On the Estimation of the Fundamental Matrix: A Convex Approach to Constrained Least-Squares

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Abstract. In this paper we consider the problem of estimating the fundamental matrix from point correspondences. It is well known that the most accurate estimates of this matrix are obtained by criteria minimizing geometric errors when the data are affected by noise. It is also well known that these criteria amount to solving non-convex optimization problems and, hence, their solution is affected by the optimization starting point. Generally, the starting point is chosen as the fundamental matrix estimated by a linear criterion but this estimate can be very inaccurate and, therefore, inadequate to initialize methods with other error criteria.
Here we present a method for obtaining a more accurate estimate of the fundamental matrix with respect to the linear criterion. It consists of the minimization of the algebraic error taking into account the rank 2 constraint of the matrix. Our aim is twofold. First, we show how this non-convex optimization problem can be solved avoiding local minima using recently developed convexification techniques. Second, we show that the estimate of the fundamental matrix obtained using our method is more accurate than the one obtained from the linear criterion, where the rank constraint of the matrix is imposed after its computation by setting the smallest singular value to zero. This suggests that our estimate can be used to initialize non-linear criteria such as the distance to epipolar lines and the gradient criterion, in order to obtain a more accurate estimate of the fundamental matrix. As a measure of the accuracy, the obtained estimates of the epipolar geometry are compared in experiments with synthetic and real data.

1 Introduction

The computation of the fundamental matrix existing between two views of the same scene is a very common task in several applications in computer vision, including calibration [15,13], reconstruction [13], visual navigation and visual servoing. The importance of the fundamental matrix is due to the fact that it represents succinctly the epipolar geometry of stereo vision. Indeed, its know-

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ledge provides relationships between corresponding points in the two images. Moreover, for known intrinsic camera parameters, it is possible to recover the essential matrix from the fundamental matrix and, hence, the camera motion, that is the rotation and translation of the camera between views [6,14].

In this paper we consider the problem of estimating the fundamental matrix from point correspondences [6,20,7]. Several techniques has been developed [9, 19,17], like the linear criterion, the distance to epipolar lines criterion and the gradient criterion [12]. The first one is a least-squares technique minimizing the algebraic error. This approach has proven to be very sensitive to image noise and unable to express the rank constraint. The other two techniques take into account the rank constraint and minimize a more indicative distance, the geometric error, in the 7 degrees of freedom of the fundamental matrix. This results in non-convex optimization problems [11,10] that present local solutions in addition to the global ones. Hence the found solution is affected by the choice of the starting point of the minimization algorithm [12]. Generally, this point is chosen as the estimate provided by the linear criterion and forced to be singular setting the smallest singular value to zero, but this choice does not guarantee to find the global minima.

In this paper we present a new method for the estimation of the fundamental matrix. It consists of a constrained least-squares technique where the rank condition of the matrix is ensured by the constraint. In this way we impose the singularity of the matrix a priori instead of forcing it after the minimization procedure as in the linear criterion.Our aim is twofold. First, we show how this optimization problem can be solved avoiding local minima. Second, we provide experimental results showing that our approach leads to a more accurate estimate of the fundamental matrix. In order to find the solution and avoiding local minima, we proceed as follows. First, we show how this problem can be addressed as the minimization of a rational function in two variables. This function is a non-convex one and, therefore, the optimization problem still presents local minima. Second, we show how this minimization can be reformulated so that it can be tackled by recently developed convexification techniques [2]. In this manner, local optimal solutions are avoided and only the global one is found.

The same problem has been studied by Hartley [9], who provided a method for minimizing the algebraic error ensuring the rank constraint, which requires an optimization over two free parameters (position of an epipole). However, the optimization stage on these two unknowns is not free of local minima in the general case.

The paper is organized as follows. In section 2, we give some preliminaries about the fundamental matrix and the estimation techniques mentioned above. In section 3, we state our approach to the problem, showing how the constrained least-squares minimization in the unknown entries of the fundamental matrix can be cast as a minimization in only two unknowns. Section 4 shows how this optimization problem can be solved using convexification methods in order to find the global minima. In section 5 we present some results obtained with our approach using synthetic and real data, and we provide comparisons with other methods. In particular, we show that our solution gives smaller geometric errors than the one provided by the linear criterion. Moreover, initializing non-linear criteria with our solution allows us to find a more accurate estimate of the fundamental matrix. Finally, in section 6 we conclude the paper.

2 Preliminaries

First of all, let us introduce the notation used in this paper.

$\mathbb{R}$: real space;
I_n: $n \times n$ identity matrix;
A^T: transpose of A;
$A > 0$ ($A \geq 0$): positive definite (semidefinite) matrix;
$(A)_{i,j}$: entry (i, j) of A;
$\|u\|_2$ ($\|u\|_{2,W}$): (weighted) euclidean norm of u;
$\det(A)$: determinant of A;
$\mathrm{adj}(A)$: adjoint of A;
$\lambda_M(A)$: maximum real eigenvalue of A;
$\mathrm{Ker}(A)$: null space of A.

