On Characterization of Discrete Triangles by Discrete Moments

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Abstract. For a given real triangle T its discretization on a discrete point set $\mathcal{S}$ consists of points from $\mathcal{S}$ which fall into T. If the number of such points is finite, the obtained discretization of T will be called *discrete triangle.*

In this paper we show that the discrete moments having the order up to 3 characterize uniquely the corresponding discrete triangle if the discretizationing set $\mathcal{S}$ is fixed.

Of a particular interest is the case when $\mathcal{S}$ is the integer grid, i.e., $\mathcal{S} = \mathbf{Z}^2$. Then the discretization of a triangle T is called *digital triangle.* It turns out that the proposed characterization preserves a coding of digital triangles from an integer grid of a given size, say $m \times m$ within an $\mathcal{O}(\log m)$ amount of memory space per coded digital triangle. That is the theoretical minimum.

Keywords. Digital triangle, digital shape, coding, moments.

1 Introduction

The basic motivation for this paper was recovering a simple and efficient characterization of digital triangles. By digital triangles we mean digital (binary) pictures of real triangles, or more formally, a *digital triangle* $D(T)$ is the set consisting of integer points which fall into a real triangle T:

$$D(T) \;=\; \{(i,j) \mid (i,j) \in T, \quad i,j \text{ are integers}\} \;=\; \{(i,j) \mid (i,j) \in T \cap \mathbf{Z}^2\}\ .$$

That is the most usual digitization scheme for planar regions.

But, sometimes the digitization (i.e., discretization) is made by using another "discretizationing" set than it is $\mathbf{Z}^2$. Some other examples of discrete presentation of real objects are: Discrete images on the hexagonal grid, radar images, images made on statistically distributed set of points, e.t.c. Because the method presented here can be applied to discretizations on different sets we start with

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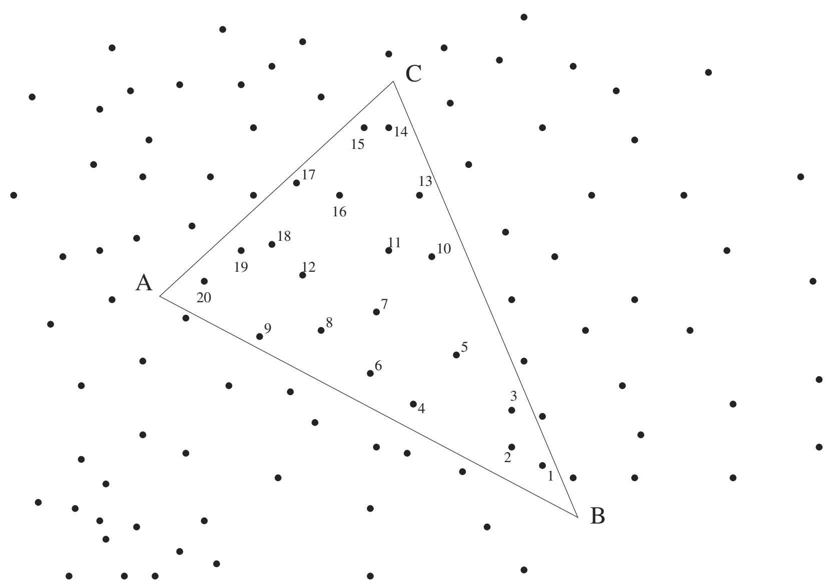

Fig. 1. A discretization of the triangle ABC consisting of 20 points is shown above.

a general definition of a *discrete triangle* $\mathbf{D}(T)$ from a fixed discrete point set $\mathcal{S}$ (see Fig. 1). Thus, we define

$$\mathbf{D}(T) \;=\; \{(x, y) \mid (x, y) \in T \cap \mathcal{S}\} \; .$$

Through the paper, it will be assumed but not mentioned, all discrete triangles consist of a finite number of points. For an illustration, the discretizations on the set consisting of all points with the coordinates which are rational numbers (i.e., $\mathcal{S} = \mathbf{Q}^2$) are not considered. It is not convenient to use a real triangle to represent its discretization. In that case any discrete triangle can be represented by infinitely many real triangles, since there is a continuum of real triangles which have the same discretization on a given set of points. Depending on $\mathcal{S}$, it can be difficult to answer which different real triangles have different discretizations. By the way, if we have a binary picture of a triangular objects, the "original triangle" is usually unknown. Consequently, the characterization of discrete triangles by a real triangle with a given discretization requires a procedure for reconstruction of an original triangle from its discretization.

