Geometric Realization of Simplicial Complexes

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Abstract. We show that an abstract simplicial complex Δ may be realized on a grid of $\mathbb{R}^{d-1}$, where $d = \dim P(\Delta)$ is the order dimension (Dushnik-Miller dimension) of the face poset of Δ.

1 Introduction

Abstract simplicial complexes are related to order dimension in Section 2 through the complex of a d-representation. This construction is analogous to the one introduced by Dushnik in [10] and similar to the one used by Scarf in mathematical economy [14] under the name of *primitive sets* and which has been applied to integer programming by Barany, Howe and Scarf [1] and to commutative algebra by Bayer, Peeva and Sturmfels [2]. In Section 3, we give a simple necessary and sufficient condition for a mapping to be a geometric realization of an abstract simplicial complex. This characterization leads in Section 4 to the geometric realization of the abstract simplicial complex defined by a d-representation on a grid in $\mathbb{R}^{d-1}$. This generalizes the result of Schnyder on planar graphs [15] (see also [3][4]). In Section 5, we prove that any abstract simplicial complex may be triangulated into a "standard" d-representation having the same face poset dimension. In Section 6, we extend the vertex shelling order introduced earlier by Fraysseix, Pach and Pollack for planar triangulations [8] and show that a complex is vertex-shellable if and only if it is shellable (in the usual sense). We also prove that a vertex shelling order of the triangulation Δ^+ mentioned above may be easily derived from the "generating" total orders, generalizing a result proved in [5]. Eventually, Theorem 7 gathers most of the main results of the paper in a single statement.

We recall some basic definitions on simplicial complexes. For further information, see [12]. In the following, we will consider only finite simplicial complexes, what will justify the following definition. An *abstract simplicial complex* Δ is a non-empty finite collection of finite sets such that $X \in \Delta, Y \subseteq X$ implies $Y \in \Delta$. The union $V(\Delta)$ of the members of Δ is the *vertex set* of Δ. The members of Δ are the *faces* of Δ. The *dimension* of a face X of Δ is $\dim X = |X| - 1$. The *dimension* $\dim \Delta$ is the maximum dimension of any face of Δ. If Δ and Δ' are abstract simplicial complexes with disjoint ground sets, we recall that their *combinatorial join* is the abstract simplicial complex $\Delta * \Delta'$ defined by $\Delta * \Delta' = \{X \cup X', \quad X \in \Delta, X' \in \Delta'\}$. By extension, $\Delta * p$ will denote $\Delta * \{\emptyset, \{p\}\}$. An abstract simplicial complex Δ is *pure* if all the maximal faces of Δ (with respect to inclusion) have the same dimension, that is if any face of Δ is included in

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a face of Δ with dimension $\dim \Delta$. The *face poset* $P(\Delta)$ of Δ is the poset which consists of all the faces of Δ ordered by inclusion. For a poset P, a *realizer* of P is a set of total orders whose intersection is P. The minimum cardinality of a realizer of poset P is its *Dushnik-Miller dimension* (or simply the *dimension*) $\dim P$ [11].

2 The Complex of a *d*-Representation

Let $d > 0$ be an integer and let V be a finite set. A *d-representation* $R = (<_1, \dots, <_d)$ of V is a set of d total orders on V whose intersection is an antichain. With respect to R, the *supremum section* $S(X)$ of a subset $X \subseteq V$ is the subset of X whose elements are the maxima of X for some linear order in R:

$$S(X) = \{x \in X, \quad \exists 1 \le i \le d, \forall y \in X - \{x\}, x >_i y\} \tag{1}$$

Given a d-representation R of V, the *complex* of R is the set $\Sigma(R)$ of all the subsets X of V such that $\forall v \in V, v \in S(X \cup \{v\})$

Lemma 1. *The complex $\Sigma(R)$ of a d-representation R of V is an abstract simplicial complex.*

Proof. It is straightforward that $S(A) \supseteq S(B) \cap A$ whenever $A \subseteq B$. Thus, if X' is a subset of a set $X \in \Sigma(R)$, we get $v \in S(X' \cup \{v\})$ for any element $v \in V$ as $v \in S(X \cup \{v\})$. □

Theorem 1. *Let (V, Δ) be an abstract simplicial complex with vertex set V.*

Then, $\dim P(\Delta)$ is the smallest integer d, such that Δ is a subcomplex of some d-representation of V.