Given a pair of images, the fundamental matrix $F \in \mathbb{R}^{3\times 3}$ is defined as the matrix satisfying the relation

$$u'^T F u = 0 \quad \forall u', u \tag{1}$$

where $u', u \in \mathbb{R}^3$ are the projections expressed in homogeneous coordinates of the same 3D point in the two images. The fundamental matrix, F has 7 degrees of freedom being defined up to a scale factor and being singular [5].
The *linear criterion* for the estimation of F is defined as

$$\min_F \sum_{i=1}^{n} (u_i'^T F u_i)^2 \tag{2}$$

where n is the number of observed point correspondences. In order to obtain a singular matrix, the smallest singular value of the found estimate is set to zero [8]. The *distance to epipolar lines criterion* and the *gradient criterion* take into account the rank constraint using a suitable parameterization for F. The first criterion defines the cost function as the sum of squares of distances of a point to the corresponding epipolar line. The second criterion considers a problem of surface fitting between the data $[(u_i')_1; (u_i')_2; (u_i)_1; (u_i)_2] \in \mathbb{R}^4$ and the surface defined by (1). These non-linear criteria result in the minimization of weighted least-squares:

$$\min_{F:\det(F)=0} \sum_{i=1}^{n} w(F, u_i', u_i)(u_i'^T F u_i)^2 \tag{3}$$

where

$$w(F, u_i', u_i) = \frac{1}{(F^T u_i')_1^2 + (F^T u_i')_2^2} + \frac{1}{(F u_i)_1^2 + (F u_i)_2^2} \tag{4}$$

for the distance to epipolar lines criterion and

$$w(F, u'_i, u_i) = \frac{1}{(F^T u'_i)_1^2 + (F^T u'_i)_2^2 + (F u_i)_1^2 + (F u_i)_2^2} \tag{5}$$

for the gradient criterion. The main problem with these non-linear criteria is the dependency of the found solution on the starting point for the optimization procedure, due to the fact that the cost funtion defined in (3) is non-convex. Experiments show a large difference between results obtained starting from the exact solution and starting from the solution provided by the linear criterion, generally used to initialize these minimizations [12].

Before proceeding with the presentation of our approach, let us review the method proposed by Hartley for the minimization of the algebraic error constrained by the singularity of the fundamental matrix [9]. In short, Hartley reduces the number of degrees of freedom from eight (the free parameters of the fundamental matrix) to two (the free parameters of an epipole) under the rank 2 constraint. Therefore, the optimization stage looks for the epipole minimizing the algebraic error. Unfortunately, this step is not free of local minima in the general case, as figure 1 shows for the statue image sequences of [9].

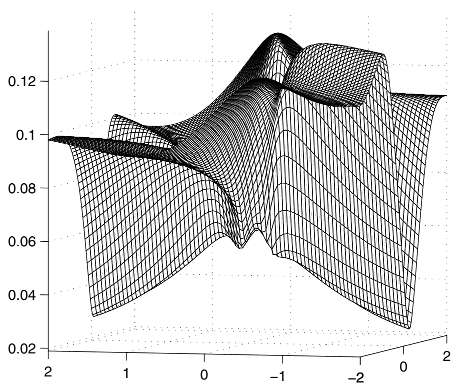

Fig. 1. Algebraic error versus position of epipole.

3 The Constrained Least-Squares Estimation Problem

The problem that we wish to solve can be written as

$$\begin{aligned} &\min_F \sum_{i=1}^{n} w_i ({u'_i}^T F u_i)^2 \\ &\text{subject to} \quad \det(F) = 0 \end{aligned} \tag{6}$$

where $n \geq 8$, $w_i \in \mathbb{R}$ are positive given weighting coefficients and the constraint ensures the rank condition of the estimation of F. Let us introduce $A \in \mathbb{R}^{n\times 8}$, $\mathrm{rank}(A) = 8$, and $b \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} w_i({u'_i}^T F u_i)^2 = \|Af - b\|_{2,W}^2 \tag{7}$$

where $f \in \mathbb{R}^8$ contains the entries of F:

$$F = \begin{pmatrix} (f)_1 & (f)_4 & (f)_7 \\ (f)_2 & (f)_5 & (f)_8 \\ (f)_3 & (f)_6 & 1 \end{pmatrix} \tag{8}$$

and $W \in \mathbb{R}^{n\times n}$ is the diagonal matrix with entries w_i. Since F is defined up to a scale factor, we have set $(F)_{3,3} = 1$ (see Remark 1). Then, (6) can be written as