In the next section we give a characterization of discrete triangles which is simple and fast for any choice of the discretizationing set $\mathcal{S}$. For such a characterization we will use, so called, discrete moments. Precisely, the *discrete moment* $\mu_{p,q}(X)$ of a finite set X is defined as:

$$\mu_{p,q}(X) \;=\; \sum_{(x,y)\in X} x^p \cdot y^q \; .$$

The moment $\mu_{p,q}(X)$ has the order $p + q$.

We prove that ten discrete moments having the order up to 3 are enough for a unique characterization of discrete triangles discretized on a fixed discrete set. In Section 3 we give some performance analysis of the proposed characterization if the discretization is made on a squared integer grid of a given size, i.e., on $\mathbf{Z}^2 \cap [0, m-1]^2$, for some integer m. $\mathbf{Z}^2 \cap [0, m-1]^2$, will be called the $m \times m$ *integer grid*. It turns out again that the use of moments is a powerful tool in image analysis ([3], [8]). Section 4 contains concluding remarks.

Through the paper a *finite set* means that the set consists of a finite number of points. Also, a *unique characterization* and *coding* will have the same meaning.

We shall say that a function $f(x, y)$ *separates* sets S_1 and S_2 if $f(x, y)$ has the different sign in the points of S_1 and S_2. For example, $(x, y) \in S_1$ implies $f(x, y) > 0$, while $(x, y) \in S_2$ implies $f(x, y) < 0$.

2 Characterization of Discrete Triangles

In this section it will be shown that the discrete moments having order up to 3 match uniquely the discrete triangles presented on a fixed set $\mathcal{S}$. We start with the following theorem.

Theorem 1. *Let S_1 and S_2 be two finite planar sets. If there exists a function of the form*

$$f(x, y) = \sum_{p+q \le k} \alpha_{p,q} \cdot x^p \cdot y^q, \qquad (1)$$

where $p, q \in \{0, 1, \ldots, k\}$ and $\alpha_{p,q}$ are arbitrary real numbers, such that $f(x, y)$ separates $S_1 \setminus S_2$ and $S_2 \setminus S_1$ then

$$\mu_{p,q}(S_1) = \mu_{p,q}(S_2) \quad \textit{for all non negative integers} \quad p,\ q, \quad \textit{with} \quad p + q \le k,$$

is equivalent to $\qquad S_1 = S_2$.

Proof. If $S_1 = S_2$ then the corresponding discrete moments are equal obviously. What we have to prove is: The equalities of the corresponded moments of the order up to k preserve $S_1 = S_2$. We prove that by a contradiction. Let

$$\sum_{(x,y) \in S_1} x^p \cdot y^q = \mu_{p,q}(S_1) = \mu_{p,q}(S_2) = \sum_{(x,y) \in S_2} x^p \cdot y^q$$

holds for all non negative integers p and q satisfying $p + q \le k$, and for some different finite sets S_1 and S_2. Since $S_1 \neq S_2$ we can assume $S_1 \setminus S_2$ is non empty, else we can start with the non empty $S_2 \setminus S_1$. Further, because there exists a function $f(x, y)$ of the form (1)

$$f(x, y) = \sum_{p+q \le k} \alpha_{p,q} \cdot x^p \cdot y^q,$$

which separates $S_1 \setminus S_2$ and $S_2 \setminus S_1$. Let be $f(x,y) > 0$ for $(x,y) \in S_1 \setminus S_2$, while $(x,y) \in S_2 \setminus S_1$ implies $f(x,y) < 0$. Then

