On the Hardness of Approximating Some NP-Optimization Problems Related to Minimum Linear Ordering Problem

(Extended Abstract)

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Abstract. We study hardness of approximating several minimaximal and maximinimal NP-optimization problems related to the minimum linear ordering problem (MINLOP). MINLOP is to find a minimum weight acyclic tournament in a given arc-weighted complete digraph. MINLOP is APX-hard but its unweighted version is polynomial time solvable. We prove that, MIN-MAX-SUBDAG problem, which is a generalization of MINLOP, and requires to find a minimum cardinality maximal acyclic subdigraph of a given digraph, is, however APX-hard. Using results of Håstad concerning hardness of approximating independence number of a graph we then prove similar results concerning MAX-MIN-VC (respectively, MAX-MIN-FVS) which requires to find a maximum cardinality minimal vertex cover in a given graph (respectively, a maximum cardinality minimal feedback vertex set in a given digraph). We also prove APX-hardness of these and several related problems on various degree bounded graphs and digraphs.

Keywords : NP-optimization problems, Minimaximal and maximinimal NP-optimization problems, Approximation algorithms, Hardness of approximation, APX-hardness, L-reduction.

1 Introduction

In this paper we deal with hardness of approximating several minimum-maximal and maximum-minimal NP-complete optimization problems on graphs as well as related maximum/minimum problems. In general, for any given instance x of such a problem, it is required to find a minimum (respectively, maximum) weight (or, cardinality) maximal (respectively, minimal) feasible solution with respect to a partial order on the set of feasible solutions of x. The terminology of minimaximal and maximinimal is apparently first used by Peters et.al. [18], though the concept has received attention of many others, specially in connection with many graph problems. For example, minimum chromatic number and its maximum version, the achromatic number [11,4]; maximum independent set and minimaximal independent set (minimum independent dominating set) [12,13];

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minimum vertex cover and maximinimal vertex cover [17,19]; minimum dominating set and maximinimal dominating set [16,5] and a recent systematic study of minimaximal and maximinimal optimization problems by Manlove [19].

We are led to investigation of several such graph problems while considering a generalization of the minimum linear ordering problem (MINLOP). Given a complete digraph $G_n = (V, A_n)$ on a set $V = \{v_1, v_2, \ldots v_n\}$ of n vertices with nonnegative integral arc weights, the MINLOP is to find an acyclic tournament [10] on V with minimum total arc weight. This is a known NP-complete optimization problem [9] and some results about approximation solutions and hardness of approximability of MINLOP have been obtained in [20]. Two problems related to MINLOP are the maximum acyclic subdigraph (MAX-SUBDAG) and the minimum feedback arc set (MIN-FAS) problems. Given a digraph $G = (V, A)$, the MAX-SUBDAG (respectively, MIN-FAS) problem is to find a subset of $B \subseteq A$ of maximum (respectively, minimum) cardinality such that (V, B) (respectively, $(V, A - B)$) is an acyclic subdigraph (SUBDAG) of G. While MAX-SUBDAG is APX-complete [17] and has a trivial $\frac{1}{2}$-approximation algorithm, MIN-FAS is not known to be in APX, though it is APX-hard [14].

A generalization of MINLOP can be formulated as follows. Note that an acyclic tournament on V is indeed a maximal SUBDAG of G_n (i.e., a SUBDAG of G_n which is not contained in any SUBDAG of G_n). Thus we generalize MINLOP as the minimum weight maximal SUBDAG (MIN-W-MAX-SUBDAG) problem which requires to find a maximal SUBDAG of minimum total arc weight in any given arc weighted digraph (which is not necessarily a complete digraph). MIN-W-MAX-SUBDAG is thus APX-hard as its special case MINLOP is so. We show that unweighted version (i.e., all arc weights 1) of MIN-W-MAX-SUBDAG, called MIN-MAX-SUBDAG, is APX-hard even though MINLOP with constant arc weight is solvable in polynomial time.

The complementary problem of MIN-MAX-SUBDAG is the maximum cardinality minimal feedback arc set (MAX-MIN-FAS) in which it is required to find a minimal feedback arc set of maximum cardinality in a given digraph. The vertex version of this is the maximum cardinality minimal feedback vertex set (MAX-MIN-FVS). An NP-optimization problem related to MAX-MIN-FVS is MAX-MIN-VC, in which it is required to find a minimal vertex cover of maximum cardinality in a given graph. Another related problem is the minimum maximal independent set (MIN-MAX-IS) problem, where one is required to find a maximal IS (or an independent dominating set) of minimum cardinality for any given graph.

Since the decision versions of these optimization problems are NP-complete, it is not possible to find optimal solutions in polynomial time, unless P=NP. So a practical alternative is to find near optimal (or approximate) solutions in polynomial time. However, it is not always possible to obtain such solutions having desired approximation properties [2,12,15,17]. Thus it is of considerable theoretical and practical interest to provide some qualitative explanation for this by establishing results about hardness of obtaining such approximate solutions.