Proof.
- Δ is a subcomplex of a $\dim P(\Delta)$-representation of V:
  Consider a realizer $\prec_1, \dots, \prec_d$ of cardinality $d = \dim P(\Delta)$ of $P(\Delta)$ and let $R = (<_1, \dots, <_d)$ be the d-representation of V induced by the restrictions on V of the d total orders $\prec_1, \dots, \prec_d$.
  Let X be an element of Δ. Then, for all $1 \le i \le d$ and all $x \in X$ we have $x \prec_i X$, as X is, by definition, greater than its elements in the face poset. Moreover, X is not comparable to any element which does not belong to X. Hence, for any $y \notin X, \exists 1 \le i \le d, \forall x \in X, \quad x \prec_i y$. Hence, $\forall y \notin X, y \in S(X + y)$. Similarly, if x belongs to X, either $X = \{x\}$ and $x \in S(X)$ or $X - x$ is a simplex which belongs to Δ and hence $x \in S(X - x + x)$ and thus $x \in S(X)$. Altogether, X belongs to $\Sigma(R)$.
- If Δ is a subcomplex of a d-representation of V, then $d \ge \dim P(\Delta)$:
  Insert in the d total orders of the representation the faces of Δ (different from vertices) the following way: In the linear order $<_i$, insert just after the vertex x all the faces including x and vertices smaller than x (with respect to linear order $<_i$), sorted by increasing size and, for a same size, in lexicographic order (with respect to $<_i$). Then, the face-inclusion of X in Y

in Δ obviously correspond to $X <_i Y$ (for all $1 \leq i \leq d$). Otherwise, if X and Y are not compared in $P(\Delta)$, there exists $a \in X \setminus Y$ and $b \in Y \setminus X$. As Δ is a subcomplex of the d-representation, there exists a linear order i such that a is greater than all the vertices of Y with respect to $<_i$. Hence, $X >_i Y$. Similarly, there exists j such that $Y >_j X$. Thus X and Y are not comparable in the intersection of the d total orders. Altogether, $P(\Delta)$ is equal to the intersection of the d total orders and hence $\dim P(\Delta) \leq d$.

□

3 Geometric Realizations of Simplicial Complexes

Consider an injective mapping $f : V(\Delta) \to \mathbb{R}^n$ (this mapping is naturally extended to map the subsets of $V(\Delta)$ to the corresponding subsets of $\mathbb{R}^n$). We shall say that f is a *geometric realization* of Δ in $\mathbb{R}^n$ if $f(\Delta)$ is a *geometric simplicial complex*, i.e. if

– for any face X of Δ, $f(X)$ is a set of affinely independent points (i.e. defines a simplex) of $\mathbb{R}^n$,
– the intersection of two faces of $f(\Delta)$ is a face of $f(\Delta)$, that is, for any faces X and Y of Δ:

$$\mathrm{Conv}(f(X)) \cap \mathrm{Conv}(f(Y)) = \mathrm{Conv}(f(X) \cap f(Y)) \tag{2}$$

where $\mathrm{Conv}(P)$ denotes the convex hull of the point set P.

If $V(\Delta)$ is a set of points in $\mathbb{R}^n$, Δ is thus a geometric simplicial complex if and only if the identity is a geometric realization of Δ. We shall say that an abstract simplicial complex Δ is *realizable* in $\mathbb{R}^n$ if there exists a geometric realization of Δ in $\mathbb{R}^n$. We remark that the first of the two conditions we gave for f to be a geometric realization implies $n \geq \dim \Delta$.