$$\begin{array}{c} \min_{f,\lambda} \|Af - b\|_{2,W}^2 \\ \text{subject to} \quad T(\lambda)f = r(\lambda) \end{array} \tag{9}$$

where $\lambda \in \mathbb{R}^2$, $T(\lambda) \in \mathbb{R}^{3\times 8}$ and $r(\lambda) \in \mathbb{R}^3$ are so defined:

$$T(\lambda) = \begin{pmatrix} I_3 & (\lambda)_1 I_3 & \begin{matrix} (\lambda)_2 & 0 \\ 0 & (\lambda)_2 \\ 0 & 0 \end{matrix} \end{pmatrix}, \tag{10}$$

$$r(\lambda) = \begin{pmatrix} 0 & 0 & -(\lambda)_2 \end{pmatrix}^T. \tag{11}$$

The constraint in (9) expresses the singularity condition on F as the linear dependency of the columns of F and hence $\det(F) = 0$. In order to solve the minimization problem (9), let us observe that the constraint is linear in f for any fixed λ. So, the problem can be solved using the Lagrange's multipliers obtaining the solution $f^*(\lambda)$:

$$f^*(\lambda) = v - ST(\lambda)P(\lambda)\left[T(\lambda)v - r(\lambda)\right] \tag{12}$$

where $v \in \mathbb{R}^8$, $S \in \mathbb{R}^{8\times 8}$ and $P(\lambda) \in \mathbb{R}^{3\times 3}$ are:

$$v = SA^T W b, \tag{13}$$

$$S = (A^T W A)^{-1}, \tag{14}$$

$$P(\lambda) = \left[T(\lambda) S T^T(\lambda)\right]^{-1}. \tag{15}$$

Now, it is clear that the minimum J^* of (9) can be computed as:

$$J^* = \min_{\lambda} J^*(\lambda). \tag{16}$$

So, let us calculate $J^*(\lambda)$. Substituting $f^*(\lambda)$ into the cost function we obtain:

$$\begin{aligned} J^*(\lambda) &= \|Af^*(\lambda) - b\|_{2,W}^2 \\ &= c_0 + \sum_{i=1}^{3} c_i(\lambda) \end{aligned} \tag{17}$$

where

$$c_0 = b^T W \left(I_n - ASA^T W \right) b, \tag{18}$$
$$c_1(\lambda) = r^T(\lambda)P(\lambda)r(\lambda), \tag{19}$$
$$c_2(\lambda) = -2v^T T^T(\lambda)P(\lambda)r(\lambda), \tag{20}$$
$$c_3(\lambda) = v^T T^T(\lambda)P(\lambda)T(\lambda)v. \tag{21}$$

The constrained problem (6) in 8 variables is equivalent to the unconstrained problem (16) in 2 variables. In order to compute the solution J^*, let us consider the form of the function $J^*(\lambda)$. The terms $c_i(\lambda)$ are rational functions of the entries of λ. In fact, let us write $P(\lambda)$ as:

$$P(\lambda) = \frac{G(\lambda)}{d(\lambda)}, \tag{22}$$
$$G(\lambda) = \text{adj}\left[T(\lambda)ST^T(\lambda) \right], \tag{23}$$
$$d(\lambda) = \det\left[T(\lambda)ST^T(\lambda) \right]. \tag{24}$$

Since $T(\lambda)$ depends linearly on λ, we have that $G(\lambda)$ is a polynomial matrix of degree 4 and $d(\lambda)$ a polynomial of degree 6. Straightforward computations allow us to show that $J^*(\lambda)$ can be written as:

$$J^*(\lambda) = \frac{h(\lambda)}{d(\lambda)} \tag{25}$$

where $h(\lambda)$ is a polynomial of degree 6 defined as:

$$h(\lambda) = c_0 d(\lambda) + r^T(\lambda)G(\lambda)r(\lambda) - 2v^T T^T(\lambda)G(\lambda)r(\lambda) + v^T T^T(\lambda)G(\lambda)T(\lambda)v. \tag{26}$$

Let us observe that the function $d(\lambda)$ is strictly positive everywhere being the denominator of the positive definite matrix $T(\lambda)ST^T(\lambda)$.

Remark 1 The parametrization of the fundamental matrix chosen in (8) is not general since $(F)_{3,3}$ can be zero. Hence, in this step we have to select one of the nine entries of F that, for the considered case, is not zero and not too small in order to avoid numerical problems. A good choice could be setting to 1 the entry with the maximum modulus in the estimate provided by the linear criterion.