In this paper, we shall establish several results about hardness of approximating such problems using the standard technique of reduction of one problem to another. Due to restriction on the number of pages, we shall give outlines of most of the lengthy proofs, details of which are in [21]. The paper is organized as follows. In Section 2, we recall the relevant concepts about graphs, digraphs, and approximation algorithms. In Section 3, we first prove APX-hardness of MIN-MAX-SUBDAG for arbitrary digraph by reducing MAX-SUBDAG to it. Then, using the results of Håstad concerning hardness of approximating MAX-IS, we prove similar results about MAX-MIN-VC for arbitrary graphs and about MAX-MIN-FVS for arbitrary digraphs. In Section 4, we prove APX-hardness of MIN-FAS and MAX-SUBDAG for k-total-regular digraphs, for all $k \geq 4$. Then we show that MIN-MAX-SUBDAG is APX-hard for digraphs of maximum total degree 12. We also prove that MAX-MIN-VC is k-approximable for all graphs without any isolated vertex and having maximum degree k, $k \geq 1$, MAX-MIN-VC is APX-complete for all graphs of maximum degree 5, and MAX-MIN-FVS is APX-hard for all digraphs of maximum total degree 10. In Section 5, we show that, MIN-FVS is APX-complete for 6-regular graphs and MAX-MIN-FVS is APX-hard for all graphs of maximum degree 9. Finally, in Section 6, we make some concluding remarks.

2 Basic Concepts

We will denote a graph (i.e. an undirected graph) by $G = (V, E)$ and a digraph (i.e. a directed graph) by $G = (V, A)$, where $V = \{v_1, v_2, \ldots v_n\}$, E is the edge set and A is the arc set. An edge between vertices v_i and v_j will be denoted by $\{v_i, v_j\}$, whereas an arc from v_i to v_j will be denoted by the ordered pair (v_i, v_j). In an undirected graph G, *degree* of a vertex v_i is denoted as $d(v_i)$ which is the number of edges incident on v_i in G, and G is called *k-regular* if each vertex in G has degree k. In a digraph G, $d^+(v_i)$ and $d^-(v_i)$ are the number of arcs in G having v_i as the initial vertex and terminal vertex, respectively, and $d(v_i)$, the *total degree* of v_i is defined as $d(v_i) = d^+(v_i) + d^-(v_i)$. A digraph G is *k-total-regular* if for each vertex v_i, $d(v_i) = k$. A *path* $P(v_1, v_t)$ in $G = (V, E)$ (respectively, *dipath* in $G = (V, A)$) is a sequence of distinct vertices $(v_1, v_2, \ldots, v_t)$ such that $\{v_i, v_{i+1}\} \in E$ (respectively, $(v_i, v_{i+1}) \in A$) for $1 \leq i < t$. A path (respectively, dipath) $P(v_1, v_t)$ is called a *cycle* (respectively, *dicycle*) if $v_1 = v_t$.

A *feedback arc set* (FAS) (respectively a *directed acyclic subgraph* (SUBDAG)) in a digraph $G = (V, A)$ is an arc set $B \subseteq A$ such that the subdigraph $(V, A - B)$ (respectively (V, B)) is acyclic. Given a digraph $G = (V, A)$, a *minimal FAS* (respectively *maximal SUBDAG*) is an FAS (respectively SUBDAG) $B \subseteq A$ which does not contain (respectively is not contained in) another FAS (respectively SUBDAG). Given a graph $G = (V, E)$, $C \subseteq V$ is called a *vertex cover* (VC) if for each edge $\{v_i, v_j\} \in E$, C contains either v_i or v_j. A VC C is called a *minimal VC* of G if no proper subset of C is also a VC of G. $S \subseteq V$ is called an *feedback vertex set* (FVS) of G if the subgraph/subdigraph $G[V - S]$ induced by the vertex set $V - S$ is acyclic. Similarly a *minimal FVS* of G is defined.

The precise formulation of the problems considered in this paper are as follows:

MAX-SUBDAG (respectively, MIN-FAS)
Instance - A digraph $G = (V, A)$.
Solution - A SUBDAG (V, B) (respectively, an FAS B) of G.
Cost - $m(x, B) = |B|$.
Goal - max (respectively, min).

MAX-SUBDAG-k (respectively, MIN-FAS-k) is the problem of MAX-SUBDAG (respectively, MIN-FAS) on k-total-regular digraphs.

MIN-W-FVS
Instance - A pair $x = (G, w)$ where G is a graph/digraph and w assigns a non-negative integer to each $v \in V$.
Solution - An FVS F of G.
Cost - $m(x, F) = \sum_{v \in F} w(v)$.
Goal - min.

MIN-FVS is the unweighted version of MIN-W-FVS, i.e. MIN-W-FAS with $w(v) = 1$ for each $v \in V$. MIN-FVS-k is the problem of MIN-FVS on k-regular (respectively k-total-regular) graphs (respectively digraphs).

MIN-MAX-SUBDAG (respectively, MAX-MIN-FAS)
Instance - Same as that of MAX-SUBDAG.
Solution - A maximal SUBDAG (V, B) (respectively, a minimal FAS B) of G.
Cost - $m(x, B) = |B|$.
Goal - min (respectively, max).

MIN-MAX-SUBDAG$\leq k$ (respectively, MAX-MIN-FAS$\leq k$) is the problem of MIN-MAX-SUBDAG (respectively, MAX-MIN-FAS) on digraphs of total degree at most k.

MAX-MIN-FVS
Instance - Same as that of MIN-FVS.
Solution - A minimal FVS B of G.
Cost - $m(x, B) = |B|$.
Goal - max.

MAX-MIN-FVS-k is the problem of MAX-MIN-FVS on k-regular (respectively k-total-regular) graphs (respectively digraphs) and MAX-MIN-FVS$\leq k$ is the problem of MAX-MIN-FVS on graphs (respectively digraphs) of degree (respectively total-degree) at most k.

MAX-MIN-VC
Instance - A graph $x = G = (V, E)$.
Solution - A minimal VC C of G.
Cost - $m(x, C) = |C|$.
Goal - max.

