# Noise-Resistant Affine Skeletons of Planar Curves

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**Abstract.** A new definition of affine invariant skeletons for shape representation is introduced. A point belongs to the affine skeleton if and only if it is equidistant from at least two points of the curve, with the distance being a minima and given by the *areas* between the curve and its corresponding chords. The skeleton is robust, eliminating the need for curve denoising. Previous approaches have used either the Euclidean or affine distances, thereby resulting in a much less robust computation. We propose a simple method to compute the skeleton and give examples with real images, and show that the proposed definition works also for noisy data. We also demonstrate how to use this method to detect affine skew symmetry.

## 1 Introduction

Object recognition is an essential task in image processing and computer vision, being the *skeleton* or *medial axis* a shape descriptor often used for this task. Thus, the computation of skeletons and symmetry sets of planar shapes is a subject that received a great deal of attention from the mathematical (see [6, 5] and references therein), computational geometry [20], biological vision [14, 16], and computer vision communities (see for example [18,21] and references therein). All this activity follows from the original work by Blum [2].

In the classical Euclidean case, *the symmetry set* of a planar curve (or the boundary of a planar shape) is defined as the set of points equidistant from at least two different points on the given curve, providing the distances are local extrema (a number of equivalent definitions exist). The skeleton is a subset of this set.

Inspired by [9,10] and [17], we define in this paper an analogous symmetry set, the *affine area symmetry set* (AASS). Instead of using Euclidean distances to the curve, we define a new distance based on the areas enclosed between the curve and its chords. We define the symmetry set as the closure of the locus of

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points equidistant from at least two different points on the given curve, provided that the distances are local minima.

As we will demonstrate, this definition based on areas makes the symmetry set remarkably noise-resistant, because the area between the curve and a chord "averages out" the noise. This property makes the method very useful to compute symmetry sets of real images without the need of denoising. In addition, being an area-based computation, the result is affine invariant. That implies that if we process the image of a planar object, the skeleton will be independent of the angle between the camera that captures the image and the scene, provided the camera is far enough from the (flat) object. For these reasons, we believe this symmetry set, in addition to having theoretical interest, has the strong potential of becoming a very useful tool in invariant object recognition.

## 2 Affine Area Symmetry Set and Affine Invariant Skeletons

In this section, we formally introduce the key concepts of *affine distance*, *affine area symmetry set* and *affine skeleton*. We begin with the following definition:
**Definition 1:** A *special affine* transformation in the plane ($\mathbf{R}^2$) is defined as

$$\tilde{X} = AX + B, \tag{1}$$

where $X \in \mathbf{R}^2$ is a vector, $A \in \mathrm{SL}_2(\mathbf{R}^2)$ (the group of invertible real $2 \times 2$ matrices with determinant equal to 1) is the affine matrix, and $B \in \mathbf{R}^2$ is a translation vector.

In this work we deal with symmetry sets which are affine invariant, in the sense that if a curve $C$ is affine transformed with Eq. (1), then its symmetry set is also transformed according Eq. (1). Affine invariant symmetry sets are not new. Giblin and Sapiro [9,10] introduced them and proposed the definition of the *affine distance symmetry set* (ADSS) for a planar curve $C(s)$. In their definition, they used affine geometry, and defined distances in terms of the affine invariant tangent of $C(s)$, which involves second order derivatives of the curve respect to an arbitrary parameter. Later they defined the ADSS analogously to the Euclidean case. there is a fundamental technical problem with this definition: in curves extracted from real images, noise is always present, and the second derivatives needed to compute the ADSS oscillate in a very wild fashion unless a considerable smoothing is performed. Giblin and Sapiro proposed a second definition, the *affine envelope symmetry set* [9,10], which still requires derivatives, and thus suffers from the same computational problem. As we will show with the new definition presented below, we do not compute derivatives at all. Consequently, the robustness of the computation is considerably higher, and shape smoothing is not required.