4 Problem Solution via Convex Programming

In this section we present a convexification approach to the solution of problem (6). The technique is based on Linear Matrix Inequalities (LMI) [1] and leads to the construction of lower bounds on the global solution of the polynomial optimization problem (6). More importantly, it provides an easy test to check whether the obtained solution is the global optimum or just a lower bound. From the previous section we have that:

$$J^* = \min_{\lambda} \frac{h(\lambda)}{d(\lambda)}. \tag{27}$$

Let us rewrite (27) as:

$$\begin{array}{rl} J^* = & \min_{\lambda,\delta} \delta \\ \text{subject to} & \dfrac{h(\lambda)}{d(\lambda)} = \delta. \end{array} \tag{28}$$

where $\delta \in \mathbb{R}$ is an additional variable. The constraint in (28) can be written as $y(\lambda, \delta) = 0$ where

$$y(\lambda, \delta) = h(\lambda) - \delta d(\lambda) \tag{29}$$

since $d(\lambda) \neq 0$ for all λ. Hence, J^* is given by:

$$\begin{array}{rl} J^* = & \min_{\lambda,\delta} \delta \\ \text{subject to} & y(\lambda, \delta) = 0 \end{array} \tag{30}$$

where the constraint is a polynomial in the unknown λ and δ.
Problem (30) belongs to a class of optimizations problems for which convexification techniques have been recently developed [2,3]. The key idea behind this technique is to embed a non-convex problem into a one-parameter family of convex optimization problems. Let us see how this technique can be applied to our case. First of all, let us rewrite the polynomials $h(\lambda)$ and $d(\lambda)$ as follows:

$$h(\lambda) = \sum_{i=0}^{6} h_i(\lambda), \tag{31}$$

$$d(\lambda) = \sum_{i=0}^{6} d_i(\lambda). \tag{32}$$

where $h_i(\lambda)$ and $d_i(\lambda)$ are homogeneous forms of degree i. Now, let us introduce the function $y(c; \lambda, \delta)$:

$$y(c; \lambda, \delta) = \sum_{i=0}^{6} \frac{\delta^{6-i}}{c^{6-i}} \left[h_i(\lambda) - c\, d_i(\lambda) \right]. \tag{33}$$

We have the following properties:

1. for a fixed c, $y(c; \lambda, \delta)$ is a homogeneous form of degree 6 in λ and δ;
2. $y(\lambda, \delta) = y(c; \lambda, \delta)$ for all λ if $\delta = c$.

Hence the form $y(c; \lambda, \delta)$ and the polynomial $y(\lambda, \delta)$ are equal on the plane $\delta = c$. In order to find J^* let us observe that $\delta \geq 0$ because J^* is positive. Moreover, since $h(\lambda)$ is positive, then $y(\lambda, \delta) \geq 0$ for $\delta = 0$. This suggests that J^* can be computed as the minimum δ for which the function $y(\lambda, \delta)$ loses its positivity, that is:

$$J^* = \min\{\delta : y(\lambda,\delta) \le 0 \quad \text{for some} \quad \lambda\}. \tag{34}$$

Hence, using the homogeneous form $y(c; \lambda, \delta)$, equation (34) becomes (see [3] for details):

$$J^* = \min\{c : y(c;\lambda,\delta) \le 0 \quad \text{for some} \quad \lambda,\delta\}. \tag{35}$$

The difference between (35) and (34) is the use of a homogeneous form, $y(c; \lambda, \delta)$, instead of a polynomial, $y(\lambda, \delta)$. Now, let us observe that $y(c; \lambda, \delta)$ can be written as:

$$y(c;\lambda,\delta) = z^T(\lambda,\delta)Y(c)z(\lambda,\delta) \tag{36}$$

where $z(\lambda, \delta) \in \mathbb{R}^{10}$ is a base vector for the forms of degree 3 in the variables $(\lambda)_1, (\lambda)_2, \delta$:

$$z(\lambda,\delta) = \begin{pmatrix} (\lambda)_1^3 & (\lambda)_1^2(\lambda)_2 \\ (\lambda)_1^2\delta & (\lambda)_1(\lambda)_2^2 \\ (\lambda)_1(\lambda)_2\delta & (\lambda)_1\delta^2 \\ (\lambda)_2^3 & (\lambda)_2^2\delta \\ (\lambda)_2\delta^2 & \delta^3 \end{pmatrix}^T \tag{37}$$

and $Y(c) \in \mathbb{R}^{10\times10}$ is a symmetric matrix depending on c. Now, it is evident that positivity of the matrix $Y(c)$ ensures positivity of the homogeneous form $y(c; \lambda, \delta)$ (see (36)). Therefore, a lower bound c^* of J^* in (35) can be obtained by looking at the loss of positivity of $Y(c)$. To proceed, we observe that this matrix is not unique. In fact, for a given homogeneous form there is an infinite number of matrices that describe it for the same vector $z(\lambda, \delta)$. So, we have to consider all these matrices in order to check the positivity of $y(c; \lambda, \delta)$. It is easy to show that all the symmetric matrices describing the form $y(c; \lambda, \delta)$ can be written as:

$$Y(c) - L \quad , \quad L \in \mathcal{L} \tag{38}$$

where $\mathcal{L}$ is the linear set of symmetric matrices that describe the null form:

$$\mathcal{L} = \left\{L = L^T \in \mathbb{R}^{10\times10} : z^T(\lambda,\delta)Lz(\lambda,\delta) = 0 \ \forall\lambda,\delta\right\}. \tag{39}$$