MAX-MIN-VC$\leq k$ is the problem of MAX-MIN-VC on graphs of degree at most k.

Given an instance x of an NP optimization problem π and $y \in sol(x)$, the performance ratio of y with respect to x is defined by $R_\pi(x,y) = \max\{\frac{m(x,y)}{m^*(x)}, \frac{m^*(x)}{m(x,y)}\}$ where $m^*(x)$ is the optimum value.

A polynomial time algorithm A for an NP optimization problem π is called an *ϵ-approximate* algorithm for π for some $\epsilon > 1$ if $R_\pi(x, A(x)) \leq \epsilon$ for any instance x of π, where $A(x)$ is the solution for x given by A. The class APX is the set of all NP optimization problems which have some ϵ-approximate algorithm.

An approximation algorithm A for an NP optimization problem π approximates the optimal cost within a factor of $f(n)$ if, for all instances x of π, it produces a solution $A(x)$ in polynomial time such that $R_\pi(x, A(x)) \leq f(|x|)$.

Among the approximation preserving reductions L-reduction [17] is the easiest one to use. π_1 is said to be *L-reducible* to π_2 [17], in symbols $\pi_1 \leq_L \pi_2$, if there exist two functions f, g and two positive constants α, β such that:

1. For any $x \in I_{\pi_1}$, $f(x) \in I_{\pi_2}$ is computable in polynomial time.
2. For any $x \in I_{\pi_1}$ and for any $y \in sol_{\pi_2}(f(x))$, $g(x,y) \in sol_{\pi_1}(x)$ is computable in polynomial time.
3. $m^*_{\pi_2}(f(x)) \leq \alpha \cdot m^*_{\pi_1}(x)$.
4. For any $x \in I_{\pi_1}$ and for any $y \in sol_{\pi_2}(f(x))$,
$|m^*_{\pi_1}(x) - m_{\pi_1}(x, g(x,y))| \leq \beta \cdot |m^*_{\pi_2}(f(x)) - m_{\pi_2}(f(x), y)|$.

We shall be using in this paper only the L-reduction though the hardness (or completeness) in the class APX is defined in terms of PTAS-reduction ($\leq_{PTAS}$) [6,2]. An NP optimization problem π is APX-hard if, for any $\pi' \in$ APX, $\pi' \leq_{PTAS} \pi$, and problem π is APX-complete if π is APX-hard and $\pi \in$ APX. However it is well known that [7,15] for any two NP optimization problems π_1 and π_2, if $\pi_1 \leq_L \pi_2$ and $\pi_1 \in$ APX, then $\pi_1 \leq_{PTAS} \pi_2$.

3 Hardness Results for Arbitrary Graphs/Digraphs

As already noted, MIN-W-MAX-SUBDAG is APX-hard, we now show that its unweighted version MIN-MAX-SUBDAG is also APX-hard, even though the unweighted version of MINLOP is solvable in polynomial time. For this it is enough to prove the following theorem, as MAX-SUBDAG is APX-complete [17].

Theorem 1. *MAX-SUBDAG $\leq_L$ MIN-MAX-SUBDAG with $\alpha = 5$ and $\beta = 1$.*

Proof. (Outline) For each instance $x = G = (V, A)$ of MAX-SUBDAG, we construct in polynomial time an instance $f(x) = G' = (V', A')$ of MIN-MAX-SUBDAG and with each feasible solution (V', S') of $f(x)$, we associate a feasible solution $g(S') = S = S' \cap A$ of x such that f and g satisfy the conditions of L-reduction with $\alpha = 5$ and $\beta = 1$.

Let $K = \{(v_i, v_j) | (v_i, v_j) \in A$ and $(v_j, v_i) \notin A\}$. For each arc $(v_i, v_j) \in K$, we introduce a new vertex v_{ij} for the construction of G'. Construct $G' = (V', A')$ as follows: $V' = V \cup \{v_{ij} | (v_i, v_j) \in K\}$ and $A' = A \cup \{(v_j, v_i), (v_j, v_{ij}), (v_{ij}, v_j) | (v_i, v_j) \in K\}$. For an example, see Figure 1. Let $k = |K|$ and p be the number of pairs of vertices $v_i, v_j \in V$ such that both $(v_i, v_j), (v_j, v_i) \in A$. Hence, $p = \frac{|A-K|}{2}$.

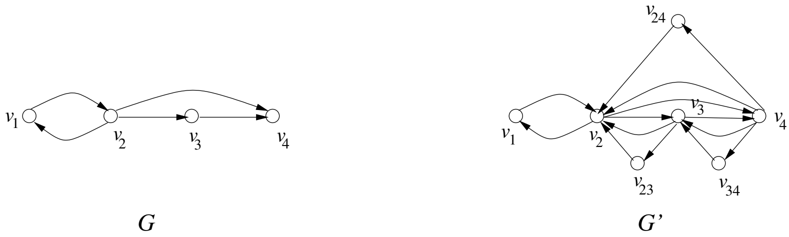

Fig. 1. A digraph G and the corresponding digraph G'

It is not difficult to establish the following claims.

Claim 1 Let (V', S') be a maximal SUBDAG of G' and $S = S' \cap A$. Then (V, S) is a SUBDAG of G and $|S'| = 3k + 2p - |S|$.

Claim 2 If (V', S'_o) is a minimum maximal SUBDAG of G', then (V, S_o) is a maximum SUBDAG of G. Also $|A| \leq 2|S_o|$.