For simplicity, here we shall always deal with simple closed curves $C(s) : [0,1] \to \mathbf{R}^2$ with a countable number of discontinuities on the derivative $C'(s)$. We start by defining the building block of our *affine area symmetry set* (AASS), the *affine distance* (inspired by [17]):

**Definition 2:** The *affine distance* between a generic point $X$ and a point of the curve $C(s)$ is defined by the area between the curve and the chord that joins $C(s)$ and $X$:

$$d(X,s) = \frac{1}{2}\int_{C(s)}^{C(s')} (C - X) \times dC \tag{2}$$

where $\times$ is the $z$ component of the cross product of two vectors,[1] the points $C(s)$ and $C(s')$ define the chord that contain $X$ and that has exactly two contact points with the curve, as shown in Fig. 1-A. This distance is invariant under the affine transformation Eq. (1), and it is independent on the parametrization of the curve.

For a simple convex curve, the function $d(X,s)$ is always defined for interior points, but for concave curves the function $d(X,s)$ may be undefined for some values of $s$ in $[0,1]$ as sketched in Fig. (1-B). When the point is exterior to the curve (as the point $Y$ in Fig. (1-A)), the distance may be undefined as well.

We can now define our affine symmetry set:

**Definition 3:** $X \in \mathbf{R}^2$ is a point in the *affine area symmetry set* of $C(s)$ (AASS) if and only if there exist two different points $s_1, s_2$ which define two different chords that contain $X$ and have equal area,

$$d(X,s_1) = d(X,s_2), \tag{3}$$

provided that $d(X,s_1)$ and $d(X,s_2)$ are defined and that they are local minima respect to $s$.

This definition is analogous to the Euclidean case and the ADSS in [9,10].

Commonly in shape analysis, a subset of the symmetry set is used, and it is denoted as *skeleton* or *medial axis*. In the Euclidean case, one possible way to simplify the symmetry set into a skeleton, inspired by original work of Giblin and colleagues, is to require that the distance to the curve is a global minimum. The affine definition is analogous:

**Definition 4:** The *affine skeleton* or *affine medial axis* is the subset of the AASS where $d(X,s_1)$ and $d(X,s_2)$ are global (not just local) minima.

There are curves for which the skeleton may be computed exactly. For example, it is easy to verify that the skeleton of a circle is its center. We can make an affine transformation to the circle and transform it into an ellipse, and by virtue of the affine invariance of our definitions, we conclude that the skeleton of an ellipse is its center too. Another important example is the triangle. In affine geometry, all triangles may be generated by affine-transforming an equilateral triangle. The skeleton of an equilateral triangle are the segments connecting the middle of the bases and the center of the figure (as can be verified by simple area computations). As a consequence, the skeleton of an arbitrary triangle are the segments connecting the middle of the sides with the center of gravity (where all medians cross).

[1] We should note that our distance is the integral of the distance used in [9,10].

An important property concerns convex curves with straight sections. If the curve contains two equal straight parallel non-collinear segments, then the AASS will contain an identical segment parallel and equidistant to the former two. For instance, the AASS of a rhombus will contain its medians.

## 2.1 Dynamical Interpretation and Affine Erosion

We now define a notion of *affine erosion* of a curve, which is analogous (but different) to that used by Moisan [17]. Let $C(s) : [0,1] \to \mathbf{R}^2$ be a simple closed curve. Let $(C(s), C(s'))$ be a chord of $C(s)$, as shown in Fig. 1-A. As before, this chord intersects the curve exactly twice The difference of this definition with respect to the definition given in [17], is that in Moisan's set-up the chord is allowed to cross the curve outside the interval $(s, s')$. For our purposes, we opted for a definition which gives a unique chord for a given parameter $s$, when this chord exists, and which also separates the curve into two disjoint parts. The connected closed set enclosed by the chord and the curve is the *chord set*, and its area is denoted by $A$.

**Definition 5:** The *minimum distance* from a point $X$ to the curve $C$ is defined by

$$f(X) = \inf(d(X,s), s \in D) \tag{4}$$

where $D$ is the domain of $d(X,s)$ for a fixed value of $X$ (see below for conditions for this for convex curves).

For interior points $X$, the minimum distance $f(X)$ is always defined because in a simple closed curve, we can always draw at least one chord that contains $X$. For exterior points it may be undefined, as for instance, in a circle.