Lemma 2. *If* $\mathrm{Conv}(f(X)) \cap \mathrm{Conv}(f(Y)) = \mathrm{Conv}(f(X) \cap f(Y))$ *holds for any two faces X and Y of an abstract simplicial complex Δ, then f is a geometric realization of Δ.*

Proof. The injectivity of f is straightforward as $\mathrm{Conv}(f(\{x\})) \cap \mathrm{Conv}(f(\{y\})) = \emptyset$ whenever $x \neq y$. Thus, we only have to prove that the image of a set $X \in \Delta$ is a set of affinely independent points. Let us prove it *ad absurdum*. Assume $x_1, \ldots, x_k$ are elements of X having linearly dependent images by f. Up to a relabeling of the x_i, we may assume that there exists an integer $1 \leq a < k$, and real numbers $\lambda_1, \ldots, \lambda_{k-1}$, such that $f(x_k) = \sum_{i=1}^{k} \lambda_i f(x_i)$, $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_1, \ldots, \lambda_a$ are negative and $\lambda_{a+1}, \ldots \lambda_{k-1}$ are positive.

Then, define $\alpha = \sum_{i=a+1}^{k} \lambda_i$. As the λ_i sum up to 1, all the λ_i are not negative and $\alpha > 0$. Hence, we have:

$$\frac{1}{\alpha} f(x_k) + \sum_{i=1}^{a} \left(-\frac{\lambda_i}{\alpha} \right) f(x_i) = \sum_{i=a+1}^{k-1} \frac{\lambda_i}{\alpha} f(x_i) \tag{3}$$

Thus, $\mathrm{Conv}(f(\{x_1,\ldots,x_a,x_k\}))\cap\mathrm{Conv}(f(\{x_{a+1},\ldots,x_{k-1}\}))$ is not empty although the faces $\{x_1,\ldots,x_a,x_k\}$ and $\{x_{a+1},\ldots,x_{k-1}\}$ of Δ (sub-faces of X) are disjoint; we are led to a contradiction. □

Lemma 3. *If* $\mathrm{Conv}(f(X))\cap\mathrm{Conv}(f(Y))$ *is empty for any two disjoint faces* X *and* Y *of an abstract simplicial complex* Δ*, then* $\mathrm{Conv}(f(X))\cap\mathrm{Conv}(f(Y)) = \mathrm{Conv}(f(X)\cap f(Y))$ *holds for any two faces* X *and* Y *of* Δ*.*

Proof. Let X and Y be any two non-disjoint faces of Δ and let π be a point in $\mathrm{Conv}(f(X))\cap\mathrm{Conv}(f(Y))$. The point π may be expressed as a weighted average of the points in $f(X)$: $\pi = \sum_{x\in X}\alpha(x)f(x)$, where α is a mapping from $V(\Delta)$ to $\mathbb{R}^+$ with sum 1 ans support $\mathrm{Supp}(\alpha)\subseteq X$. Similarly, $\pi = \sum_{y\in Y}\beta(y)f(y)$, where β is a function from $V(\Delta)$ to $\mathbb{R}^+$ with sum 1 and support $\mathrm{Supp}(\beta)\subseteq Y$. Let $\lambda(z)=\min(\alpha(z),\beta(z))$. This function is positive and has support $\mathrm{Supp}(\lambda)\subseteq X\cap Y$. Let s be its sum: $s=\sum_{z\in X\cap Y}\lambda(z)$. Let us show that s cannot be different from 1: otherwise,

$$\sum_{x\in X}\left(\frac{\alpha(x)-\lambda(x)}{1-s}\right)f(x) = \sum_{y\in Y}\left(\frac{\beta(y)-\lambda(y)}{1-s}\right)f(y) \tag{4}$$

and, if X' denotes the support of $\alpha-\lambda$ and Y' denotes the support of $\beta-\lambda$, $\mathrm{Conv}(f(X'))\cap\mathrm{Conv}(f(Y'))$ is not empty although the faces $X'\subseteq X$ and $Y'\subseteq Y$ are disjoint. Hence, $s=1$, what may only be achieved by $\alpha=\beta$. Thus, π belongs to $\mathrm{Conv}(f(X)\cap f(Y))$. □