$$
\begin{aligned}
0 < \sum_{(x,y)\in S_1\setminus S_2} f(x,y) &= \sum_{(x,y)\in S_1\setminus S_2} \left(\sum_{p+q\le k} \alpha_{p,q}\cdot x^p \cdot y^q \right) \\
&= \sum_{p+q\le k} \left(\sum_{(x,y)\in S_1\setminus S_2} \alpha_{p,q}\cdot x^p \cdot y^q \right) + \sum_{p+q\le k} \left(\sum_{(x,y)\in S_1\cap S_2} \alpha_{p,q}\cdot x^p \cdot y^q \right) \\
&- \sum_{p+q\le k} \left(\sum_{(x,y)\in S_1\cap S_2} \alpha_{p,q}\cdot x^p \cdot y^q \right) \\
&= \sum_{p+q\le k} \left(\alpha_{p,q}\cdot \sum_{(x,y)\in S_1} x^p \cdot y^q \right) - \sum_{p+q\le k} \left(\alpha_{p,q}\cdot \sum_{(x,y)\in S_1\cap S_2} x^p \cdot y^q \right) \\
&= \sum_{p+q\le k} \left(\alpha_{p,q}\cdot \sum_{(x,y)\in S_2} x^p \cdot y^q \right) - \sum_{p+q\le k} \left(\alpha_{p,q}\cdot \sum_{(x,y)\in S_1\cap S_2} x^p \cdot y^q \right) \\
&= \sum_{(x,y)\in S_2\setminus S_1} \left(\sum_{p+q\le k} \alpha_{p,q}\cdot x^p \cdot y^q \right) = \sum_{(x,y)\in S_2\setminus S_1} f(x,y) \le 0.
\end{aligned}
$$

The contradiction $0 < 0$ finishes the proof. []

Now, we can prove the main result of the paper.

Theorem 2. *Let a discrete point set* $\mathcal{S}$ *be given. Then, any discrete triangle* $\mathbf{D}(T)$ *from* $\mathcal{S}$ *is uniquely determined by ten discrete moments*

$$\mu_{p,q}(\mathbf{D}(T)), \qquad \textit{where} \quad p+q \le 3, \quad \textit{and} \quad p,\ q \quad \textit{are non negative integers.}$$

Proof. We will use the previous theorem specifying that $k = 3$ and S_1 and S_2 are discrete triangles $\mathbf{D}(T)$ and $\mathbf{D}(T_1)$.

If we consider T and T_1 as planar compact regions then the set intersection $T \cap T_1$ is either the empty set or the convex region whose boundary is the polygon having 6 vertices at most. In the case of $T \cap T_1 = \emptyset$ the proof is trivial because there always exists a straight line which separates T and T_1 and consequently the same line separates $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$. If the mentioned line is defined by $\alpha \cdot x + \beta \cdot y + \gamma = 0$ then (due to Theorem 1) the proof follows by setting $f(x,y) = \alpha \cdot x + \beta \cdot y + \gamma$.

If $T \cap T_1 \neq \emptyset$ then it is easily to conclude that all possible situations are characterized by Figures 2-9. Namely, if we consider the number of edges (i.e., the number of vertices) of $T \cap T_1$ and mutual order between edges which belong to the boundary of T and those which belong to the boundary of T_1 we distinguish the following cases:

i) if $T \cap T_1$ has 6 vertices, the situation corresponds to Figures 2 and 3;
ii) if $T \cap T_1$ has 5 vertices we recognize two possibilities corresponded to Figures 4 and 5;
iii) if $T \cap T_1$ has 4 vertices, we recognize three essentially different situations corresponded to Figures 6, 7, and 8;
iv) if $T \cap T_1$ has 3 vertices we have one nontrivial situation presented on Fig. 9.

At all Figures 2-9, the triangle T with vertices A, B, and C is fixed, while the triangle T_1 varies. Triangle T_1 is drown by dashed lines.

The statement follows due to Theorem 1, since the existence of a function of the form (1) which separates the set differences of any two different discrete triangles $\mathbf{D}(T)$ and $\mathbf{D}(T_1)$ is shown in all possible situations.

For a straight line l, let $\alpha_l \cdot x + \beta_l \cdot y - \gamma_l = 0$ be an equation which defines l. In the capture of any figure, a function $f(x,y)$ which separates $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ is described by suitable chosen lines and the product of their equations.