Moisan defines the affine erosion of the curve $C$ as the set of the points of the interior of $C$ which do not belong to any *positive* chord set with area less than $A$. Here we define the affine erosion in terms of our affine distance:

**Definition 6:** The *affine erosion* $E(C,A)$ of the shape enclosed by a curve $C$,

by the area $A$, is the set of the points $X$ of the interior of $C$ that satisfy

$$E(C,A) := \{X \in \mathbf{R}^2 : f(X) \geq A \geq 0\} \tag{5}$$

Roughly speaking, it is the area bounded by the envolvent of all the possible chords of area $A \geq 0$. We also define the *eroded curve* $C(A)$ as the "boundary" of the affine erosion

$$C(A) := \{X \in \mathbf{R}^2 : f(X) = A\}. \tag{6}$$

Note that if we consider the area to be a time parameter, $t = A$, the distance $f(X)$ represents the time that the eroded curve $C(A)$ takes to reach the point $X$. Initially, when $A = 0$, we have the initial curve. At later times ($A > 0$), the curve $C(A)$ will be contained inside the original curve.

There is a fundamental relationship between the affine erosion of a curve and the skeleton, namely, a shock point $X$ is an skeleton point.

**Definition 7:** A *shock point* $X$ is a point of the eroded curve $C(A)$ where two different chords $(C(s_1), C(s_1'))$ and $(C(s_2), C(s_2'))$ of equal area $A$ intersect. Clearly, the distance from $X$ to these two points are equal, $d(X, s_1) = d(X, s_2)$, and on the other hand, the distance is a global minimum at $s_1$ because as $X$ belongs to $C(A)$ (see definitions 5 and 6), $f(X) = \inf(d(X, s), s \in D) = A$. Thus, a shock point $X$ is an skeleton point.

## 2.2 Basic Properties of the Affine Area Distance and Symmetry Set

In this section we shall present some theoretical results concerning the AASS, mostly without proof. Further results and details will appear elsewhere.[2] We shall show that the AASS has connections with our previously defined *affine envelope symmetry set* or AESS [10],[3] as well as with the *affine distance symmetry set* (ADSS) mentioned before. As mentioned above, it is not clear to us how far the area definition can be extended. That is, it is not yet clear whether we can define a *smooth* family of functions

$$d: \ C \times U \to \mathbf{R},$$

associating to $(s, X)$ the 'area $d(s, X)$ of the sector of $C$ determined by $C(s)$ and $X$', for all points $X$ inside some reasonably large set $U$ in the plane $\mathbf{R}^2$. We have seen that when $C$ is convex then the function is well-defined and smooth for all $X$ inside the curve $C$. We shall assume this below.

We have

$$2d(s, X) = \int_s^{t(s)} [C(s) - X, C'(s)]ds = \int_s^{t(s)} F(s, X)ds,$$

for any regular parametrization of $C$, where $[\ ,\ ]$ means the determinant of the two vectors inside the square brackets, $'$ means $\frac{d}{ds}$ and $t(s)$ is the parameter value of the other point of intersection of the chord through $X$ and $C(s)$ with the curve. Using standard formulae for differentiation of integrals,

$$2d_s := 2\frac{\partial d}{\partial s} = F(t(s), X)t'(s) - F(s).$$

We evaluate $t'(s)$ by using the fact that $C(s), X$ and $C(t(s))$ are collinear:

$$C(t(s)) - X = \lambda(C(s) - X) \text{ for a scalar } \lambda, \text{ i.e. } [C(s) - X, \ C(t(s)) - X] = 0.$$

Differentiating the last equation with respect to $s$ we obtain

$$t'(s) = [C(t(s)) - X, \ C'(s)]/[C(s) - X, \ C'(t(s))],$$

[2] Some of the results below have also been obtained by Paul Holtom [12].