Theorem 2 (Folklore). *Let* Δ *be an abstract simplicial complex and let* $f : V(\Delta)\to\mathbb{R}^n$ *be a mapping. Then,* f *is a geometric realization of* Δ *in* $\mathbb{R}^n$ *if and only if* $\mathrm{Conv}(f(X))$ *and* $\mathrm{Conv}(f(Y))$ *are disjoint, for any two disjoint faces* X, Y *of* Δ*.*

Proof. If f is a geometric realization then disjoint sets in Δ are maped into disjoint simplices. Conversely, if disjoint faces are maped into set having disjoint convex hulls, then the intersection of the convex hulls of the images of any two faces X and Y of Δ is the convex hull of the image of their intersection, according to Lemma 3. Then, according to Lemma 2, f is a geometric realization of Δ in $\mathbb{R}^n$. □

4 Geometric Realization of a *d*-Representation

In the following, we consider a d-representation $R=(<_1,\ldots,<_d)$ of a set V and a mapping f from V to $\mathbb{R}^d$, such that f_i is strictly positive and strictly increasing with respect to $<_i$ (for $1\le i\le d$). We denote by P the image of V and $\Delta(P)$ the image of $\Sigma(R)$. As f is clearly injective, $\Delta(P)$ and $\Sigma(R)$ are isomorphic.

Let $X \subseteq P$ be a subset of points, $\sigma(X)$ denotes the point with coordinates $\sigma_i(X) = \max_{\pi \in X} \pi_i$ and $\Theta(X)$ denotes the closed set $\Theta(X) = \{\pi, \quad \forall 1 \leq i \leq n, \pi_i \leq \sigma_i(X)\}$. According to these definitions and according to the definition of $\Sigma(R)$, a point set $X \subseteq P$ is a face of $\Delta(P)$ if and only if

– any point in X belongs to the frontier of $\Theta(X)$,
– no point of P belongs to the interior of $\Theta(X)$.

Theorem 3. *Let $R = (<_1, \ldots, <_d)$ be a d-representation of a set V. Let f be any mapping from V to $\mathbb{R}^d$, such that f_i is strictly positive and increasing with respect to $<_i$ sufficiently fast (that is: the ratio of consecutive values of f_i is bigger than $A \geq 1 + \sqrt{d}$), let $(\lambda_1, \ldots, \lambda_d)$ be d-uple of positive real numbers different from $(0, \ldots, 0)$ and let H be the hyperplane of $\mathbb{R}^d$ defined by $\sum_i \lambda_i x_i = 1$.*

Then, the mapping $\phi : V \to H$ defined by $\phi_i(x) = \frac{f_i(x)}{\sum_j \lambda_j f_j(x)}$ is a geometric realization of $\Sigma(R)$ in $H \approx \mathbb{R}^{d-1}$.

Proof. We shall first prove that $\Delta(P) * O$ is a geometric simplicial complex, that is that $\mathrm{Conv}(X) \cap \mathrm{Conv}(Y)$ is empty for any two disjoint sets of $\Delta(P) * O$ (according to Theorem 2). As, for any face $X \in \Delta(P) * O$, $X \cup \{O\}$ is also a face of $\Delta(P) * O$, it is necessary and sufficient to prove that $\mathrm{Conv}(X \cup \{O\}) \cap \mathrm{Conv}(Y)$ is empty for any two faces X, Y of $\Delta(P)$ or, equivalently, that there exists an hyperplane H passing through O and which separates X from Y. As no point of X belongs to $\Theta(Y)$, each point $x \in X$ has a coordinate bigger than the corresponding one of $\sigma(Y)$. Hence, the set $I = \{i, \exists x \in X, x_i > \sigma_i(Y)\}$ is not empty. Similarly, the set $J = \{j, \exists y \in Y, y_i > \sigma_i(X)\}$ is also not empty. Let $a(x) = \sum_{i \in I} \frac{x_i}{\sigma_i(Y)}$ and $b(x) = \sum_{j \in J} \frac{x_j}{\sigma_j(X)}$. We have: $a(x) \geq A$ for any $x \in X$ as there exists an index i, for which $x_i > 1$ and hence $x_i \geq A$; To the opposite, $a(y) < 1 + \frac{d-1}{A}$ for $y \in Y$, as only one coordinate of y may reach the maximum value. Similarly, $b(x) < 1 + \frac{d-1}{A}$ for any $x \in X$ and $b(y) \geq A$ for any $y \in Y$. As $A \geq 1 + \sqrt{d}$, we have $A > 1 + \frac{d-1}{A}$. Thus, the hyperplane defined by $a(x) = b(x)$ passes through O and separates X (for which $a(x) > b(x)$ from Y (for which $a(x) < b(x)$) and $\Delta(P) * O$ will be a geometric simplicial complex.

