Market-Based Interest Rates: Deterministic Volatility Case *

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Abstract. Central banks issue often many kinds of bonds to guide their benchmark interest rates. Their market data are thought of to reflect current state of the countries financial system. Then at least how much data is needed? Based on the framework of HJM model, We prove that the amount of the data needed is related to the form of the volatility function of forward rates, and then the initial forward rate curve is not essential.

1 Introduction

In some emergent financial markets, benchmark interest rates are policy-based and determined by their governments. While their central banks issue often many kinds of interest rate instruments to make their benchmark interest rates market-based.

Spot rate models are popular in theory and in practice. However, spot rates are not observable on the market. So we have to recalibrate the term structure of interest rate, given initial market data. Then forward rate models, such as Heath-Jarrow-Morton model(HJMM)[1], arise, which eliminate the disadvantages of spot rate models and fit initial forward rate curve naturally. However, there is one natural problem, that is, finite amount of data, but one initial curve, is presented in the market. Does the initial curve is essential for curve fitting problem, or finite amount of data is sufficient? This problem can be also stated as follows: can some group of market data be treated as state variable as forward rate curve does?

From a geometric view of the interest rate theory, [2] proved that in deterministic volatility models of HJM framework, forward rate curves can be represented as a function of a state variable of finite dimension. [3] presented detailed steps to construct this function. [4] and [5] discussed some cases of stochastic volatility and proved similar results as deterministic cases. This paper will prove that forward rate curve is not essential to describe the system state, while several distinct groups of market data can represent the same system state.

2 Musiela Parameterized HJM Model

Zero-coupon bonds are the representative interest rate instruments in the bond market, which is sold in the price less than their face values and bought back in the price of face values, and no interests is paid in the duration of their existence. Without loss of generality, their face values are assumed to be one unit.

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First of all, some important definitions are given as follows.

$P(t,x)$: the price at time t of a zero-coupon bond maturing at time $t+x$, $t,x \geq 0$;

$r(t,x)$: the forward rate at time t of a riskless loan which is contracted at time $t+x$ and matured instantaneously, $t,x \geq 0$;

$R(t)$: spot rate at time t, $t \geq 0$.

We assume that the market is frictionless and the bonds are perfectly divisible. There are some explicit relationships among the bonds prices, forward rates and spot rates:

$$r(t,x) = -\frac{\partial \log P(t,x)}{\partial x} \tag{1}$$

$$P(t,x) = \exp\{-\int_0^x r(t,s)ds\} \tag{2}$$

$$R(t) = r(t,0) \tag{3}$$

We assume that the bond market is a filtered probability space $(\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t\geq 0})$, where Ω is the sample space, Q is the martingale measure defined on Ω, $\mathcal{F}$ is the filtration, and $\{\mathcal{F}_t\}$ is the time t filtration. Let $W(t)$ be a m-dimensional standard Wiener process. Assume that $R(t)$ is $\{\mathcal{F}_t\}$-measurable, that is , it is known at time t and $R(t)$ is a one-to-one mapping between $\{\mathcal{F}_t\}$ and its value set. In probability theory, $\{\mathcal{F}_t\}$ is the information set of the probability system at time t and its element means system event. Above all, the value of stochastic variable $R(t)$ reflects the system state. $R(t)$ is called the state variable of the system, and the evolution of the process $\{R(t), \forall t \geq 0\}$ reflects the transformation of the system state in the state space. The forward rates $r_t(\cdot)$ and bond prices $P(t,\cdot)$ are also state variables ([6]). Then we can use several distinct state variables to reflect the same system state and there must be some homeomorphic mapping between corresponding state spaces.

In the HJM model, the evolution of the forward rates is a stochastic differential equation:

$$\begin{aligned} dr(t,x) &= \mu(t,x)dt + \sigma(t,x)dW_t, \\ r(0,x) &= r_0^*(x), \forall x \geq 0. \end{aligned} \tag{4}$$

where, $\mu(t,x)$ and $\sigma(t,x)$ are 1-dimensional drift coefficient and d-dimensional volatility coefficient, respectively, and $r_0^*(\cdot)$ is the initial forward curve. Under the arbitrage-free condition, HJM proves that the drift coefficient must satisfy

$$\mu(t,x) = \frac{\partial}{\partial x} r(t,x) + \sigma(t,x)\int_0^x \sigma(t,s)ds. \tag{5}$$

So, given the initial forward rate curve $r_0^*(\cdot)$ and particular forms of volatility coefficient $\sigma(t,x)$, we can get the forward rates in the future with help of dynamics of the forward rate.