[3] This set is basically defined as the closure of the center of conics having three-point contact with at least two points on the curve.

and from this it follows quickly that, provided the chord is not tangent to $C$ at $C(s)$ or $C(t(s))$, $d_s = 0$ if and only if $\lambda = \pm 1$. But $\lambda = 1$ means $C(s)$ and $C(t(s))$ coincide so for us the interesting solution is $\lambda = -1$, which means that $X$ is the mid-point of the segment (this result has also been obtained in [17]): *The area function $d$ has a stationary point at $(s, X)$, i.e. $\partial d/\partial s = 0$, if and only $X$ is the midpoint of the segment from $C(s)$ to $C(t(s))$.*

One immediate consequence of this is: *The envelope of the chords cutting off a fixed area from $C$ (the affine eroded set) is also the locus of the midpoints of these chords.*

Further calculations on the same lines show that: *The first two derivatives of $d$ with respect to $s$ vanish at $(s, X)$ if and only if $X$ is the midpoint of the chord and also the tangents to $C$ at the endpoints of the chord are parallel.*

In mathematical language this means that the 'bifurcation set' of the family $d$ is the set of points $X$ which are the midpoints of chords of $C$ at the ends of which the tangents to $C$ are parallel. This set is also the envelope of lines parallel to such parallel tangent pairs and halfway between them, and has been called the *midpoint parallel tangent locus* (MPTL) by Holtom [12]. Various facts are known about the MPTL, for example it has an odd number of cusps, and these cusps coincide in position with certain cusps of the AESS [11]. The cusps in question occur precisely when the point $X$ is the center of a conic having 3-point contact with $C$ at two points where the tangents to $C$ are parallel: *The first three derivatives of $d$ with respect to $s$ vanish at $(s, X)$ if and only if $X$ is the midpoint of the chord, the tangents to $C$ at the endpoints are parallel, and there exists a conic with center $X$ having 3-point contact with $C$ at these points.*

The *full bifurcation set* of the family $d$ consists of those points $X$ for which (i) $d$ has a degenerate stationary point ($d_s = d_{ss} = 0$) for some $s$ or else (ii) there are two distinct $s_1, s_2$ and $d$ has an ordinary stationary point ($d_s = 0$) at each one, and the same value there: $d(s_1, X) = d(s_2, X)$. The latter is precisely the AASS as defined 2 above. Mathematically the AASS and the MPTL 'go together' in the same way that the classical symmetry set and evolute go together, or the ADSS and the affine evolute go together: in each case the pair makes up a single mathematical entity called a *full bifurcation set.* A good deal is known about the structure of such sets, including the structure of full bifurcation sets arising from *families* of curves. See [7]. For instance, the symmetry set has endpoints in the cusps of the evolute, and in the same way the AASS has endpoints in cusps of the MPTL.[4]

When $X$ lies on the AASS there are two chords through $X$, with $X$ the mid-point of each chord, and the areas defined by $d$ are equal. From the midpoint conditions alone it follows that the four endpoints of the chords form a *parallelogram.* The tangent to the AASS at $X$ is in fact parallel to two sides of this parallelogram.

[4] The ADSS as defined in [10] also has endpoints and these are in the cusps of the *affine evolute.* The endpoints of the AASS are by contrast in the cusps of the *midpoint parallel tangent locus.*

Consider the AASS of for example a triangle, as in Section 2, where it was noted that the affine medial axis comes from the center of a side and stops at the centroid of the triangle. The 'full' AASS, allowing for non-absolute minima of $d$, stops at the point *halfway* up the median, as can be verified by an elementary calculation with areas. Presumably this is a highly degenerate version of a cusp on the AASS. With the triangle, the two branches of the cusp are overlaid on each other.

We mention finally one curious phenomenon connected with the AASS. Given a point $C(s_1)$ there will generally be an *area-bisecting chord* through this point. That is, the two areas on either side of the chord and within $C$ are equal. In that case let $X$ be the midpoint of the chord, and let $C(t(s_1))$ be the other end of the chord. Let $s_2 = t(s_1)$: then $t(s_2) = s_1$ by construction and $X$ satisfies the conditions to be a point of the AASS. That is, the midpoints of all area-bisecting chords automatically appear in the AASS. These points are in some sense anomalous: the 'genuine' AASS consists of the other points satisfying the defining condition.