Now, consider the hyperplane H' defined by $\sum_i \lambda_i x_i = \epsilon$, where ϵ is a sufficiently small positive real number, so that H' separates O from P. Then, the intersection of $\Delta(P) * O$ and H' is a geometric realization of $\Delta(P) \approx \Sigma(R)$ and so is its homothetic image defined by the image of $\Sigma(R)$ by the mapping ϕ. □

5 Triangulation

Lemma 4. *Let $R = (<_1, \ldots, <_d)$ be a d-representation of V with complex $\Sigma(R)$ and let x be the maximum of $<_k$. For $i \neq k$, let $<'_i$ be the total order on V where x precedes all the elements of $V \setminus \{x\}$ and the element of $V \setminus \{x\}$ are ordered by $<_i$. Then $R' = (<'_1, \ldots, <'_{k-1}, <_k, <'_{k+1}, \ldots, <_d)$ is a d-representation which complex $\Sigma(R')$ includes $\Sigma(R)$.*

Proof. Denote by S' the supremum section corresponding to R'. Let X be a subset of V, then $S(X) \subseteq S'(X)$:

- If $x \notin X$, $\quad S'(X) = S(X)$,
- If $x \in X$, $\quad S'(X) = \{x\} \cup S(X \setminus \{x\}) \supseteq S(X)$

Thus, any element X of $\Sigma(R)$ belongs to $\Sigma(R')$: for all $v \in V, v \in S(X \cup \{v\}) \subseteq S(X \cup \{v\}$ and hence $X \in \Sigma(R')$. □

A d-representation $R = (<_1, \ldots, <_d)$ on a set V is *standard* if $|V| \geq d$ and, for all $i \neq j$, the maximum element of $<_i$ is one of the $d-1$ smallest elements of $<_j$. The maxima of the orders of a standard d-representation are the *exterior* elements of this representation. The other elements of V are the *interior* elements. An abstract simplicial complex is *standard* if it is the complex of some $\dim P(\Delta)$-representation. Notice that, for a standard abstract simplicial complex Δ, we have: $\dim \Delta = \dim P(\Delta) - 1$.

Given any subset X of V and a d-representation R of V, we define the *shade function* I_X of X on $V \setminus X$ as $I_X(u) = \{i \in [1, d], \forall x \in X, u >_i x\}$. Given any subset X of V and a d-representation R of V, we define the *shading order* $<_X$ of X on $V \setminus X$ as follows: $u <_X v \iff (I_X(u) \subseteq I_X(v))$ and $(\forall i \in I_X(u), u <_i v)$

Lemma 5. *Any standard representation will be pure. More precisely, let* $R = (<_1, \ldots, <_d)$ *be a standard representation. If* X *is a face of* $\Sigma(R)$ *which maximal element* x_1 *(with respect to* $<_1$*) is not an exterior element of* R*, then, for every* $\dim X < k < d$ *there exists a* k*-dimensional face which includes* X *and has* x_1 *as a maximum element with respect to* $<_1$*.*

Proof. Let X be an element of $\Sigma(R)$ different from the $|X|$ first elements of $<_1$ and assume $|X| < d$.

If X is included into the set $\{v_1, \ldots, v_k\}$ of the external elements of R, the addition of any external element smaller or equal to x_1 with respect to $<_1$ will do.