Now $|S'_o| = 3k + 2p - |S_o| \leq 3(k + p) - |S_o| \leq 6|S_o| - |S_o| = 5|S_o|$. Also for any maximal SUBDAG (V', S') of G', $|S_o| - |S| = |S'| - |S'_o|$. □

Next we prove results about hardness of approximating MAX-MIN-VC and MAX-MIN-FVS, using reducibility arguments and the results of Håstad [12] concerning MAX-IS stated bellow.

Theorem 2. *[Håstad] Unless NP=ZPP (respectively P=NP), for any* $\epsilon > 0$ *there exists no polynomial time algorithm to approximate MAX-IS within a factor of* $n^{1-\epsilon}$ *(respectively* $n^{\frac{1}{2}-\epsilon}$*), where* n *is the number of vertices in an instance.*

Regarding MAX-MIN-VC we have

Theorem 3. *Unless NP = ZPP (respectively P=NP), for any* $\epsilon > 0$ *there exists no polynomial time algorithm to approximate MAX-MIN-VC within a factor of* $\frac{1}{2}n^{\frac{1}{2}-\epsilon}$ *(respectively* $\frac{1}{2}n^{\frac{1}{4}-\epsilon}$*), where* n *is the number of vertices in an instance.*

Proof. (Outline) Given an instance $G = (V, E)$ of MAX-IS, we construct an instance $G' = (V', E')$ of MAX-MIN-VC, where $V' = V \cup [\cup_{v \in V} \{v^1, v^2, \ldots v^{n+1}\}]$ and $E' = E \cup \{\{v, v^1\}, \{v, v^2\}, \ldots \{v, v^{n+1}\} | v \in V\}$. In other words, G' is obtained from G by introducing for each vertex $v \in V$, $n+1$ additional vertices $v^1, v^2, \ldots, v^{n+1}$ and adding $(n+1)$ additional edges $\{v, v^1\}, \{v, v^2\}, \ldots \{v, v^{n+1}\}$ to the graph G.

We can establish the following claims without much difficulty.

Claim 1 A vertex cover $S' \subseteq V'$ of G' is a minimal VC iff (a) for $v \in S' \cap V$, $v^i \notin S'$, for any $1 \leq i \leq n+1$, and (b) for $v \in V - S'$, $\{v^1, v^2, \ldots v^{n+1}\} \subseteq S'$.

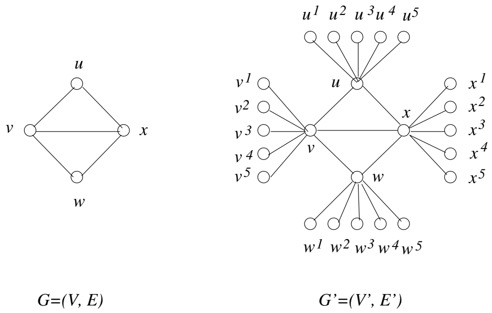

Fig. 2. An instance G of MAX-IS and the corresponding instance G' of MIN-MAX-VC

From the Claim 1 it follows that for any minimal VC $S' \subseteq V'$ of G', there exists a set $S \subseteq V$ such that $S' = (V - S) \cup [\cup_{v \in S}\{v^1, v^2, \ldots, v^{n+1}\}]$.

Claim 2 Let S be a maximal IS of G. Then $S' = (V-S) \cup [\cup_{v \in S}\{v^1, v^2, \ldots, v^{n+1}\}]$ is a minimal VC of G'.

Claim 3 Let S' be a minimal VC in G'. If $V - S'$ is not a maximal IS of G, then there exists a minimal VC S'' of G' such that $V - S''$ is a maximal IS of G and moreover,
(a) $|S''| > |S'|$
(b) $|V - S''| > |V - S'|$ and
(c) $|S''| = n(|V - S''| + 1)$.
Proof. Note that, for any VC S' of G', $V \cap S'$ is a VC of G. Hence $V - S' = V - (V \cap S')$ is an independent set of G. Let S' be a minimal VC of G' for which $V - S'$ is not a maximal independent set of G. Then we can always extend $(V - S')$ to a unique maximal IS S of G (in polynomial time) by introducing vertices of G one by one in the order $v_1, v_2, \ldots, v_n$ while mentaining the independence property. Hence $S \supset (V - S')$. By Claim 2, $S'' = (V - S) \cup [\cup_{v \in S}\{v^1, v^2, \ldots, v^{n+1}\}]$ is a minimal VC of G' and $|S''| = n(|S| + 1)$. Now we show that $S = V - S''$. For this first note that $S \subseteq V$ as S is a maximal independent set of G. Next, let $u \in S$, then from the definition of S'' it follows that $u \notin S''$, so $u \in V - S''$. Hence $S \subseteq V - S''$. Also, if $u \in V - S''$, then $u \notin S''$, i.e. $u \notin V - S$, so $u \in S$. Hence $S \supseteq V - S''$. Thus $S = V - S''$. From this it follows that $V - S''$ is a maximal independent set of G.

From Claim 1, we have $|S'| = |V \cap S'| + (n+1)|V - (V \cap S')| = n(n+1) - n|V \cap S'| = n + n|V - S'| = n(|V - S'| + 1)$. Since $|S| > |V - S''|$, it follows that $|S''| > |S'|$. Also (b) and (c) follow from the fact that $S = V - S''$. □

Claim 4 $S \subseteq V$ is a maximum IS of G iff $S' = (V-S) \cup [\cup_{v \in S}\{v^1, v^2, \ldots, v^{n+1}\}]$ is a maximum minimal VC of G'.
Proof. Let S be a maximum IS of G. By Claim 2, S' is a minimal VC of G'. If S' is not a maximum minimal VC of G', then using Claim 3 there exists a minimal VC S'' of G' such that $|S''| > |S'|$, $S = V - S''$ is a maximal IS of G and $|S''| = n(|S| + 1)$. As $|S'| < |S''|$, $|S'| = n + n|S|$ and $|S''| = n + n|\overline{S}|$, it follows

that $|S| < |\overline{S}|$, which is a contradiction. Hence S' is a maximum cardinality minimal VC in G.