## 3 Robust Numerical Implementation for Discrete Curves

Inspired by [1], we propose the following algorithm to compute the affine skeleton:

**1) Discretization of the curve:** Discretize $C(s) = (x(s), y(s))$ with two vectors for the points $C_k = (x_k, y_k)$ with $1 \leq k \leq M$ (see Fig. 1-C).

**2) Discretization of the rectangular domain:** Discretize the domain that contains the curve, of dimensions $L_x \times L_y$, with a uniform grid of $N_x \times N_y$ points. Each point $X_{ij}$ of the grid will have coordinates $X_{ij} = (i\Delta x, j\Delta y)$, where $\Delta x = L_x/N_x$, $\Delta y = L_y/N_y$ (See Fig. 1-C) and $0 \leq i \leq N_x$, $0 \leq j \leq N_y$.

Now, for each point $X_{ij}$ of the grid we perform the following steps:

**3a) Compute the chord areas:** With Eq. 2, compute the areas between $X_{ij}$ and each point in the curve $C_k$:

$$d(X_{ij}, C_k) = \frac{1}{2}\int_{C_k}^{C_k^*} (C - X_{ij}) \times dC \tag{7}$$

for $k = 1, \ldots, M$. The integral is computed by approximating the curve with a polygon that interpolates the points $C_k$, and computing the point $C_k^*$ as the intersection between the curve and the line joining $C_k$ and $X_{ij}$. As mentioned before, the distance is not defined if the chord does not touch exactly two points on the curve. The points have to be labeled with a logical vector $E_k$ indicating whether or not the distance is defined at $C_k$. Here we detect these singular points just by scanning around the curve and counting the crossings between the line that contains $C_k$ and $X_{ij}$. Then we store the areas in a vector $d_k = d(X_{ij}, C_k)$, with $k = 1, \ldots, M$.

**3b) Search for local minima of the chord areas,** approximated by the local minima of the set $d_k$, with $k = 1, \ldots, M$. When $d_{k-1}$, $d_k$ and $d_{k+1}$ are defined, the local minimum condition is $d_{k-1} \geq d_k \leq d_{k+1}$. When $d_k$ and $d_{k\mp 1}$

are defined but $d_{k\pm1}$ is not defined, the condition is simply $d_k \leq d_{k\mp1}$. Now, for each point of the grid $X_{ij}$ we shall have after this step the set of $l$ local minima distances $(d_1^*, d_2^*, ..., d_l^*)$ corresponding to the points $(C_1^*, C_2^*, ..., C_l^*)$. All these quantities are functions of $X_{ij}$.

**3c) Approximate AASS computation:** Compute the differences

$$D(X_{ij}) = d_p^* - d_q^* \qquad p = 1, \ldots, l, \quad q = 1, \ldots, l, \quad r \neq q, \tag{8}$$

$p$ and $q$ represent the indexes of the local minimum distances of *different* chords. If this difference is smaller in magnitude than a given tolerance $\epsilon$, we can consider $X_{ij}$ to be an approximate point of the area symmetry set (see Fig. (1-C)). Then, as a first approximation, *add to the AASS all the points* $X_{ij}$ *that satisfy*

$$\left| d_p^* - d_q^* \right| < \epsilon \tag{9}$$

The tolerance $\epsilon$ has to be of the order of the variation of the distance difference along a cell in the discretized domain, i.e.

$$\epsilon \approx \max \left( \left| \frac{\partial D}{\partial x} \right| \Delta x, \left| \frac{\partial D}{\partial y} \right| \Delta y \right) \tag{10}$$

The partial derivatives are taken respect to the components of $X_{ij}$.