Otherwise, let x be an element of x which is maximal in X with respect to two different total orders and let $<_k$ be one of them for which $k \neq 1$. Obviously, x is not an external element of R as an external element is greater than an internal with respect to exactly one total order. For the same reason, v_k does not belong to X and is smaller or equal to x_1 with respect to $<_1$. Let α be a minimal element of the set $\{v \notin X, v <_1 x_1 \text{ and } v <_X v_k\} \cup \{v_k\}$. As $I_X(\alpha) \neq \emptyset$ (otherwise, $\alpha \notin S(X+\alpha)$ would contradict $X \in \Sigma(R)$) and as $I_X(\alpha) \subseteq I_X(v_k)$ by construction, we get: $I_X(\alpha) = \{k\}$. This ensures that $S(X+\alpha) = X+\alpha$. Assume there exists an element $y \notin X + \alpha$, such that $y \notin S(X + \alpha + y)$. As $X \in \Sigma(R)$, the element y belongs to $S(X + y)$. As $I_X(\alpha) = \{k\}$, the only possibility for y not to belong to $S(X + \alpha + y)$ corresponds to the situation where $I_X(y) = \{k\}$ and $y <_k \alpha$, that is: $y <_X \alpha$. Moreover, y is smaller or equal to x_1 with respect to $<_1$ (as $1 \notin I_X(y)$) and this contradicts the minimality of α. □

Theorem 4. *Any abstract simplicial* Δ *complex may be triangulated into a standard representation having the same face poset dimension.*

Proof. As a direct consequence of Theorem 1, Δ is the subcomplex of some d-representation R_0. By successive applications of Lemma 4, there exists a standard d-representation R, such that $\Sigma(R)$ includes $\Sigma(R_0)$ and hence includes Δ. According to Lemma 5, $\Sigma(R)$ is pure of dimension $d-1$ and hence is a triangulation of Δ. □

6 Shellability of Standard Complexes

Shellability of pure simplicial complexes has been extensively studied. We shall introduce here the *vertex shellability*, which is a generalization of the concept introduced in [8] and which has proved its efficiency through numerous applications in quite different kind of problems related to planarity (see, for instance, [6][7][9][13]). We shall prove that the concepts of shellability and vertex-shellability are actually equivalent. Thatfor, recall that a pure abstract simplicial complex Δ of dimension $d-1$ is *shellable* if all its $(d-1)$-faces (that is: all its elements of cardinality d) can be listed $F_1, \ldots, F_s$ in such a way that $\left(\bigcup_{i=1}^{j-1} \overline{F_i}\right) \cap \overline{F_j}$ is pure of dimension $d-2$ for every $1 < j \leq s$ (where $\overline{F_i}$ is defined by: $\overline{F_i} = \{X \in \Delta, X \subseteq F_i\}$) or, equivalently, for $1 \leq i < j \leq s$, any subset X of F_i and F_j, there exists a $(d-2)$-dimensional face $Y \supseteq X$ and a $(d-1)$-dimensional face F_h (for some $h < j$) such that Y is included in both F_h and F_j.

A pure abstract simplicial complex Δ of dimension $d-1$ is said to be *vertex shellable* if all its vertices can be listed $v_1, \ldots, v_n$ in such a way that:

- $\{v_1, \ldots, v_d\} \in \Delta$,
- for all face $\sigma \in \Delta$ with maximum vertex v_k $(k > d)$, there exists $j < k$, such that $v_j \notin \sigma$ and $\sigma - v_k + v_j \in \Delta)$,
- for all vertex v_k $(k \geq d)$, the abstract simplicial complex $\Delta_k = \{\sigma - v_k, \sigma \in \Delta \text{ and } \sigma \subseteq \{v_1, \ldots, v_k\}\}$ is a pure $d-2$ dimensional shellable simplicial complex.

Lemma 6. *Let Δ be a pure $d-1$ dimensional shellable simplicial complex and let $F_1, \ldots, F_s$ be a shelling order.*

Assume there exists $d < a < s$, such that F_a is not included in $\bigcup_{i<a} F_i$ but F_{a+1} is. Then, $F_1, \ldots, F_{a-1}, F_{a+1}, F_a, F_{a+2}, \ldots, F_s$ is also a shelling order.