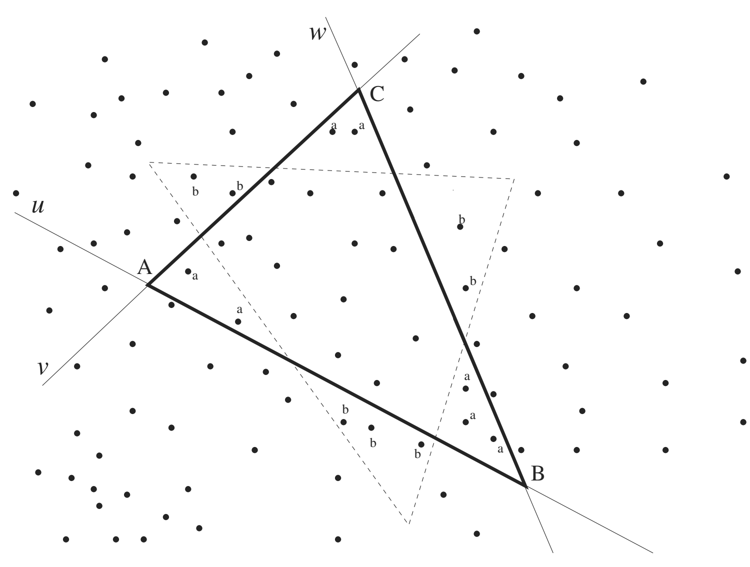

Fig. 2. $f(x,y) = (\alpha_u \cdot x + \beta_u \cdot y - \gamma_u) \cdot (\alpha_v \cdot x + \beta_v \cdot y - \gamma_v) \cdot (\alpha_w \cdot x + \beta_w \cdot y - \gamma_w)$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b).

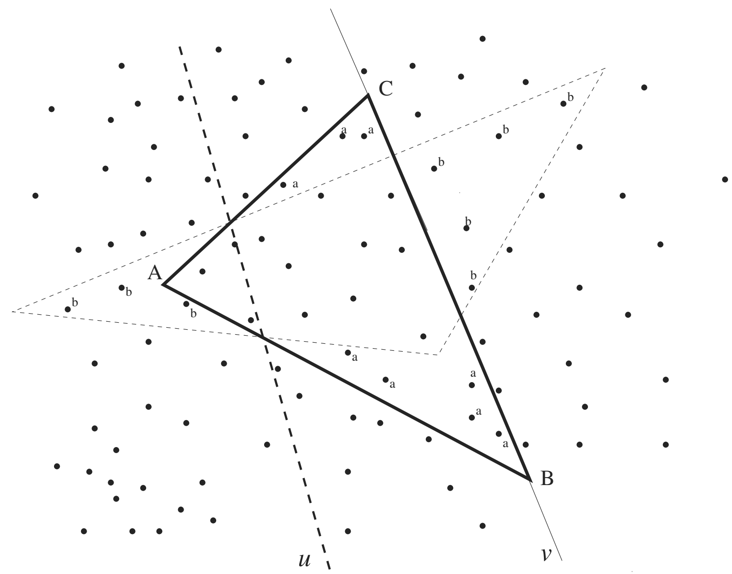

Fig. 3. $f(x, y) = (\alpha_u \cdot x + \beta_u \cdot y - \gamma_u) \cdot (\alpha_v \cdot x + \beta_v \cdot y - \gamma_v)$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b).