There is not a simple general formula for this expression. However, by using the distance definition Eq. (2) and by restricting ourselves to *regular* local minima which satisfy $d_s(s^*) = 0$, we can demonstrate that $\nabla d_p^*(X_{ij}) = (-\Delta y_p^*, \Delta x_p^*)$, where $(\Delta x_p^*, \Delta y_p^*)$ are the components of the chords $C_p^{*\prime} - C_p^*$ corresponding to local minimum area. This formula is not general, since we may have non-regular minima, as for example, points where $d(X, s)$ has a discontinuity respect to $s$. However we still use this expression because we only need one order of magnitude for the tolerance. Then, we define the tolerance to be $\epsilon = \max\left(|\Delta y_p^* - \Delta y_q^*|\Delta x, |\Delta x_p^* - \Delta x_q^*|\Delta y\right)$.

**3d) Focusing of the AASS:** At this point, the skeleton is quite crude, and the branches have an spatial error of the order of the discretization of the domain $(\Delta x, \Delta y)$. We can compute with negligible computational cost the (small) correction vector $\Delta X_{ij} = (u, v)$ to the position $X_{ij}$ that makes the difference of distances exactly equal to zero at $X_{ij} + \Delta X_{ij}$:

$$d(X_{ij} + \Delta X_{ij}, C_p^*) = d(X_{ij} + \Delta X_{ij}, C_q^*)$$

(see Fig 1-C). At first order, we must solve

$$D(X_{ij}) + \Delta X_{ij} \cdot \nabla D(X_{ij}) = 0 \tag{11}$$

with $\Delta X_{ij}$ parallel to the gradient of $D$. We get

$$u = \frac{(\Delta y_p^* - \Delta y_q^*) D}{G^2} \qquad v = \frac{-(\Delta x_p^* - \Delta x_q^*) D}{G^2} \tag{12}$$

$$G^2 = (\Delta y_p^* - \Delta y_q^*)^2 + (\Delta x_p^* - \Delta x_q^*)^2 \tag{13}$$

After this step, the AASS for a discrete image was computed.

**3e) Pruning:** If one of the distances $d_p^*$ or $d_q^*$ is not a positive global minimum, then discard the corresponding point. In this way we obtain the affine skeleton. We also have to discard points which are originated very close each other at the curve in such a way that they are effectively undistinguishable. Here we discard points such that the area of the triangles $A(C_p, X_{ij}, C_q)$ are smaller than the areas of the triangles defined by the discretization of the curve $A(C_p, X_{ij}, C_{p+1}) + A(C'_p, X_{ij}, C'_{p+1})$ and the area defined by the discretization of the domain $\Delta x L_y + \Delta y L_x$, as indicated in Fig. 1-D.

## 4 Examples

In the following examples, we compute the skeleton in a domain discretized with $N_x = N_y = 200$ points.

Skeletons may be used to detect symmetries, topic that has been the subject of extensive research in the computer vision community, e.g., [3,4,19,22]. In particular, affine skeletons may be used to detect skew symmetries. Numerical experiments show that if the skeleton contains a straight branch then a portion of the curve has skew symmetry respect to this line. [5] This is illustrated now. In Fig. 2-A we show the original figure with its corresponding skeleton, while in Fig. 2-B we show the figure affine transformed by the matrix $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. In Fig. 2-C we corrupted the shape by adding to each point of the discrete curve a random number of amplitude 0.025. The corresponding skeleton remains almost unchanged.

We now show how to compute skeletons from real data. First we need to obtain the points of the curve $C$ which define the shape. If the points are to be extracted from a digital image with a good contrast, they may be extracted by thresholding the image in binary values and then getting the boundary with a boundary-following algorithm [13]. This procedure was performed with the shape of the right in Fig. 2-D and with the tennis racket of Fig. 2-E. When the border of the shape is more complex or fuzzier, as in the shape of the left in Fig. 2-D , more sophisticated techniques can be used. Here we extracted the contour with the "snakes" algorithm as formulated in [8,15]. The resulting skeletons are shown in Fig. 2-D. In Fig. 2-E, the image has an approximate skew symmetry, and note that the skeleton contains a short straight branch.