Let S' be a maximum minimal VC of G'. Then by Claim 3, $S = V - S'$ is a maximal IS of G and $|S'| = n(|S| + 1)$. We claim that S is a maximum IS in G. Suppose there exists a maximal IS $S^* \subseteq V$ of G with $|S^*| > |S|$. By Claim 2, $\hat{S} = (V - S^*) \cup [\cup_{v \in S^*}\{v^1, v^2, \ldots, v^{n+1}\}]$ is a minimal VC in G' and $|\hat{S}| = n(|S^*| + 1)$. Since $|S^*| > |S|$, it follows that $|\hat{S}| > |S'|$, which is a contradiction. Hence, S is a maximum IS of G. □

Let $\alpha(G)$ denote the independence number and $\beta(G)$ denote the size of a maximum minimal VC in G. Hence, from Claim 4, we have $\beta(G') = n(\alpha(G)+1)$. Now let S' be any minimal VC of G'. If $V - S'$ is a maximal IS of G then to S' we associate $S = V - S'$ as the feasible solution of MAX-IS for G. If $V - S'$ is not a maximal IS of G then let S'' be the minimal VC of G' corresponding to S' as in Claim 3, so that $S = V - S''$ is a maximal IS of G and $|S'| < |S''| = n(|S| + 1)$. To this minimal VC S' of G' we associate S as the feasible solution of MAX-IS for G. Hence for any minimal VC S' of G' we have

$$\begin{aligned}
\frac{\alpha(G)}{|S|} &= \frac{n\alpha(G)}{n|S|} = \frac{\beta(G') - n}{|S''| - n} = \frac{\beta(G')}{|S''| - n} - \frac{n}{|S''| - n} \\
&= \frac{\beta(G')}{|S''|} \cdot \frac{|S''|}{|S''| - n} - \frac{1}{|S|} = \frac{\beta(G')}{|S''|} \cdot \frac{n(|S| + n)}{n|S|} - \frac{1}{|S|} \\
&= \frac{\beta(G')}{|S''|} + \frac{1}{|S|}(\frac{\beta(G')}{|S''|} - 1) \\
&\le \frac{\beta(G')}{|S''|} + \frac{\beta(G')}{|S''|} - 1 \qquad (\text{since } \frac{\beta(G')}{|S''|} \ge 1 \text{ and } |S| \ge 1) \\
&< 2\frac{\beta(G')}{|S''|} \le 2\frac{\beta(G')}{|S'|}
\end{aligned}$$

Let N be the number of vertices in G'. Since $N = n^2 + 2n$ and $N \le 2n^2$, for $n > 2$. Now, for any $\epsilon > 0$, $n^{1-\epsilon} \ge \frac{N^{\frac{1}{2}(1-\epsilon)}}{2^{\frac{1}{2}(1-\epsilon)}} \ge \frac{1}{2}N^{\frac{1}{2}(1-\epsilon)} \cdot 2^{\frac{1}{2}+\frac{\epsilon}{2}} \ge \frac{1}{2}N^{\frac{1}{2}(1-\epsilon)}$, and $n^{\frac{1}{2}-\epsilon} \ge \frac{1}{2}N^{\frac{1}{4}(1-2\epsilon)}$. Hence by, Theorem 2, the result follows. □

Regarding MAX-MIN-FVS, we have similar results.

Theorem 4. *Unless NP=ZPP (respectively P=NP), for any $\epsilon > 0$, there exists no polynomial time algorithm to approximate MAX-MIN-FVS within a factor of $\frac{1}{4}n^{\frac{1}{2}-\epsilon}$ (respectively $\frac{1}{4}n^{\frac{1}{4}-\epsilon}$), where n is the number of vertices in an instance.*

Proof. (Outline) We prove this by a reduction from MAX-MIN-VC to MAX-MIN-FVS as follows.

Let $G = (V, E)$ be a graph (an instance of MAX-MIN-VC). Construct an instance $G' = (V', A')$ of MAX-MIN-FVS from G with $V' = \cup_{v_i \in V}\{v_i^1, v_i^2\}$ and $A' = [\cup_{v_i \in V}\{(v_i^1, v_i^2)\}] \cup [\cup_{\{v_i, v_j\} \in E}\{(v_i^2, v_j^1), (v_j^2, v_i^1)\}]$. In other words, for each $v_i \in V$, G' has 2 vertices v_i^1, v_i^2 and an arcs (v_i^1, v_i^2). Also for each $\{v_i, v_j\} \in E$ G' has (v_i^2, v_j^1) and (v_j^2, v_i^1). Hence, G' has $2n$ vertices and $n + 2m$ arcs.

We can easily establish the following claims.