3 Finite-Dimensional Realization (FDR)

Given real numbers $\beta > 1$ and $\gamma > 0$, let $\mathcal{H}_{\beta,\gamma}$ be the Hilbert space of all differential functions $r: \mathbf{R}_+ \rightarrow \mathbf{R}$ satisfying the norm condition

$$\| r \|_\gamma < \infty.$$

Here the norm is defined by

$$\| r \|_{\beta,\gamma}^2 = \sum_{n=0}^{\infty} \beta^{-n} \int_0^{\infty} (\frac{d^n r(x)}{dx^n})^2 \exp\{-\gamma x\} dx.$$

Because the following results are independent of the values of β, γ, we write $\mathcal{H}$ for $\mathcal{H}_\gamma$. Surely the term forward rate curve simply refers to a point in $\mathcal{H}$.

We assume that the volatility function σ has the form

$$\sigma : \mathcal{H} \times \mathbf{R}_+ \to \mathbf{R}^m.$$

Here, each component of $\sigma(r, x) = (\sigma_1(r, x), \ldots, \sigma_m(r, x))$ is a functional of the infinite dimensional r-variable, and a function of the real variable x. We can rewrite the forward rate curve $\{r(t, x), \forall x \geq 0\}$ at time t as the form of r_t, which is a function-valued stochastic variable. Thus r_t satisfies the following Stratonovich differential equation

$$\begin{aligned} dr_t &= \mu(r_t)dt + \sigma(r_t) \circ dW_t, \\ r_0 &= r_0^*, \end{aligned} \tag{6}$$

where

$$\mu(r) = \frac{\partial}{\partial x} r(t, x) + \sigma(t, x) \int_0^x \sigma(t, s) ds - \frac{1}{2} \sigma_r^{'}[\sigma(r)], \tag{7}$$

and $\sigma_r^{'}$ is a Frechet derivative of the volatility function $\sigma(r)$ with respect to r, $\circ$ means the Stratonovich integral.

Due to the special properties of Stratonovich integral, there is no second-order items here and then we can rewrite the above Stratonovich stochastic differential equation by

$$\frac{dr_t}{dt} = \mu(r_t) + \sigma(r_t) \cdot \nu_t, \tag{8}$$

where ν_t denotes noise.

Given the dynamics of the above forward rate curves, and a special form of the functional σ, we refer $\{\mu, \sigma\}_{LA}$ to the Lie algebra generated by $\{\mu, \sigma\}$.

Assume f is a smooth mapping on the space $\mathcal{H}$, and x is a fixed point in $\mathcal{H}$, we write the solution of the following equation as $x_t = \exp\{ft\}x$:

$$\frac{dx_t}{dt} = f(x_t), \;\; x_0 = x. \tag{9}$$

When the dimension of $\{\mu, \sigma\}_{LA} = d < \infty$ is finite, this Lie algebra can be spanned by a set of d smooth mappings $f_1, \ldots, f_d$. Let

$$G(z) = \exp\{f_1 z_1\} \cdots \exp\{f_d z_d\} r_0^*, \; z = (z_1, \ldots, z_d) \in \mathbf{R}^d \tag{10}$$

[2] proves that when the Lie algebra $\{\mu, \sigma\}_{LA}$ has a finite dimension d, there exists an invariant submanifold ς near the initial point r_0^*, such that $\{\mu, \sigma\}_{LA}$ is contained in a tangent space of the manifold ς , and r_t can be realized as

$$\begin{cases} r_t = G(z_t), \\ dz_t = a(z_t)dt + b(z_t) \circ dW_t, \\ z_0 = z_0^*, \end{cases} \tag{11}$$

where $a(z_t), b(z_t)$ and z_0^* are respectively drift coefficient, volatility coefficient of the process z_t and its initial point, and G is a diffeomorphism. They can be deduced by the relationship G between r_t and z_t. In other words, the infinite-dimensional forward rate curve r_t can be realized as an invertible function G of a finite-dimensional diffusion process z_t, which is called the finite-dimensional realization of the forward rate curve r_t.

4 Minimal State Variables

The value of the forward rate curve r_t reflects the system state at time t. However, infinite amount of data is needed to obtain the forward rate curve, while finite amount of data is available in the market. Then a natural problem arises: Does there exist a set of finite data which is sufficient to reflect the system state?