Since $\mathcal{L}$ is a linear set, every element L can be linearly parametrized. Indeed, let $L(\alpha)$ be a generic element of $\mathcal{L}$. It can be shown that $\mathcal{L}$ has dimension 27 and hence

$$L(\alpha) = \sum_{i=1}^{27} \alpha_i L_i \tag{40}$$

for a given base $L_1, L_2, \ldots, L_{27}$ of $\mathcal{L}$. Hence, (38) can be written as:

$$Y(c) - L(\alpha) \quad , \quad \alpha \in \mathbb{R}^{27}. \tag{41}$$

Summing up, a lower bound c^* of J^* can be obtained as:

$$\begin{aligned} c^* &= \min_{c,\alpha} c \\ \text{subject to} \quad & \min_{\alpha} \lambda_M \left[L(\alpha) - Y(c)\right] > 0. \end{aligned} \tag{42}$$

This means that c^* can be computed via a sequence of convex optimizations indexed by the parameter c. Indeed, for a fixed c, the minimization of the maximum eigenvalue of a matrix parametrized linearly in its entries is a convex optimization problem that can be solved with standard LMI techniques [16,1]. Moreover, a bisection algorithm on the scalar c can be employed to speed up the convergence.
It remains to discuss when the bound c^* is indeed equal to the sought optimal J^*. It is obvious that this happens if and only if $y(c^*;\lambda,\delta)$ is positive semidefinite, i.e. there exists λ^* such that $y(c^*;\lambda^*,c^*)=0$. In order to check this condition, a very simple test is proposed. Let $\mathcal{K}$ be defined as:

$$\mathcal{K} = \mathrm{Ker}\left[L(\alpha^*) - Y(c^*)\right] \tag{43}$$

where α^* is the minimizing α for the constraint in (42). Then, $J^* = c^*$ if and only if there exists λ^* such that $z(\lambda^*, c^*) \in \mathcal{K}$. It is possible to show that, except for degenerate cases when $\dim(\mathcal{K}) > 1$, the last condition amounts to solving a very simple system in the unknown λ^*. In fact, when $\mathcal{K}$ is generated by one only vector k, then λ^* is given by the equation:

$$z(\lambda^*, c^*) = \frac{{c^*}^3}{(k)_{10}} k. \tag{44}$$

In order to solve the above equation, it is sufficient to observe that if (44) admits a solution λ^* then:

$$\begin{aligned} (\lambda^*)_1 &= c^* \frac{(k)_6}{(k)_{10}}, \\ (\lambda^*)_2 &= c^* \frac{(k)_9}{(k)_{10}}. \end{aligned} \tag{45}$$

Now, we have just to verify if λ^* given by (45) satisfies (44). If it does then c^* is optimal and the fundamental matrix entries f^* solution of (9) are given by:

$$\begin{aligned} f^* &= f^*(\lambda^*) \\ &= v - ST(\lambda^*)P(\lambda^*)\left[T(\lambda^*)v - r(\lambda^*)\right]. \end{aligned} \tag{46}$$

Whenever c^* be not optimal, standard optimization procedures starting from the value of λ given by (45) can be employed for computing J^*. This is expected to prevent the achievement of local minima. However, in our experiments we did not experience any case in which c^* is strictly less than J^*.

Remark 2 In order to avoid numerical problems due to too small values of the parameter c in (33), the procedure described above can be implemented replacing δ in (28) by $\delta - 1$. This change of variable ensures $c \geq 1$.

5 Experiments and Results

In this section we present some results obtained by applying our method for solving problem (6). The goal is to investigate its performance with respect to the linear criterion. To evaluate the algorithm, we generated image data from different 3D point sets and with different camera motions, and also applied the algorithm to real image data from standard sequences. In both cases, we scaled the image data in order to work with normalized data.

In the sequel, we will refer to the estimate of the fundamental matrix given by the linear criterion with F_l; to the estimate provided by our method, *constrained least-squares criterion*, with F_{cls}; and to the estimate provided by the distance to epipolar lines criterion with F_d when initialized by F_l and with $\bar{F}_d$ when initialized by F_{cls}. The algorithm we use to compute F_{cls} is summarized below.

