then output T; else go to step 1.

Output: $T = (t_x, t_y) \in NDP_p$.

Algorithm 2 for Finding a Non-Double Point ($E_p(a,0) \cong \mathbf{Z}_{(p+1)/2} \times \mathbf{Z}_2$)

Input: p (prime), $E_p(a,0)$, N_p, $Q \in E_p(a,0)$.

Step 1. Choose a random point $T = (t_x, t_y)$ on the curve $E_p(a,0)$ such that $e_n(Q,T) = 1$.

Step 2. If T is a non-double point, that is, $(p+1)/4 \cdot T \neq \mathcal{O}$ over $E_p(a,0)$, then output T; else go to step 1.

Output: T such that $T = (t_x, t_y) \in NDP_p$, and $T \in E'$, where E' is a cyclic subgroup of $E_p(a, 0)$ with the maximum order of $(p+1)/2$ which includes point Q.

There exists a polynomial time general algorithm for finding a point on the elliptic curve [9]. In case 1, for any $y \in \mathbf{Z}_p$, the point $((y^2 - b)^{1/3}, y)$ is on the curve. Since $3 \not| p-1$, the value of $(y^2-b)^{1/3}$ can be easily computed by $(y^2-b)^{\beta} \bmod p$, where $3\beta \equiv 1 \bmod (p-1)$. In case 2, for any $x \in \mathbf{Z}_p$, the point $(x, (x^3+ax)^{1/2})$ is on the curve. Since $p = 4k+3$ (k : integer), the value of $(x^3+ax)^{1/2}$ can be easily computed by $(x^3+ax)^{k+1} \bmod p$.

7 Security

The security of the proposed Type 1 scheme and Type 2 scheme over elliptic curves is based on the difficulty of factoring n. In this section, we discuss the security of these schemes from various viewpoints.

7.1 Solving the Order

The original RSA and Rabin schemes can be broken if one can determine order of the multiplicative groups. It is known that finding $\phi(n) = (p-1)(q-1)$ is computationally equivalent to factoring n. That is, the former is polynomially reducible to the latter, and vice versa. In our proposed schemes (Types 1 and 2 in the cases of Lemmas 1 and 2), a similar relationship holds.

Theorem 8. *Let* N_n *be* $\mathrm{lcm}(\#E_p(a,b), \#E_q(a,b)) = \mathrm{lcm}(p+1, q+1)$. *Finding* N_n *is computationally equivalent to factoring the composite number* n.

7.2 Finding the Secret Key

The security of the original RSA scheme is also based on the difficulty of finding the secret exponent key. The security of the Type 1 scheme is also based on the difficulty of finding the secret multiplier key d. We have the following relationship.

Theorem 9. *Solving a secret key* d *from public keys* e *and* n *is computationally equivalent to factoring a composite number* n.

7.3 Complete Breaking

Completely breaking Type 1 and 2 schemes means to recover both m_x and m_y from any ciphertext pair (c_x, c_y) and the public keys. It is well known that completely breaking the original Rabin cryptosystem is as hard as factoring the composite n used as the modulus. For the Type 2 scheme, we have the following theorem.

Theorem 10. *Completely breaking the Type 2 scheme is computationally equivalent to factoring* n.

Proof: It is clear that if once the factors of n are known, plaintext (m_x, m_y) can easily be computed from ciphertext (c_x, c_y) and public keys (a, n). Conversely, if there is an Algorithm A1, given P on $E_n(a,b)$ ($E_n(0,b)$ or $E_n(a,0)$), to output Q satisfying $P = 2 \cdot Q$ with non-negligible probablity, then we can construct an expected polynomial-time algorithm B to factor n, using A1 as an oracle. First, B chooses a random point $R =$

(r_x, r_y) $(r_x, r_y \in \mathbf{Z}_n)$, and multiplies it by 2, asks A1 to halve this point, and B obtains R' satisfying $P = 2 \cdot R'$ with non-negligible probablity. Then B computes $R_0 = R - R'$. Since $2 \cdot R_0 = \mathcal{O}$, and R_0 over $E_p(a, b)$ (R_0 over $E_q(a, b)$) is $\mathcal{O}_p$ ($\mathcal{O}_q$), then R_0 is an undefined point with probability 1/2. If R_0 is undefined, B can compute a non-trivial factor of n by the extended Euclidean algorithm used for division modulo n. Clearly, the expected running time of B is polynomial-time in $\log n$. □

In the Type 1 scheme, the equivalence between completely breaking this scheme and factoring n is *not* known. This situation is the same as the original RSA scheme.