Claim 1 For any $C \subseteq V$,
(1) C is a VC of G iff $F = \{v_i^1 | v_i \in C\}$ is an FVS of G'.
(2) C is a minimal VC of G iff F is a minimal FVS of G'.
Claim 2 Let F be any minimal FVS of G'. Then
(1) for any $v_i \in V$, $F \cap \{v_i^1, v_i^2\}$ is either empty or singleton.
(2) for any $v_i \in V$ such that $F \cap \{v_i^1, v_i^2\} \neq \phi$, $F' = F - \{v_i^1, v_i^2\} + v_i^1$ is also a minimal FVS of G'.
(3) There is a minimal FVS F' of G' such that $|F'| = |F|$ and $F' = \{v_i^1 | v_i \in C\}$ for some minimal VC C of G such that $|C| = |F'|$.

Now let F_o be a maximum minimal FVS of G' and F be any minimal FVS of G'. By Claim 2, without loss of generality we can assume that every vertex in F_o (respectively, in F) is v_i^1 for some $v_i \in V$. Also by Claim 2, $C_o = \{v_i | v_i^1 \in F_o\}$, (respectively, $C = \{v_i | v_i^1 \in F\}$) is a maximum minimal VC (respectively, mininal VC) of G, and $|C_o| = |F_o|$ (respectively, $|C| = |F|$). Hence $\frac{|C_o|}{|C|} = \frac{|F_o|}{|F|}$.

Let $N = |V'|$. Then $N = 2n$. Now $\frac{1}{2} n^{\frac{1}{2}-\epsilon} = \frac{1}{2} \frac{(2n)^{\frac{1}{2}-\epsilon}}{2^{\frac{1}{2}-\epsilon}} = \frac{1}{4} N^{\frac{1}{2}-\epsilon} \cdot 2^{\frac{1}{2}+\epsilon} \geq \frac{1}{4} N^{\frac{1}{2}-\epsilon}$. Hence by, Theorem 3, the result follows. □

4 Hardness Results for Bounded Degree Digraphs

We know that MIN-FAS is APX-hard [14] and MAX-SUBDAG is APX-complete [17] for general digraphs. In this section, we show that these problems remain APX-hard even for k-total-regular digraphs for all $k \geq 4$. We also show that MIN-MAX-SUBDAG (respectively, MAX-MIN-VC) is APX-hard for digraphs of maximum total degree 12 (respectively, graphs of maximum degree 5). Regarding MIN-FAS, we first prove the following.

Lemma 1. *MIN-FAS-k $\leq_L$ MIN-FAS-($k+1$), for all $k \geq 1$.*

Proof. We construct in polynomial time, from a k-total-regular digraph $G = (V, A)$, a $(k+1)$-total-regular digraph $G' = (V', A')$ where $V' = V^1 \cup V^2$ where $V^i = \{v^i | v \in V\}$ for $i = 1, 2$ and $A' = A^1 \cup A^2 \cup B$ where $A^i = \{(u^i, v^i) | (u, v) \in A\}$ for $i = 1, 2$ and $B = \{(v^1, v^2) | v \in V\}$. From a minimal FAS S' of G' construct a minimal FAS S of G as follows: $S = \{(u, v) | (u^1, v^1) \in S^1\}$ where without loss of generality we assume that $S' = S^1 \cup S^2$ with S^1 and S^2 are minimal FASs of $G^1 = (V^1, A^1)$ and $G^2 = (V^2, A^2)$ respectively and $|S^1| \leq |S^2|$. It is easy to see that, if S'_o is a minimum FAS of G', then the corresponding S_o is a minimum FAS of G and $|S'_o| = 2|S_o|$. Further, for any minimal FAS $S' = S^1 \cup S^2$ of G', with $|S^1| \leq |S^2|$, $|S'| - |S'_o| = |S'| + |S_o| - 2|S_o| \geq 2(|S'| - |S_o|)$ so that $|S| - |S_o| \leq \frac{1}{2}(|S'| - |S'_o|)$. Thus, the two inequalities of L-reduction hold with $\alpha = 1$ and $\beta = \frac{1}{2}$. □

We now have the following.

Theorem 5. *MIN-FAS-k is APX-hard for all $k \geq 4$.*

Proof. (Outline) By Lemma 1, it is enough to show that MIN-FAS-4 is APX-hard. For this we show that MIN-VC-3 $\leq_L$ MIN-FAS-4.

We construct in polynomial time, from any 3-regular graph $G = (V, E)$ a 4-total-regular digraph $G' = (V', A')$ as defined in the proof of Theorem 4. For any FAS F of G', we associate a VC C of G defined as $C = \{v|$ either $(u^2, v^1) \in F$ or $(v^1, v^2) \in F\}$.

Further, C is a VC of G with $|C| \leq |F|$. For every edge $\{u, v\} \in E$, as $(u^1, u^2, v^1, v^2, u^1)$ is a cycle in G', F must contain at least one arc from this cycle, and so, C must contain either u or v. Hence, C is a VC of G, and by the construction of C from F, $|C| \leq |F|$.

Also, it can be easily shown that if F_o is a minimum FAS of G', then the associated VC C_o of G is a minimum VC of G and $|F_o| = |C_o|$, and for any FAS F of G', $|C| - |C_o| \leq |F| - |F_o|$. So the transformation from G to G' is an L-reduction with $\alpha = 1$ and $\beta = 1$. □

Similarly, for MAX-SUBDAG, we first prove the following.

Lemma 2. *MAX-SUBDAG-k* $\leq_L$ *MAX-SUBDAG-(k+1).*

Proof. Similar to the proof of Lemma 1. □

We now prove the following.

Theorem 6. *MAX-SUBDAG-k is APX-complete for any* $k \geq 4$.