4.1 Main Results

$\{G(z), \forall z \in \mathbf{R}^d\}$ is a parameterized form of the manifold ς, there exists an open neighborhood U of the point 0 in the space $\mathbf{R}^d$, and an open neighborhood V of the point r_0^* in the manifold ς, such that

$$V = G(U),$$

where $G^{-1}(V)$ denotes the coordinate system of the manifold ς at r_0^*, or G is a homeomorphic mapping from $\mathbf{R}^d$ to ς. The inverse mapping G^{-1} of G exists, and there exists one-to-one relationship between $\mathbf{R}^d$ and ς. It means that z_t is also a state variable as r_t does. We can refer to the stochastic variables r_t, z_t and $R(t)$ as homeomorphic mappings between the filtration $\mathcal{F}_t$ and $\mathbf{R}^\infty$($\mathbf{R}^d$,$\mathbf{R}^1$, respectively), their state spaces are of the same structures with infinite-dimension, d-dimension, and 1-dimension, respectively.

Theorem 1. The total forward rate curve is not essential to describe the current system state, while a set of market data with d dimension is sufficient.

Proof. A homeomorphic mapping is of full order, so the order of the homeomorphic mapping G^{-1} equals to the order of the space of z_t, which is exactly d.

Given a set of maturity $\{x_1, x_2, \cdots, x_d\}$, we have from equation (11) that

$$r_t(x_i) = G(x_i, z_t), \quad i = 1, 2, \cdots, d \tag{12}$$

The Jaccobi matrix of the function G has a same order as G, which equals to d. Then the above equations has a unique solution.

Above all, if we have d units of market forward rates $(r_t(x_1), r_t(x_2), \ldots, r_t(x_d))$, the value of z_t can be computed. It means that in the space of the forward rates, d units of separate points is sufficient to describe the current system state. □

Theorem 1 shows that the total forward rate curve is not essential to describe the current system state, we need only d units of market data, which is identical to the dimension of the Lie algebra $\{\mu, \sigma\}_{LA}$, and depend on the form of the volatility function σ.

Theorem 2. At time t, two sets of market data will generate the same state of system.

Proof. we have proved in theorem 1 that the function $G : \mathbf{R}^d \to \mathcal{H}$ is a homeomorphic mapping, which implies that there is a one-to-one relationship between the space

$\mathbf{R}^d$ and $\mathcal{H}$. From equation (11), we conclude that given a forward rate curve, there is only one unique value of z_t corresponding to it. On the basis of theorem 1, the same system state will be obtained, given any two sets of market data. □

When we choose a model for the real world, or the form of the volatility function, the dimension of the Lie algebra $\{\mu, \sigma\}_{LA}$ can be deduced with the help of the theory of algebra, then acquire the form of the function G, at last, choosing a set of d units market data, the current system state can be well described.

Proposition 4.4 in [2] shows that we can also directly choose d units data of the forward rates as the state of system. [2] also proves that, when $d \leq 2$, the forward rate process can be realized as a spot rate process. In this case, we can say that the market data reflect the benchmark interest rates of the central bank.

4.2 Construction of the State Variable

[2] gives the construction of the state variable z_t as follows.

(1) choose a collection $f_1, f_2, \ldots, f_d$ of smooth mapping which spans $\{\mu, \sigma\}_{LA}$;

(2) compute the form of $G(z_t) = \exp\{f_1 z_1\} \cdots \exp\{f_d z_d\} r_0^*$;

(3) Due to the Stratonovich differential equations of r_t and z_t, the following relationships hold,

$$G^{'} a = \mu, G^{'} b = \sigma. \tag{13}$$

Thus a, b can be from the above equations.

4.3 Identification of Error-Priced Market Data

When error-priced data exists in the market, we can use the following method to identify them based on the above results.

(1) When there is only one error data, from theorem 2, we arrange all market data available into three sets, and from equation (11), three units value of z_t is obtained, then the only one which differs from other two means that this set contains the error data. Second, arranging this set into two subsets, together with other true-priced data, we form two new sets of d units data, and find the set containing the error data. Go on until the error data is determined.

(2) The case of two error data: the method is similar to the above case.

As long as the market data available is enough,error data should be identified by using the above method.

5 Conclusion

From a geometric view of interest rate theory, we study the existence problem of minimal state variables by using some important results of FDR problem of [2]. In the framework of HJM model, we find that in the case of deterministic volatility, the system state can be well described based on the available information in the current market, and any set of market data generates the unique state of system.

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