7.4 Homomorphism Attacks and Their Countermeasures

The encryption-decryption functions $\mathbf{E}(\cdot)$ and $\mathbf{D}(\cdot)$ for Type 1 and 2 schemes are homomorphic for addition as

$$\mathbf{E}(M_1 + M_2) = \mathbf{E}(M_1) + \mathbf{E}(M_2) \quad \text{and} \quad \mathbf{D}(M_1 + M_2) = \mathbf{D}(M_1) + \mathbf{D}(M_2),$$

for any points M_1 and M_2 on the *same* elliptic curve. This kind of homomorphic property is the basis for some attacking methods proposed [7] against the original RSA and Rabin schemes.

The probability that randomly chosen integer pairs M_1 and M_2 are on the same elliptic curve is as negligiblly small. Thus, passive attacks using homomorphism seem to be ineffective against Type 1 and 2 schemes.

Consider an active attack (a chosen-plaintext attack) using homomorphism. Suppose an attacker A wants to make a victim B sign a plaintext $M = (m_x, m_y)$ without B's consent. A generates another message M' with B's public keys $(e_B,\ n_B)$ and random integer r,

$$M' = M + e_B \cdot (r \cdot M) \text{ over } E_{n_B},$$

and sends M' to B. B makes a signature S' for M' with his secret key d_B:

$$S' = d_B \cdot M' = d_B \cdot (M + e_B \cdot (r \cdot M)) \text{ over } E_{n_B}.$$

Then, A computes a signature S for M from S' by

$$S = S' - r \cdot M \text{ over } E_{n_B}.$$

Using this technique, A can forge B's signatures without B's secret key. To counter this attack, a randomization of a plaintext with a hashing function h should be applied before the application of the function D. This method is similar to that required for the original RSA scheme.

7.5 Isomorphism Attacks and Their Countermeasures

Definition Let $n = pq$ (p, q: prime), and E_n^1 and E_n^2 be elliptic curves such that

$$E_n^1 :\ y^2 = x^3 + a_1 x + b_1 \bmod n, \quad E_n^2 :\ y^2 = x^3 + a_2 x + b_2 \bmod n.$$

E_n^1 and E_n^2 are isomorphic if there exist $u_p \in \mathbf{Z}_p^*$ and $u_q \in \mathbf{Z}_q^*$ such that

$$a_2 \equiv u_p^4 a_1 \bmod p, \quad b_2 \equiv u_p^6 b_1 \bmod p,$$

$$a_2 \equiv u_q^4 a_1 \bmod q, \quad b_2 \equiv u_q^6 b_1 \bmod q.$$

Then the following isomorphic property of the elliptic curves over a ring is shown by using the property of the elliptic curves over a finite field and the Chinese Remainder Theorem.

Lemma 11. Let E_n^1 and E_n^2 be elliptic curves such that

$$E_n^1 : \ y^2 = x^3 + a_1 x + b_1 \bmod n, \quad E_n^2 : \ y^2 = x^3 + a_2 x + b_2 \bmod n.$$

Let $M_1 = (m_{1x}, m_{1y})$, $C_1 = (c_{1x}, c_{1y}) \in E_n^1$ and $M_2 = (m_{2x}, m_{2y})$, $C_2 = (c_{2x}, c_{2y}) \in E_n^2$ where

$$C_1 = e \cdot M_1 \text{ over } E_n^1, \quad C_2 = e \cdot M_2 \text{ over } E_n^2.$$

Then the following statements are equivalent:

(i) E_n^1 and E_n^2 are isomorphic.