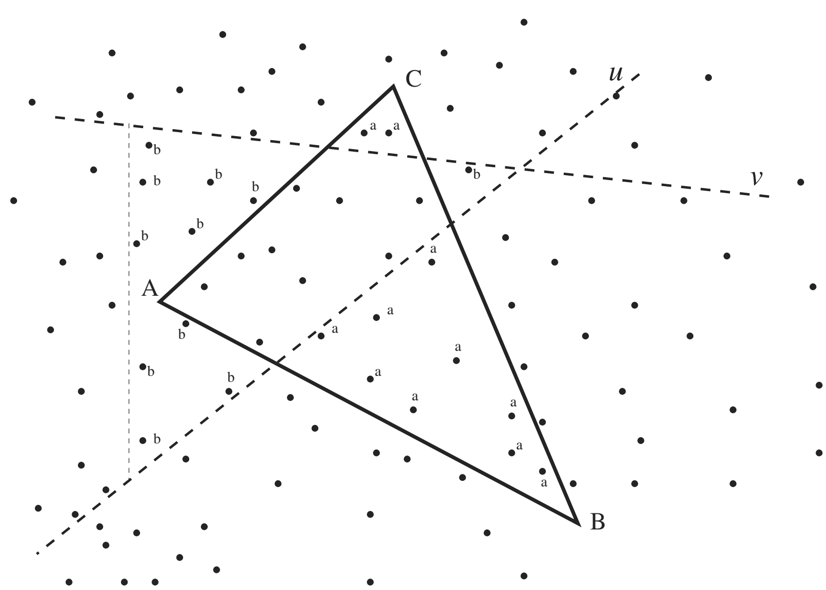

Fig. 4. $f(x, y) = (\alpha_u \cdot x + \beta_u \cdot y - \gamma_u) \cdot (\alpha_v \cdot x + \beta_v \cdot y - \gamma_v)$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b).

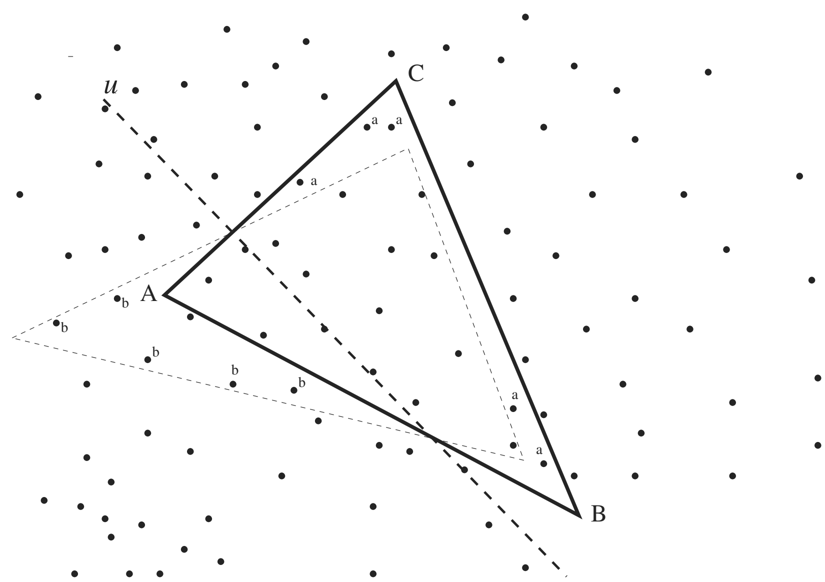

Fig. 5. $f(x,y) = \alpha_u \cdot x + \beta_u \cdot y - \gamma_u$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b) .

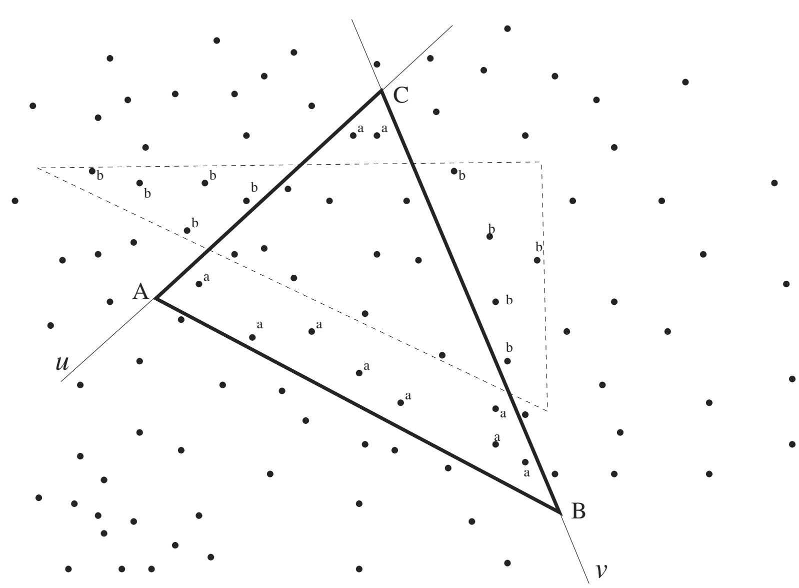

Fig. 6. $f(x,y) = (\alpha_u \cdot x + \beta_u \cdot y - \gamma_u) \cdot (\alpha_v \cdot x + \beta_v \cdot y - \gamma_v)$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b) .