Algorithm for computing F_{cls}

1. Given the point correspondences u_i', u_i, form the polynomials $h(\lambda)$ and $d(\lambda)$ as shown, respectively, in (26) and (24).
2. Build a symmetric matrix function $Y(c)$ satisfying (36).
3. Solve the sequence of LMI problems (42).
4. Retrieve λ^* as shown in (45) and check for its optimality.
5. Retrieve f^* as shown in (46) and form F_{cls}.

First, we report the results obtained with synthetic data. The points in 3D space have been generated randomly inside a cube of size 40cm and located 80cm from the camera centre. The focal length is 1000 pixels. The first image is obtained by projecting these points onto a fixed plane. The translation vector t and the rotational matrix R are then generated randomly, obtaining the projection matrix P of the second camera and, from this, the second image of the points. The camera calibration matrix is the same in all the experiments. In order to use corrupted data, Gaussian noise has been added to the image point coordinates (in the following experiments we refer to image noise as the standard deviation of the normal distribution). These experiments have been repeated fifty times and the mean values computed. The weighting coefficients w_i in (6) have been set to 1.

In the first experiment, a comparison of the mean algebraic error e_a defined as:

$$e_a = \frac{1}{n} \sum_{i=1}^{n} (u_i'^T F u_i)^2 \tag{47}$$

for the linear criterion (F_l) and the constrained least-squares criterion (F_{cls}) has been performed. The goal is to show that imposing the rank constraint a priori gives very different results with respect to setting the smallest singular value to zero. Figure 2 shows the behaviour of the logarithm of e_a for the two methods. In the second experiment we consider the properties of estimated epipolar geometry. Specifically, we compare the mean geometric error e_g defined as:

$$e_g = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \left(\frac{1}{(F^T u_i')_1^2 + (F^T u_i')_2^2} + \frac{1}{(Fu_i)_1^2 + (Fu_i)_2^2} \right) (u_i'^T F u_i)^2} \tag{48}$$

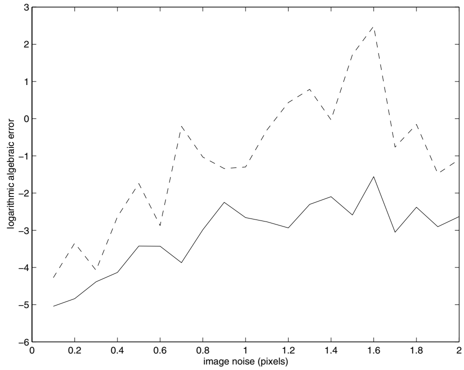

Fig. 2. Logarithmic algebraic error ($\log_{10}(e_a)$) for linear (dashed) and constrained least-squares criterion (solid).

that is the mean geometric distance between points and corresponding epipolar lines. Figure 3 shows the behaviour of the logarithm of e_g for linear and constrained least-squares criterion. As we can see, the error e_g achieved by F_{cls} is clearly less than the one achieved by F_l for all image noises considered. Now,

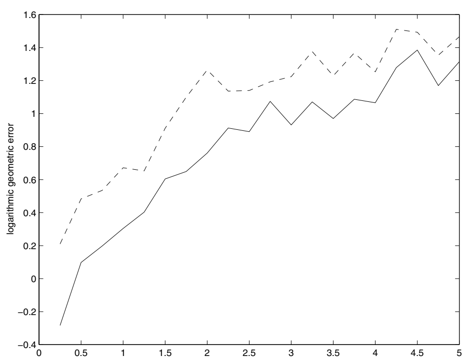

Fig. 3. Logarithmic geometric error ($\log_{10}(e_g)$) for linear (dashed) and constrained least-squares criterion (solid).

let us see the results obtained with real data. Figure 4 shows two typical views used to estimate the fundamental matrix. The forty point correspondences are found by a standard corner finder. Table 1 shows the geometric error e_g given by linear and constrained least-squares criterion, and by the distance to epipolar lines criterion initialized by F_l (F_d) and by F_{cls} ($\bar{F}_d$). As we can see, the geometric error achieved by F_{cls} is clearly less than the one achieved by F_l.
Table 2 shows the geometric error obtained for the views of figure 5. Here, the

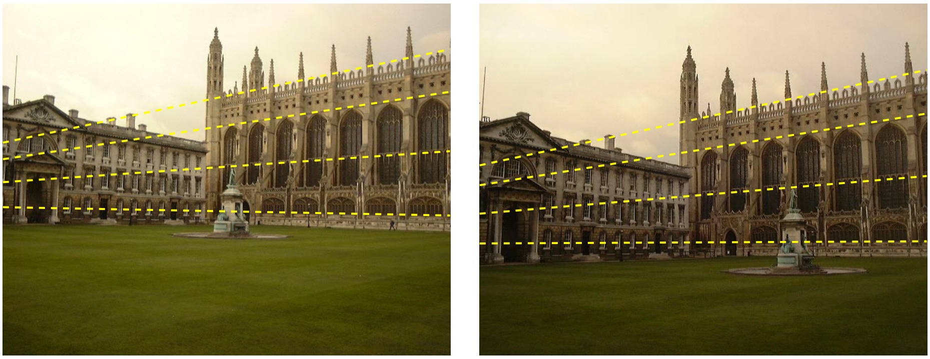

Fig. 4. King's College sequence (epipoles outside the image). The epipolar lines are given by $\bar{F}_d$ after optimization using F_{cls} solution as the starting point.