(ii) $a_2 \equiv u^4 a_1 \bmod n, \quad b_2 \equiv u^6 b_1 \bmod n \quad \exists u \in \mathbf{Z}_n^*.$ (1)

(iii) $c_{2x} \equiv u^2 c_{1x} \bmod n, \quad c_{2y} \equiv u^3 c_{1y} \bmod n \quad \exists u \in \mathbf{Z}_n^*.$ (2)

(iv) $m_{2x} \equiv u^2 m_{1x} \bmod n, \quad m_{2y} \equiv u^3 m_{1y} \bmod n \quad \exists u \in \mathbf{Z}_n^*.$ (3)

If C_1, C_2 and M_1 satisfying congruence (2) are given, then M_2 can be easily found by computing congruence (3). Notice that it is easy to check whether or not congruence (2) holds. If M_1 and M_2 are randomly chosen, then the probability that there exists u satisfying congruence (2) is a negligibly small $1/n$ for large n. Thus, passive attacks using isomorphism seem to be difficult for Types 1 and 2 schemes.

Consider an active attack (a chosen-plaintext attack) based on the isomorphic property of the elliptic curves. Suppose an attacker A wants to make a victim B sign a plaintext $M = (m_x, m_y)$ without B's consent. A generates another message M' with B's public key n_B and random integer u:

$$M' = (u^2 m_x \bmod n_B, \ u^3 m_y \bmod n_B),$$

and sends M' to B. B makes a signature $S' = (s'_x, \ s'_y)$ for M' with his secret key d_B:

$$S' = d_B \cdot M' \text{ over } E'_{n_B}.$$

Then, A computes a signature $S = (s_x, \ s_y)$ for M from S' by

$$S = (s_x, \ s_y) = (u^{-2} s'_x \bmod n_B, \ u^{-3} s'_y \bmod n_B).$$

Note that the curve E_{n_B} containing points $(M, \ S)$ and the curve E'_{n_B} containing points (M', S') are isomorphic. Using this technique, A can forge B's signatures without B's secret key. To counter this attack, the same technique described in Section 7.4 can be applied.

An attacker may try to forge a signature by using both homomorphism and isomorphism shown above. However, such combined attacks can also be prevented by randomization with the hash function h.

7.6 Security for Low Multiplier Attack

Hastad [6] showed a low exponent attack on the original RSA and Rabin schemes when the same message is encrypted with several distinct moduli. He considered the problem of solving systems of congruences $P_i(m) \equiv 0 \pmod{n_i}$ $i = 1, ..., k$, where P_i are polynomial of degree e and the n_i are distinct relatively prime numbers and $m < \min n_i$. He proved that if $k > \frac{e(e+1)}{2}$, then m can be recovered in polynomial time. Thus, he pointed out that enciphering linearly related messages with the RSA scheme with low exponent or the Rabin scheme is insecure. For the original RSA scheme, let $c = m^e \bmod n$, $c_i = m_i^e \bmod n_i$, and $n = n_1 \cdot n_2 \cdots n_k$. In Hastad's algorithm, c is first obtained from c_i using the Chinese Remainder Theorem. Next, m can be efficiently calculated from $c = m^e$ (with neglecting n), provided that $m^e < n$. For our proposed Types 1 and 2 schemes, let $C = e' \cdot M$ over $E_n(a, b)$, $C_i = e' \cdot M$ over E_{n_i}, where $C = (c_x, c_y)$, $M = (m_x, m_y)$, $C_i = (c_{ix}, c_{iy})$. The value of (c_x, c_y) is also obtained from (c_{ix}, c_{iy}). However, it is difficult to solve (m_x, m_y) from (c_x, c_y) because c_x and c_y are expressed by *rational* equations in m_x and m_y. Since the rational equations include divisions modulo n, if we transform the rational form relation into the polynomial form relation, the size of the coefficient of the polynomial form is of the order of n's size. Therefore, it seems impossible to solve the rational or polynomial form relation by neglecting modulus n. Thus, even if the multiplier e' is small, a Hastad-like attack does not seem to work against the elliptic curve cryptosystems.