Proof. By Lemma 2, it is enough to show that MAX-SUBDAG-4 is APX-hard. For this we show that MIN-VC-3 $\leq_L$ MAX-SUBDAG-4 and the reduction given in the proof of Theorem 5 is in fact an L-reduction from MIN-VC-3 to MAX-SUBDAG-4 with $\alpha = 1$ and $\beta = 1$. □

Regarding MIN-MAX-SUBDAG, we have the following easy theorem.

Theorem 7. *MIN-MAX-SUBDAG≤12 is APX-hard.*

Proof. In the proof of Theorem 1, we constructed an instance G' of MIN-MAX-SUBDAG from an instance G of MAX-SUBDAG in such a way that if G is 4-regular then, every vertex in G' is of total degree at most 12. Since MAX-SUBDAG-4 is APX-complete, the result follows. □

Next we shall consider MAX-MIN-VC. First we have the following two simple lemmas.

Lemma 3. *For any 3-regular graph* $G = (V, E)$ *and any maximal IS* I *in* G, $|I| \geq \frac{1}{4}|V|$.

Lemma 4. *MAX-MIN-VC is* k*-approximable for graphs of maximum degree* k, $k \geq 1$*, and having no isolated vertex.*

Proof. Any minimal VC for such a graph is k-approximable. □

Now we have

Theorem 8. *MAX-MIN-VC≤5 is APX-complete.*

Proof. Since MAX-MIN-VC is in class APX for bounded degree graphs (Lemma 4) and MAX-IS-3 is APX-complete [1], it is enough to show that MAX-IS-3 $\leq_L$ MAX-MIN-VC$\leq$5.

Let $G = (V, E)$ be a 3-regular graph. From G construct $G' = (V', E')$ of degree at most 5 as follows: $V' = V \cup [\cup_{v \in V} \{v^1, v^2\}]$ and $E' = E \cup [\cup_{v \in v} \{\{v, v^1\}, \{v, v^2\}\}]$.

By using the arguments given in the proof of Theorem 3, it can be proved that any minimal VC C of G' is of the form $C = (V - I) \cup [\cup_{v \in I} \{v^1, v^2\}]$, for some IS I of G where $I = V - (C \cap V)$ and $|C| = |I| + n$. Also, C_o is a maximum minimal VC of G' iff the associated I_o is a maximum IS of G, with $|C_o| = |I_o| + n$.

Now, $|C_o| = |I_o| + n \leq |I_o| + 4|I_o| = 5|I_o|$ (by Lemma 3), so that, the first inequality of L-reduction holds with $\alpha = 5$. Next, for any minimal VC C of G', $|C_o| - |C| = |I_o| + n - |I| - n = |I_o| - |I|$, so that, the second inequality of L-reduction holds with $\beta = 1$. □

Theorem 9. *MAX-MIN-FVS≤10 is APX-hard.*

Proof. In the proof of Theorem 4, we constructed an instance G' of MAX-MIN-FVS from an instance G of MAX-MIN-VC in such a way that if G is of degree at most 5, then G' is of total-degree at most 10. Since MAX-MIN-VC$\leq$ 5 is APX-complete it follows that MAX-MIN-FVS$\leq$10 is APX-hard. □

5 Hardness Results for Bounded Degree Graphs

In this section we establish APX-hardness of MIN-FVS and MAX-MIN-FVS for certain restricted class of undirected graphs. Regarding MIN-FVS, it is known that it can be solved in polynomial time for all graphs of maximum degree 3 [22], but it is not known whether MIN-FVS is NP-complete for graphs of maximum degree 4 or 5. However, it is easy to show that [8] MIN-W-FVS$\leq$4 is NP-complete and also APX-complete.

Next we show that MIN-FVS-6 is APX-complete.

Theorem 10. *MIN-FVS-6 is APX-complete.*

Proof. (Outline) As MIN-FVS is in class APX [3], it is enough to show that MIN-FVS-6 is APX-hard. Towards this we will show that MIN-VC-3 $\leq_L$ MIN-FVS-6.

Let $G = (V, E)$ be a 3-regular graph. From G construct a 6-regular graph $G' = (V', E')$ as follows: For every edge $\{v_i, v_j\} \in E$, let $V_{ij} = \{v_{ij}^1, v_{ij}^2, v_{ij}^3, v_{ij}^4, v_{ij}^5, v_{ij}^6, v_{ij}^7\}$ be the set of seven new vertices and $H_{ij} = (V_{ij}, E_{ij})$ be the graph obtained from the complete graph on V_{ij} by removing the edge $\{v_{ij}^1, v_{ij}^7\}$. Now $V' = V \cup [\cup_{\{v_i, v_j\} \in E} V_{ij}]$ and $E' = E \cup_{\{v_i, v_j\} \in E} [E_{ij} \cup \{\{v_i, v_{ij}^1\}, \{v_{ij}^7, v_j\}\}]$, see Figure 3. Clearly G' is 6-regular.

Let F be an FVS of G'. Then F contains at least 4 vertices from V_{ij}. The following claims can be easily established.

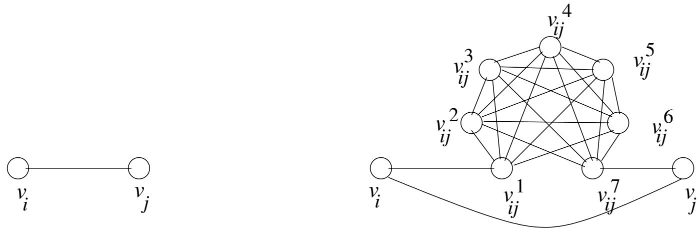

Fig. 3. An edge $\{v_i, v_j\} \in E$ and corresponding subgraph in G'.