Proof. As $\left(\bigcup_{i<a} \overline{F_i}\right) \cap \overline{F_a}$ is a pure $d-2$ dimensional simplicial complex and as there exists $\alpha \in F_a \backslash \bigcup_{i<a} F_i$, we get that there exists $i < a$, such that $F_a - \alpha \subset F_i$. As $\left(\bigcup_{i<a} F_i\right) \cup F_{a+1}$ does not include α, we get that $\left(\bigcup_{i<a} \overline{F_i} \cup \overline{F_{a+1}}\right) \cap \overline{F_a}$ is a pure $d-2$ simplicial complex.

Moreover, let X be a face of both F_{a+1} and F_j $(j < a)$, with dimension strictly less than $d-2$. Then, as $F_1, \ldots, F_s$ is a shelling order, there exists a $(d-2)$-dimensional face $Y \supseteq X$ which is a face of F_{a+1} and F_i (with $i < a+1$). If $i = a$ then, as $Y \subset F_{a+1}$, the vertex α does not belong to Y and $Y = F_a - \alpha$. Hence, there exists $j < a$ such that $Y \subset F_j$. Altogether, $\left(\bigcup_{i<a} \overline{F_i}\right) \cap \overline{F_{a+1}}$ is a pure $(d-2)$-dimensional complex. □

Lemma 7. *Let Δ be a pure $(d-1)$-dimensional abstract simplicial complex. If Δ is shellable, then it is vertex shellable.*

Proof. Assume Δ is shellable. According to Lemma 6, there exists a shelling order $F_1, \ldots, F_s$ such that, for all $d < a < s$, if F_a is not included in $\bigcup_{i<a} F_i$, no F_b (with $b > a$) is included in $\bigcup_{i<a} F_i$. Hence, as $F_i \setminus \left(\bigcup_{j<i} F_j\right)$ includes at most one element, we can list the vertices of Δ the following way:

– begin with a list including the vertices of F_1 and let $a = 1$.
– while $a < s$, add to the list the vertex (if any) of F_{a+1} which is not in the list and let $a \leftarrow a + 1$.

Eventually, we get a list $v_1, \ldots, v_n$ of all the vertices of Δ, such that if $i < j$, all the $(d-1)$-faces of Δ with maximum element v_i precedes all the $(d-1)$-faces including v_j in the shelling order. Thus, we get:

– $\{v_1, \ldots, v_d\} \in \Delta$.
– for all face $\sigma \in \Delta$ with maximum vertex v_k $(k > d)$, there exists a face σ' that precedes σ in the shelling order, such that $\sigma - v_k \in \sigma'$. As σ' precedes σ and does not include v_k, its maximal element is a vertex v_i with $i < k$. Hence, the vertex v_j such that $\sigma' = \sigma - v_k + v_j$ is such that $j < k$.
– for all vertex v_k $(k \geq d)$, let Δ_k be the abstract simplicial complex $\Delta_k = \{\sigma - v_k, \quad \sigma \in \Delta \text{ and } \sigma \subseteq \{v_1, \ldots, v_k\}\}$. Let F_a and F_b be the first and last $(d-1)$-dimensional face of the shelling order having v_k as maximal element. We shall prove that Δ_k is a pure $(d-2)$-dimensional abstract simplicial complex having $F_a - v_k, \ldots, F_b - v_k$ as a shelling order: Consider in Δ_k a face X of $F_i - v_k$ and $F_j - v_k$ with $a \leq i < j \leq b$. Then, $X + v_k$ is a face of F_i and F_j in Δ. Thus there exists a $(d-2)$-dimensional face $Y \supseteq X + v_k$ and $h < j$, such that $Y \subseteq F_h$. As v_k belongs to F_h, we get $h \geq a$. Hence, there exists a face $F_h - v_k$ of Δ_k having a $(d-3)$-dimensional face $Y - v_k$ including X and included in F_j, what ends our proof.

□

Theorem 5. *Let Δ be a pure $(d-1)$-dimensional abstract simplicial complex. Then, Δ is shellable if and only if it is vertex shellable.*

Proof. According to Lemma 7, we only have to prove that the vertex shellability of Δ will imply its shellability.