8 Performance

An elliptic curve addition $P_1 + P_2$ on $E_n(a, b)$ requires one division, one squaring operation and one general multiplication in $\mathbf{Z}_n$ when $P_1 \neq P_2$, and an extra squaring when $P_1 = P_2$. (The much faster additions and subtractions in $\mathbf{Z}_n$ are neglected for the sake of simplicity). Surprisingly, as opposed to $\mathbf{Z}_n$ where squaring can be performed faster than a general multiplication, doubling a point on an elliptic curve is computationally more costly than adding two different points. This means that in order to compute a multiple $c \cdot P$ of a point P, an irregular addition chain for c avoiding doubling operations should be used. When neglecting the fact that squaring in $\mathbf{Z}_n$ can be implemented somewhat faster than a general multiplication, elliptic curve addition and doubling operations require about 2 and 3 multiplications in $\mathbf{Z}_n$ and one division in $\mathbf{Z}_n$, respectively.

Division in $\mathbf{Z}_n$ can be implemented by the generalized Euclidean algorithm for computing greatest common divisors. The most efficient algorithm for computing multiplicative inverses, however, is that invented by Massey [17], which is a generalization of Stein's algorithm [25]. However, a division in $\mathbf{Z}_n$ seems to be less efficient than a multiplication in $\mathbf{Z}_n$.

On the other hand, if we calculate the addition on $E_n(a, b)$ in homogeneous coordinates, we can avoid the division in $\mathbf{Z}_n$ (except the final stage of the addition chain), although we must perform more multiplications.

Let $P_1 = (x_1, y_1, z_1) \in E_p(a, b)$, $P_2 = (x_2, y_2, z_2) \in E_p(a, b)$, and suppose that $P_1, P_2 \neq \mathcal{O}$, $P_1 \neq P_2$ and $P_1 \neq -P_2$. The addition formula [9] for $E_p(a, b)$ to find $P_3 = P_1 + P_2 =$

(x_3, y_3, z_3) is given by

$$\begin{cases} x_3 &= v\{z_2(u^2 z_1 - 2v^2 x_1) - v^3\} \mod p, \\ y_3 &= z_2(3uv^2 x_1 - v^3 y_1 - u^3 z_1) + uv^3 \mod p, \\ z_3 &= v^3 z_1 z_2 \mod p, \end{cases}$$

where $u = y_2 z_1 - y_1 z_2 \mod p$, $v = x_2 z_1 - x_1 z_2 \mod p$.
The doubling formula [9] for $E_p(a, b)$ to find $P_3 = 2 \cdot P_1$ is given by

$$\begin{cases} x_3 &= 2y_1 z_1 (w^2 - 8x_1 y_1^2 z_1) \mod p, \\ y_3 &= 4y_1^2 z_1 (3wx_1 - 2y_1^2 z_1) - w^3 \mod p, \\ z_3 &= 8y_1^3 z_1^3 \mod p, \end{cases}$$

where $w = 3x_1^2 + az_1^2 \mod p$.

One addition over $E_n(a, b)$ requires 12 multiplications in $\mathbf{Z}_n$, and one doubling over $E_n(a, b)$ requires 10 multiplications in $\mathbf{Z}_n$, if $a = 0$.

Therefore, in the affine coordinates, the computation required for our scheme (Scheme 1) is about $(2 + c)$ times as much as that for the RSA scheme, where c is the ratio of the computation amount of division in $\mathbf{Z}_n$ to that of multiplication in $\mathbf{Z}_n$. On the other hand, in the homogeneous coordinates, the computation required for encryption with our scheme is about 11 times as much as that for the RSA scheme. Since in our elliptic curve system a message consists of two elements of $\mathbf{Z}_n$ compared to only one in the RSA system, the computation speed of our scheme is about $2/(2 + c)$ or $1/6$ of the speed of RSA.

9 Conclusions

We have proposed new public key cryptosystems based on elliptic curves modulo n, where n is a product of two large primes. Furthermore, we have given some analysis of the security of these systems. For the proposed Type 1 scheme, the master key concept [10] and the blind signature concept [2] are similarly applicable (using the combined techniques of Sections 7.4 and 7.5).

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