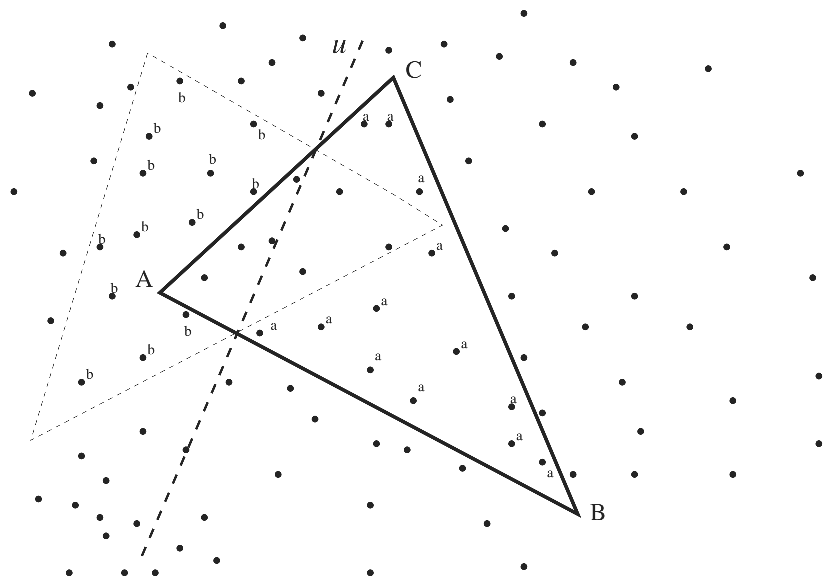

Fig. 7. $f(x, y) = \alpha_u \cdot x + \beta_u \cdot y - \gamma_u$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b) .

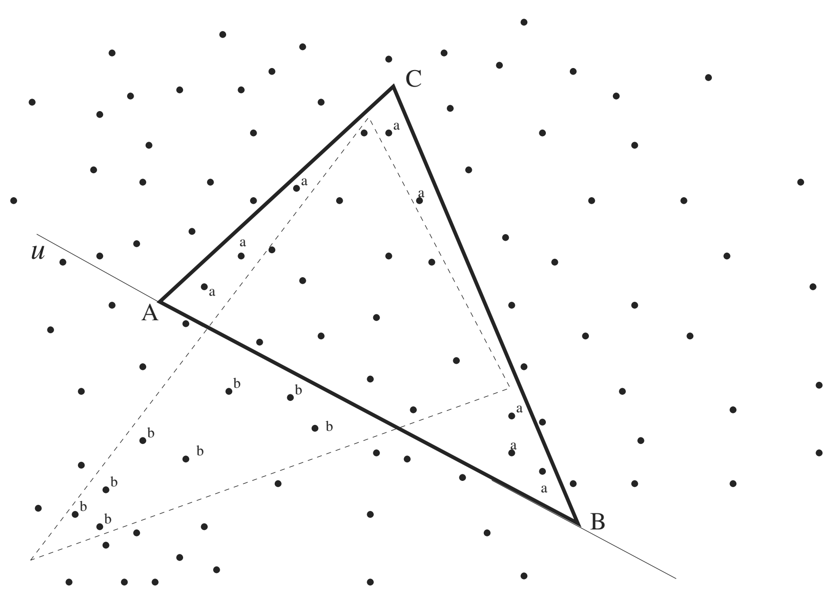

Fig. 8. $f(x, y) = \alpha_u \cdot x + \beta_u \cdot y - \gamma_u$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b) .