Let $v_1, \ldots, v_n$ be a vertex shelling order. We list the $(d-1)$-faces of Δ the following way:

– let $F_1 = \{v_1, \ldots, v_d\}$ and let $a = d$.
– while $a < s$ we add to the list the faces of Δ having v_k as a maximal element in the order induced by the shelling order of Δ_k.

Let $1 \leq i < j \leq s$ and let X be a subset of F_i and F_j and let v_k be the maximal element of F_j.

– If $v_k \notin X$, then there exists a $(d-1)$-face F_h including $F_j - v_k$ with $h < j$, according to the definition of a vertex shelling order. Obviously, $X \subseteq F_j - v_k$.
– If $v_k \in X$, then there exists a $(d-3)$-face $Y \supseteq X - v_k$ of Δ_k and a face F_h having v_k as a maximal element, such that $h < j$ and $F_h - v_k \supseteq Y$, according to the shelling order of Δ_k. Hence, $Y + v_k$ is a $(d-2)$-face included in both F_h and F_j (with $h < j$).

□

Let $R = (<_1, \ldots, <_d)$ be a d-representation. Let X and Y be two $(d-1)$-dimensional simplices of $\Sigma(R)$ and let x_i (resp. y_i) be the maximal element of X (resp. Y) with respect to $<_i$ (notice that $X = \{x_1, \ldots, x_d\}$ and $Y = \{y_1, \ldots, y_d\}$), the *d-order* is the total order on the $(d-1)$-dimensional simplexes of $\Sigma(R)$ which is defined by:

$$\forall X \neq Y \in \Sigma(R), \quad (X < Y \iff x_k <_k y_k, \text{ where } k = \min\{i, x_i \neq y_i\}) \quad (5)$$

Theorem 6. *Any standard abstract simplicial complex will be shellable.*

More precisely, let $R = (<_1, \ldots, <_d)$ be a standard representation. Then, the d-order is a shelling order of $\Sigma(R)$ and any of the $<_i$ is a vertex-shelling order.

Proof. We shall only prove that the d-order is a shelling order, as this obviously implies that $<_1$ is a vertex shelling order (according to the definitions of a vertex shelling order and the fact that the d-order is a lexicographic order starting with $<_1$) and hence that $<_i$ is a vertex shelling order (by symmetry).

We shall prove that the d-order is a shelling order by induction over the dimension d of the simplicial complex.

Let F_i be the ith face ($i > 1$) and let x be its maximal element with respect to $<_1$. As F_1 is the only face including the $d-1$ exterior elements $v_2, \ldots, v_k$, there exists, according to Lemma 5 a maximal face F_j including $F_i - x$ and which maximal element (with respect to $<_1$) is strictly smaller than x and hence $j < i$. Let F_a ($a \leq i$) be the first simplicial complex containing x, $\left(\bigcup_{j=1}^{a-1} \overline{F_j}\right) \cap \overline{F_i} = \overline{F_i - x}$ and, according to the induction on the dimension, if $a < i$, $\left(\bigcup_{j=a}^{i-1} \overline{F_j - x}\right) \cap \overline{F_i - x}$ is a pure simplicial complex of dimension $d-3$ and hence, $\left(\bigcup_{j=a}^{i-1} \overline{F_j}\right) \cap \overline{F_i}$ is a pure simplicial complex of dimension $d-2$. Altogether, $\left(\bigcup_{j=1}^{i-1} \overline{F_j}\right) \cap \overline{F_i}$ is a pure simplicial complex of dimension $d-2$. □

Theorem 7. *Let Δ be an abstract simplicial complex and $d = \dim P(\Delta)$. Then, there exists a standard d-representation $R = (<_1, \ldots, <_d)$ $V(\Delta)$ defining a triangulation Δ^+ of Δ, which is shellable and realizable in $\mathbb{R}^{d-1}$.*

Proof. According to Theorem 4, Δ may be triangulated into $\Delta^+ = \Sigma(R)$, where R is a standard d-representation $V(\Delta)$. Δ^+ is shellable, according to Theorem 6 and its geometric realization follows from Theorem 3, using any sufficiently fast increasing functions. □

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