## 5 Conclusions and Open Questions

In this paper, we introduced a new definition of affine skeleton and a robust method to compute it. The definition is based on areas, making the skeleton

[5] Note that although the Euclidean skeleton of a symmetric shape contains a straight line, this is not true anymore after the shape is affine transformed, obtaining skew symmetry.

remarkably insensitive to noise, and thus, useful for processing real images. There are still theoretical problems that have to be solved in order to build a consistent theory to support our definition:

**(a)** The skeleton in the Euclidean case may be found by detecting the shocks in the solutions of the Hamilton-Jacobi equation (this is equivalent to the Huygens principle in Blum's method). We do not know at this point whether there is a differential equation which would allow us to compute our affine skeleton in an analogous way.

**(b)** Additional properties of the area-distances $d(X, s)$ and AASS, analogous to those for the ADSS and AESS, are to be further investigated.

**(c)** The extension of the definition to multiply connected curves would be of paramount importance in practical applications.

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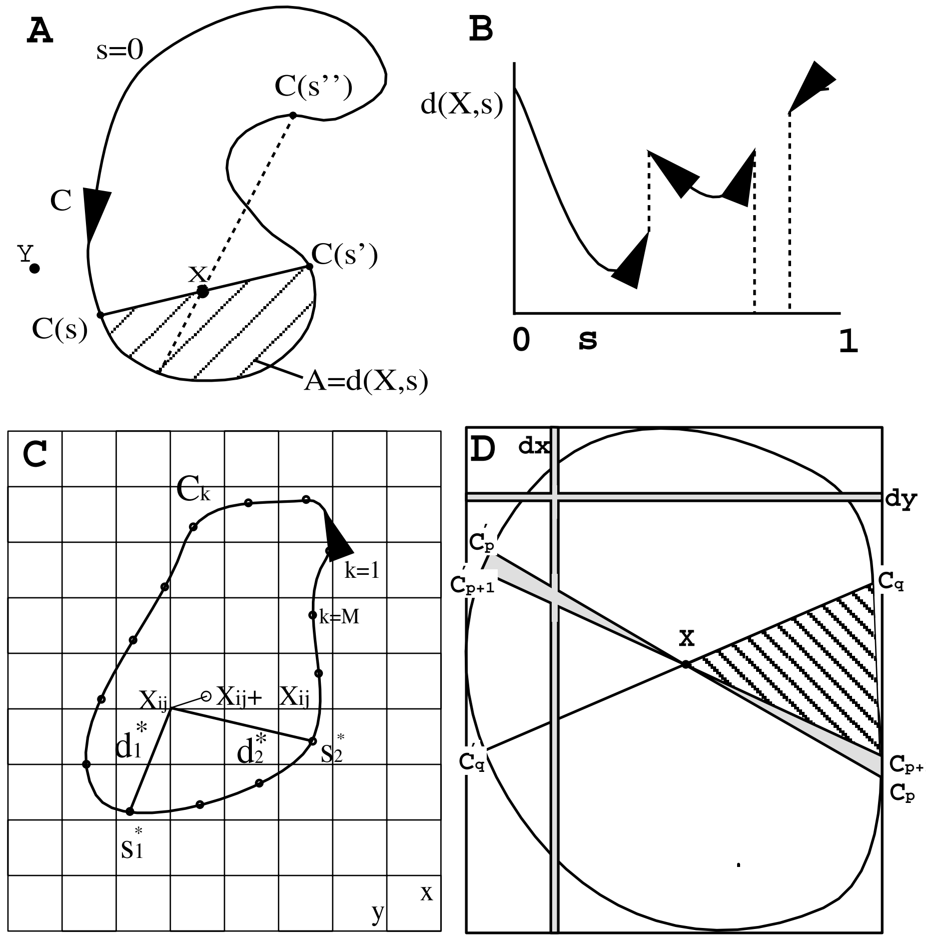


**Fig. 1.** A) We define the affine distance from $X$ to the curve as the area between the curve $C$ and the chord $(C(s), C(s'))$. The chord touches the curve exactly twice. For example, there is no "legal" chord which contains the pair of points $(C(s''), X)$. As a consequence, the function $d(X, s)$ may have discontinuities as shown in B. C) Discretization of the curve and the domain. D) We consider two points $p, q$ "undistinguishable" if the area of the triangle $A(C_p, X, C_q)$ (dashed) is smaller than the area defined by the discretization of the curve $A(C_p, X, C_{p+1}) + A(C'_p, X, C'_{p+1})$, and the discretization of the domain (shaded regions).