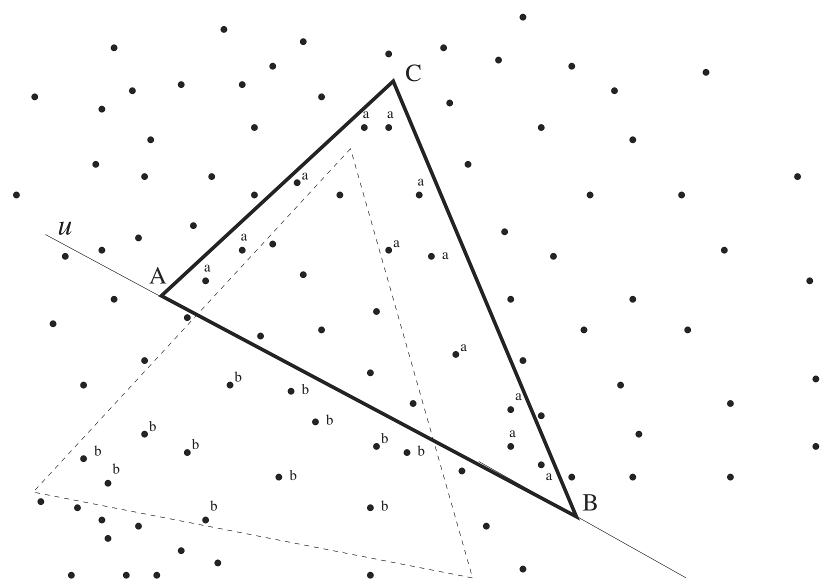

Fig. 9. $f(x,y) = \alpha_u \cdot x + \beta_u \cdot y - \gamma_u$ is a separating function for $\mathbf{D}(T) \setminus \mathbf{D}(T_1)$ (points labeled by a), and $\mathbf{D}(T_1) \setminus \mathbf{D}(T)$ (points labeled by b). $\square$

3 On Digital Triangles

As it is mentioned, we use a digital triangle instead of a discrete triangle if the discretization is made on the integer grid, i.e., $\mathcal{S} = \mathbf{Z}^2$. In this section we analyze the storage complexity if the proposed characterization (coding scheme) is applied. We have the following theorem.

Theorem 3. *Any digital triangle presented on the* $m \times m$ *integer grid can be coded by*

$$\mathcal{O}(\log m)$$

bits. That is the asymptotic minimum.

Proof. The proof follows from the simple fact $\mu_{p,q}(D(T)) \leq (m-1)^5$, which implies that less than $10 \cdot \log(m-1)^5 = \mathcal{O}(\log m)$ bits are enough for the storage of the discrete moments $\mu_{p,q}(D(T))$ under the assumptions: $p+q \leq 3$ and $D(T)$ is a digital triangle from the $(m \times m)$-integer grid.

On the other side, since any of m^2 pixels can be a discretization of a real triangle with a sufficiently small area, it follows that $\mathcal{O}(\log m)$ bits per coded digital triangle is a lower bound for the storage complexity. $\square$

Remark. An equivalent formulation of the previous theorem is: *If a real triangle is presented on a digital picture of a given resolution* r *(i.e., there are* r *pixels per unit) then* $\mathcal{O}(\log r)$ *bits are sufficient for the storage.*

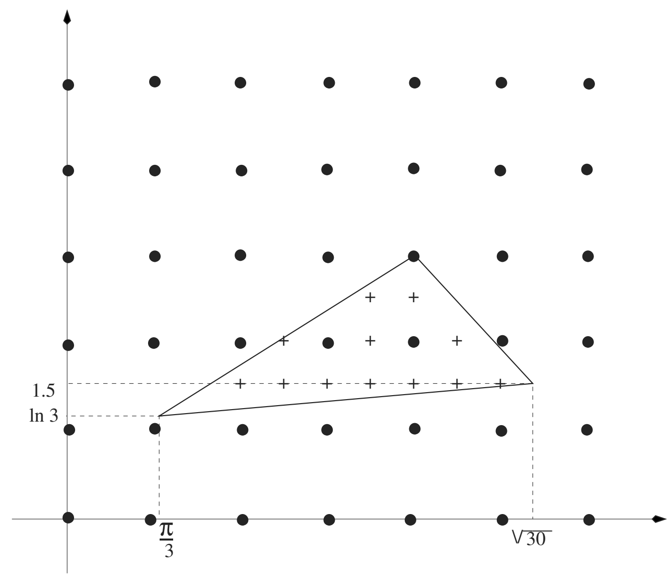

Fig. 10. Digitalization of the triangle T having the vertices $(\frac{\pi}{3}, \ln 3), (4, 3)$, and $(\sqrt{30}, \frac{3}{2})$ by applying the resolutions r=1 and $r = 2$. The "new" points appearing in $D(T)$ after the increase of resolution are denoted by $+$.