Claim 1 Let F be any FVS of G' containing exactly 4 vertices from V_{ij} for some $\{v_i, v_j\} \in E$. Then F must contain either v_i or v_j.

To an FVS F of G', we associate the set C of vertices in G defined as $C = (F \cap V) \cup \{v_i \mid |F \cap V_{ij}| \geq 5 \text{ and } i < j\}$.

Claim 2 C is a VC of G and $|F| \geq |C| + 4|E| = |C| + 6n$.

Proof. If C is not a VC of G, then there exists $\{v_i, v_j\} \in E$ such that $C \cap \{v_i, v_j\} = \phi$. By the definition of C, it follows that $|F \cap V_{ij}| \leq 4$ and $F \cap \{v_i, v_j\} = \phi$. If $|F \cap V_{ij}| < 4$, then F is not an FVS of G', so $|F \cap V_{ij}| = 4$. By Claim 1, F must contain either v_i or v_j. Otherwise F can not be an FVS of G'. This contradicts that $F \cap \{v_i, v_j\} = \phi$. Hence, C is a VC of G.

Now $|F| = 4|E| + |F \cap V| + |\{v_i \mid |F \cap V_{ij}| \geq 5, i < j\}|$, as F contains at least 4 vertices from V_{ij} for each $\{v_i, v_j\} \in E$, and for the edges $\{v_i, v_j\} \in E$ such that $|F \cap V_{ij}| \geq 5$, F contains at least one more vertex from V_{ij} in addition to 4 vertices already considered. Hence, $|F| \geq |C| + 4|E| = |C| + 6n$ as G is a 3-regular and $|E| = \frac{3}{2}n$. □

Claim 3 For any VC C in G, the set $F = C \cup \{v_{ij}^2, v_{ij}^3, v_{ij}^4, v_{ij}^5\} | \{v_i, v_j\} \in E\}$ is an FVS of G' such that $C = F \cap V$ and $|F| = |C| + 6n$.

Claim 4 If F_o is a minimum FVS of G', then the associated set C_o is a minimum VC of G and $|F_o| = |C_o| + n$.

Now, $|F_o| = |C_o| + 6n \leq |C_o| + 12|C_o| = 13|C_o|$ (as any VC in a 3-regular graph contains at least $\frac{n}{2}$ vertices). Hence, the first inequality of L-reduction holds with $\alpha = 13$. Next, for any FVS F of G', $|F| - |F_o| \geq |C| + 6n - |C_o| - 6n = |C| - |C_o|$. So the second inequality of L-reduction holds with $\beta = 1$. □

Next we shall consider MAX-MIN-FVS. Before that we note the following.

Lemma 5. *For any FVS F of a 6-regular graph $G = (V, E)$, $|F| > \frac{2}{5}n$.*

Finally, we have,

Theorem 11. *MAX-MIN-FVS≤9 is APX-hard.*

Proof. Let $G = (V, E)$ be a 6-regular graph. Construct a graph $G' = (V', E')$ of degree at most 9 as follows: $V' = V \cup \{v^1, v^2, v^3 | v \in V\}$ and $E' = E \cup \{(v, v^1), (v, v^2), (v, v^3), (v^1, v^2), (v^1, v^3) \mid v \in V\}$ (see Figure 4). Let F be any minimal FVS of G'. Note that, for any $v \in V - F$, F contains either v^1 or both v^2 and v^3. Further, if $v \in F \cap V$, then $F \cap \{v^1, v^2, v^3\} = \phi$. To F we associate

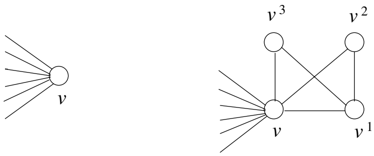

Fig. 4. a vertex v in G and its corresponding neighbors in G'

$C = F \cap V$, which is clearly an FVS of G. Note that $|F| \leq |C| + 2|V - F| = |C| + 2|V - C| = 2n - |C|$.

Let F_o be a maximum minimal FVS of G'. Then $|F_o| = 2n - |C_o|$ where $C_o = F_o \cap V$. For, if $|F_o| < 2n - |C_o|$, then $F = C_o \cup \{\{v^2, v^3\}|v \in V - C_o\}$ is a minimal FVS of G' with $|F| = 2n - |C_o| > |F_o|$ contradicting our assumption that F_o is a maximum minimal FVS of G'. Also note that C_o is a minimum FVS of G.

Now, $|F_o| = 2n - |C_o| < 5|C_o| - |C_o| = 4|C_o|$, (by previous Lemma). So, the first inequality of L-reduction holds with $\alpha = 4$. Next, for any minimal FVS F of G', $|F_o| - |F| \geq 2n - |C_o| - 2n + |C| = |C| - |C_o|$. So the second inequality of L-reduction holds with $\beta = 1$. □

6 Concluding Remarks

In this paper we have established hardness results for several NP-optimization problems related to MINLOP. These problems are variations or generalizations of well-known NP-optimization problems on graphs/digraphs. While for MAX-MIN-VC and MAX-MIN-FVS we have established strong results like those of Hȧstad [12] concerning MAX-IS and MAX-CLIQUE, for others we have just shown them to be APX-hard. Whether strong results about hardness of approximating such problems can be obtained is worth investigating. Despite such negative results, efforts may be made to obtain useful positive results giving efficient algorithms which may be $f(n)$-approximate for suitable function $f(n)$. Also, we do not have any results about MAX-MIN-FAS problem similar to MAX-MIN-FVS. These and other relavent issues concerning these problems are being pursued.

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