Table 1. Geometric error e_g obtained with the image sequence of figure 4.

Criterion	Geometric error e_g
F_l	1.733
F_{cls}	0.6578
F_d	0.6560
$\bar{F}_d$	0.6560

point correspondences used are 27. Again, F_{cls} achieves a significant improvement with respect to F_l.

Table 2. Geometric error for the image sequence of figure 5.

Criterion	Geometric error e_g
F_l	1.255
F_{cls}	0.6852
F_d	0.5836
$\bar{F}_d$	0.5836

Finally, table 3 and table 4 show the geometric error obtained for the well known examples used in [9] and shown in figures 6 and 7. The point correspondences used are 100 for the first example and 128 for the second one. Observe that this time, not only does F_{cls} achieve a smaller geometric error than F_l, but also $\bar{F}_d$ produces a better result than F_d, indicating the presence of different local minima. Moreover, in the calibration jig example, F_{cls} provides better results even than F_d.

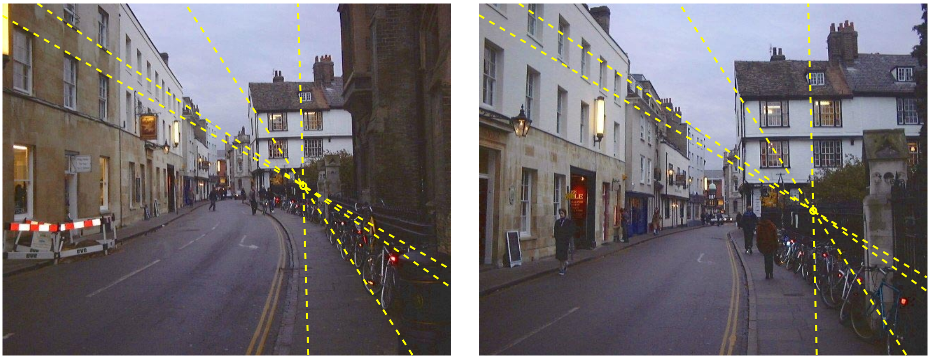

Fig. 5. Cambridge street sequence (epipoles in the image).

Table 3. Geometric error for the views of figure 6.

Criterion	Geometric error e_g
F_l	0.4503
F_{cls}	0.4406
F_d	0.1791
$\bar{F}_d$	0.1607

As we can see from the results, the solution provided by our method gives smaller algebraic and geometric errors with respect to the linear criterion, with both synthetic and real data. Moreover, initializing non-linear criteria with our solution allows us to achieve more accurate estimates of the fundamental matrix.

6 Conclusions

In this paper, we have proposed a new method for the estimation of the fundamental matrix. It consists of minimizing the same algebraic error as that used in the linear criterion, but taking into account explicitly the rank constraint. We have shown how the resulting constrained least-squares problem can be solved using recently developed convexification techniques. Our experiments show that this method provides a more accurate estimate of the fundamental matrix compared to that given by the linear criterion in terms of epipolar geometry. This suggests that our estimation procedure can be used to initialize more complex non-convex criteria minimizing the geometric distance in order to obtain better results.

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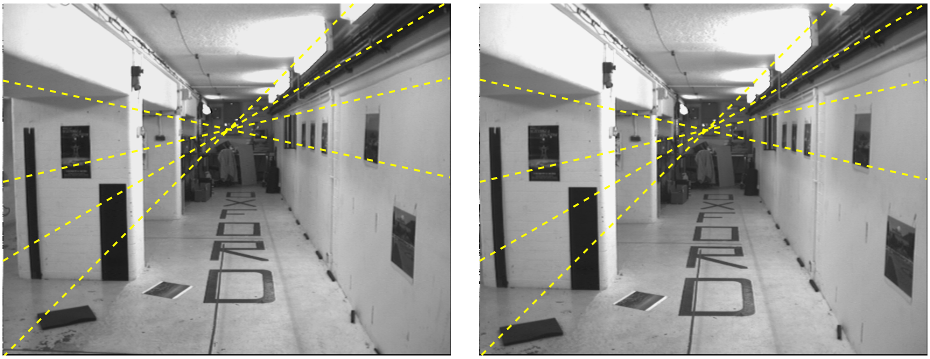

Fig. 6. Oxford basement sequence (epipoles in the image).