Of course, this is much better than storage of all $\approx area\_of(T) \cdot r^2 = \mathcal{O}(r^2)$ pixels belonging to $D(T)$, but also, it is better than the storage by using the Freeman 8-chain code. Namely, it is straightforward that $\mathcal{O}(r)$ bits are necessary for the storage of $D(T)$ by the Freeman code of the "digital" boundary of $D(T)$. A few examples given in Table 1, confirm that. From the table, it can be seen that for a relatively small resolution $r = 1$, i.e., one pixel is the measure unit (see Figure 10), the proposed code requires 76 bits (since 19 characters from the set $\{0, 1, 2, \ldots 9\}$ should be stored) while 18 bits (for the storage of 6 characters from $\{0, 1, 2, \ldots, 7\}$) are sufficient for Freeman coding (i.e., 8-chain coding). The situation is changing if the resolution increase. For example, if $r = 100$, i.e., there are 100 pixels per measure unit, the proposed code (99 characters from $\{0, 1, 2, \ldots 9\}$) requires 396 bits while the Freeman code consisting of 884 characters from $\{0, 1, 2, \ldots 7\}$ requires 2652 bits. If $r = 500$ the amounts of bits are are 504 and 13284, for $r = 1000$, they are 556 and 35432, respectively. Of course, the dominance of the new code is more obvious for higher values of r and in the limit case it is in accordance with the previous asymptotic estimates.

4 Concluding Remarks

In this paper an efficient characterization of digital triangles is given. The described characterization is simple and asymptotically optimal with respect to the

Table 1. The triangle T with vertices $(\frac{\pi}{3},\ \ln 3)$, $(4,\ 3)$, and $(\sqrt{30},\ \frac{3}{2})$ is presented on digital pictures having different resolutions. The code of $D(T)$ presented here and the size of the Freeman code of the boundary of $D(T)$ are given.

applied resolution r	the proposed code $\mu_{0,0}, \mu_{1,0}, \mu_{0,1}, \mu_{2,0}, \mu_{1,1}, \mu_{0,2}$, $\mu_{3,0}, \mu_{2,1}, \mu_{1,2}, \mu_{0,3}$	Freeman code length
1	3, 11, 7, 41, 26, 17 155, 98, 64, 43	6
2	15, 107, 57, 803, 410, 229 6281, 3082, 1664, 975	14
5	87, 1507, 815, 27807, 14435, 8013 538081, 269513, 144561, 82559	42
$3 \cdot e$	241, 6906, 3688, 210812, 107977, 59234 6751896, 3334301, 1766307, 996778	70
10	356, 12480, 6640, 466850, 237545, 129750 18352200, 8976191, 4721145, 2652358	86
25	2257, 197697, 105150, 18513157, 9412325, 5133818 1822886751, 891082875, 467844501, 262200330	218
$\sqrt{1000}$	3620, 401408, 213717, 47566142, 24222891, 13226957 5924570114, 2902540215, 1526584563, 856530219	278
100	36185, 12693880, 6752390, 4759918368, 2420790884, 1320567596 1876452768418, 917747474042, 482006394242, 270164847422	884

used memory space. The method can be applied to discretization of triangles on different "discretizationing" sets, as well. The corresponding discrete moments having order at most 3 are sufficient for a unique determination of the given discrete triangle.

Let us mention that the problem of efficient representation is already studied for digital straight line segments ([1], [7]), digitized circular arcs ([11]), digital ellipses ([13]), and digital polynomial segments ([12]). So, by the coding scheme proposed here a fast comparison of digital triangles and an efficient storage of them are preserved. The recognition and reconstruction problems are not studied here.

For digital n-gons the recognition problem is considered in the literature but only for digital squares ([2], [9]). The problem of optimal coding of digital polygons for an arbitrary n seems to be difficult one even in the case of digital convex polygons. Namely, is it known ([4]) that any optimal coding is within

$\mathcal{O}(m^{2/3})$ bits per coded digital convex polygon from the $m \times m-$integer grid but such a coding scheme is still unknown.

The problem of a fast computation of the discrete moments for digital triangles is omitted from the paper because a number of papers were already devoted to developing fast algorithms of moment computation for $2D$ objects ([5], [6]). Also, a general approach for moment calculation for polygons and line segments is given in [10].

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