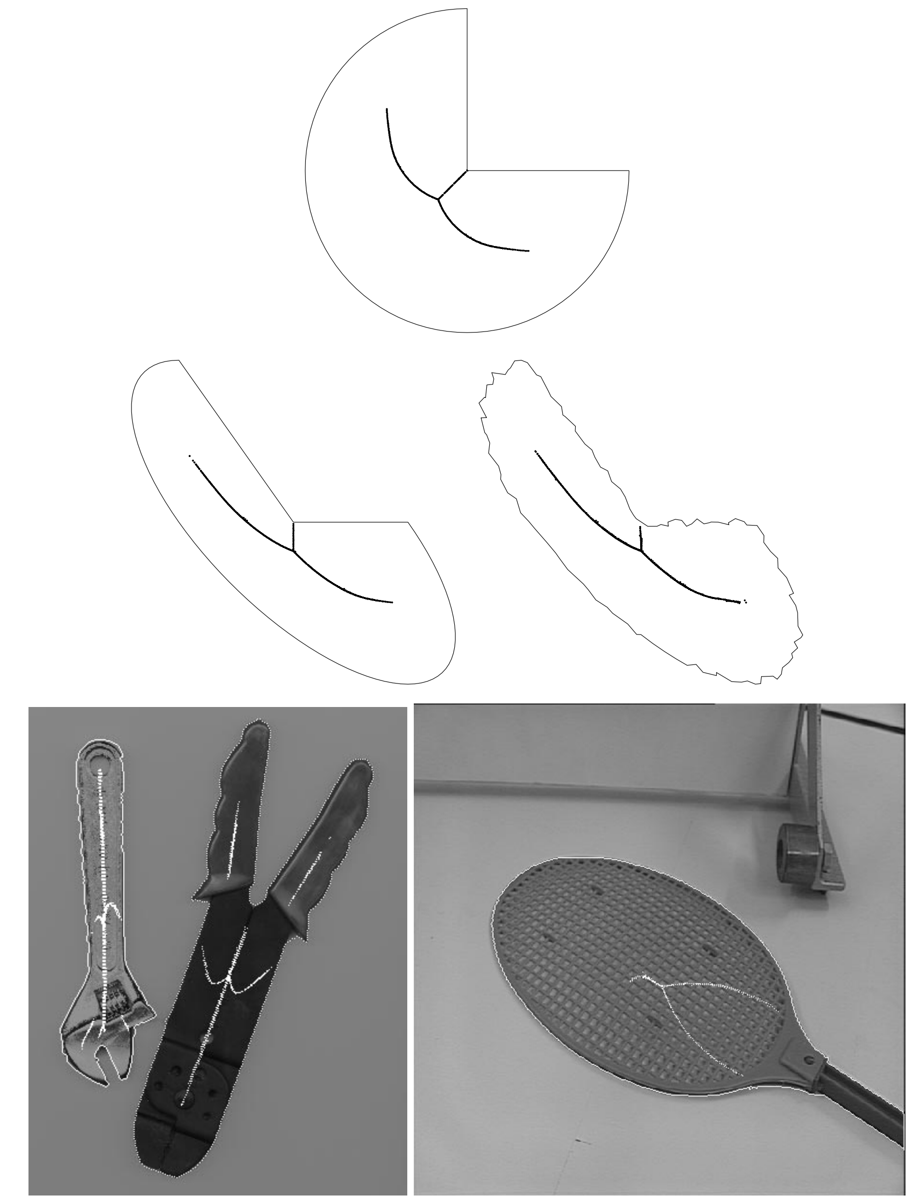

**Fig. 2.** (In lexicographic order) A) A concave curve and its skeleton. B) After an affine transformation, the skeleton keeps the information about the original symmetry. C) When noise corrupts the curve, the main branches of the skeleton may still be recognized (in this computation, $M = 115$ and $N = 200$). D) Affine skeletons of shapes extracted from digital images. E) A figure with approximate skew symmetry.