Table 4. Geometric error for the views of figure 7.

Criterion	Geometric error e_g
F_l	0.4066
F_{cls}	0.1844
F_d	0.1943
$\bar{F}_d$	0.1844

in [9]. We are also grateful to the anonymous referees of this paper for their useful comments.

References

1. S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan: *Linear Matrix Inequalities in System and Control Theory*. SIAM, Philadelphia (Penn.), 1994.
2. G. Chesi, A. Tesi, A. Vicino, and R. Genesio: A convex approach to a class of minimum norm problems. *Robustness in Identification and Control*, pages 359–372, A. Garulli, A. Tesi, A. Vicino, Eds., Springer-Verlag, London (UK), 1999.
3. G. Chesi, A. Tesi, A. Vicino, and R. Genesio: On convexification of some minimum distance problems. *Proc. 5th European Control Conference*, Karlsruhe (Germany), 1999.
4. R. Deriche, Z. Zhang, Q.-T. Luong, and O. Faugeras: Robust recovery of the epipolar geometry for an uncalibrated stereo rig. *Proc. 3rd European Conference on Computer Vision*, Stockholm (Sweden), 1994.
5. O. Faugeras: What can be seen in three dimensions with an uncalibrated stereo rig? *Proc. 2nd European Conference on Computer Vision*, pages 563–578, Santa Margherita Ligure (Italy), 1992.
6. O. Faugeras: *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Cambridge (Mass.), 1993.
7. R. Hartley: Estimation of relative camera positions for uncalibrated cameras. *Proc. 2nd European Conference on Computer Vision*, pages 579–587, Santa Margherita Ligure (Italy), 1992.

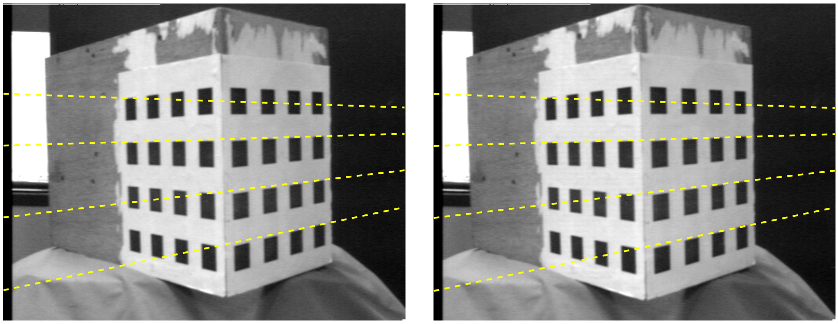

Fig. 7. Calibration Jig sequence (epipoles outside the image). The matched points are localized accurately.

8. R. Hartley: In defence of the 8-point algorithm: *Proc. International Conference on Computer Vision*, pages 1064–1070, 1995.
9. R. Hartley: Minimizing algebraic error in geometric estimation problem. *Proc. 6th International Conference on Computer Vision*, pages 469–476, Bombay (India), 1998.
10. K. Kanatani: *Geometric computation for machine vision*. Oxford Univeristy Press, Oxford (UK), 1992.
11. D. Luenberger: *Linear and Nonlinear Programming*. Addison-Wesley, USA, 1984.
12. Q.-T. Luong and O. Faugeras: The fundamental matrix: theory, algorithms, and stability analysis. *The International Journal of Computer Vision*, 17(1):43–76, 1996.
13. Q.-T. Luong and O. Faugeras: Camera calibration, scene motion and structure recovery from point correspondences and fundamental matrices. *The International Journal of Computer Vision*, 22(3):261–289, 1997.
14. S. Maybank: Properties of essential matrices. *International Journal of Imaging Systems and Technology*, 2:380–384, 1990.
15. P. Mendonca and R. Cipolla: A simple techinique for self-calibration. *Proc. IEEE Conference on Computer Vision and Pattern Recognition*, pages 500–505, Fort Collins (Colorado), 1999.
16. Y. Nesterov and A. Nemirovsky: *Interior Point Polynomial Methods in Convex Programming: Theory and Applications*. SIAM, Philadelphia (Penn.), 1993.
17. P. Torr and D. Murray: The development and comparison of robust methods for estimating the fundamental matrix. *International Journal of Computer Vision*, 3(24):271–300, 1997.
18. P. Torr and A. Zisserman: Performance characterizaton of fundamental matrix estimation under image degradation. *Machine Vision and Applications*, 9:321–333, 1997.
19. P. Torr, A. Zisserman, and S. Maybank: Robust detection of degenerate configurations for the fundamental matrix. *Computer Vision and Image Understanding*, 71(3):312–333, 1998.
20. Z. Zhang: Determining the epipolar geometry and its uncertainty - a review. *The International Journal of Computer Vision*, 27(2):161–195, 1998.