Recognition of Digital Naive Planes and Polyhedrization

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Abstract. A digital naive plane may be seen as a repetition of (n, m)-cubes, set composed of $n \times m$ adjacent voxels or more generally sets of p voxels. In a previous works [VC99a], we have shown how to link the parameters of a naive plane to the different configurations of voxels sets by the construction of the associated Farey net. We propose an algorithm to recognize any set of coplanar voxels. This algorithm will be used for the polyhedrization of voxel objects. This is an original contribution offering a new method for digital plane recognition.

Keywords: Digital naive plane - Polyhedrization - Recognition - Equivalence classes

1 Introduction

The characterization of digital naive planes is now solved. Effectively, naive planes have been studied through their configurations of tricubes [Sch97,VC97], of (n, m)-cubes [VC99b] and connected or not connected voxels set [VC99a,Gér99]. The link between the normal equation of a plane and configuration of voxels set has been studied by the construction of the corresponding Farey net [VC99a].

We can find many references about the recognition of digital planes. Some algorithms were related to the construction of the convex hull of the studied voxels set [KS91,KR82]. Other approaches use linear programming[ST91], mean square approximation [BF94] or Fourier-Motzkin transform [FP99,FST96,Vee94].

The first algorithms entirely discrete were to recognize rectangular pieces of naive planes [Deb95,DRR94,VC99b]. In this paper, we propose an incremental algorithm to recognize any coplanar voxels set.

2 Definitions

Let a, b, c, r be four integers such as a, b, c are not null all together and verify $\gcd(a, b, c) = 1$.

G. Borgefors, I. Nyström, and G. Sanniti di Baja (Eds.): DGCI 2000, LNCS 1953, pp. 296–307, 2000.

The digital naive plane $\mathcal{P}(a, b, c, r)$, where (a, b, c) is its normal vector and r its translation parameter, is the set of points (x, y, z) in $\mathbb{Z}^3$ verifying:

$$0 \leq ax + by + cz + r < \max(|a|, |b|, |c|)$$

We will limit our study to naive planes $\mathcal{P}(a, b, c, r)$ in the 48th part of space such as $0 \leq a \leq b \leq c$ and $c \neq 0$. These planes are `functional` in $0xy$. For each point (x, y) in $\mathbb{Z}^2$, we have only one point (x, y, z) in $\mathbb{Z}^3$ belonging to the naive plane. Let us notice $f(a, b, c, r)$ the function from $\mathbb{Z}^2$ to $\mathbb{Z}$ defined by:

$$f(a, b, c, r)(u, v) = -\left[\frac{au + bv + r}{c}\right]$$

where $[w]$ denotes the integer part of the real number w, then $z = f(a, b, c, r)(x, y)$.

The points (x, y, z) of the naive plane $\mathcal{P}(a, b, c, r)$ which verify $ax + by + cz + r = 0$ (resp. $ax + by + cz + r = c - 1$) are the `lower` (resp. `upper`) `leaning points` of the naive plane.

3 Equivalence Class of a Voxels Set

Let n in $\mathbb{N}^*$ and $V = \{(i_1, j_1), \cdots, (i_n, j_n)\}$ a set of n points of $\mathbb{Z}^2$.

The cluster of voxels $S(a, b, c, r)(x, y)$ of the naive plane $\mathcal{P}(a, b, c, r)$ indexed by V and with origin (x, y) is defined by:

$$S(a, b, c, r)(x, y) = \bigcup_{q=1}^{n} \{(x + i_q, y + j_q, z + k_q) \mid (i_q, j_q) \in V,\ k_q = f(a, b, c, r)(x + i_q, y + j_q) - z\}$$

where $z = f(a, b, c, r)(x, y)$.

$S(a, b, c, r)(x, y)$ is the part of the naive plane going through the n voxels $(x + i_q, y + j_q, z + k_q)$, $q = 1, \cdots, n$.

Let (x_l, y_l, z_l) be a lower leaning point from $\mathcal{P}(a, b, c, r)$. It verifies $ax_l + by_l + cz_l + r = 0$. The point (x, y, z) can be written as $(x_l + u, y_l + v, z_l + w)$ with (u, v, w) in $\mathbb{Z}^3$. As it belongs to the naive plane $\mathcal{P}(a, b, c, r)$, it verifies $0 \leq a(x_l + u) + b(y_l + v) + c(z_l + w) + r < c$. Let us notice r' equal to $au + bv + cw$. For $q = 1, \cdots, n$, the point $(x + i_q, y + j_q, z + k_q)$ verifies $0 \leq ai_q + bj_q + ck_q + r' < c$. Consequently, $S(a, b, c, r)(x, y)$ can be written as:

$$S(a, b, c, r)(x, y) = \{(x, y, z)\} \oplus S(a, b, c, r')(0, 0)$$

where $A \oplus B$ is the Minkowski sum between the sets A and B.

All real planes for which the discretization by the *object boundary quantization* method on the set $\{(x, y)\} \oplus V$ is the set $S(a, b, c, r)(x, y)$ have to go through the point (x_l, y_l, z_l). Moreover on the point (x, y) the discretization must be the voxel (x, y, z). A first equivalence class of $S(a, b, c, r)(x, y)$ is the set of parameters (α, β) with $0 \leq \alpha \leq \beta \leq 1$ of the real plane $\alpha(x - x_l) + \beta(y - y_l) + z - z_l = 0$ verifying $0 \leq \alpha u + \beta v + w < 1$. We notice this equivalence class by the set:

$$\overline{S}(r)(x,y) = \bigcap_{q=1}^{n} \{(\alpha,\beta) \mid 0 \le \alpha \le \beta \le 1,$$
$$(x+i_q, y+j_q, z+k_q) \in S(a,b,c,r)(x,y), -1 < i_q\alpha + j_q\beta + k_q < 1\}$$

For each integer point (i,j), we have $f(a,b,c,r)(x_l+i, y_l+j) = f(a,b,c,r)(x_l,y_l) + f(a,b,c,0)(i,j)$. For $q = 1,\cdots,n$, the integer k_q satisfies $k_q = f(a,b,c,0)(u+i_q, v+j_q) - f(a,b,c,0)(u,v)$. So $\overline{S}(r)(x,y)$ is equal to $\overline{S}(0)(u,v)$.

Let $\mathcal{D}(i,j,k)$ be the line in the parametric space $W = \{(\alpha,\beta),\ 0 \le \alpha \le \beta \le 1\}$ with equation $i\alpha + j\beta + k = 0$. Let $B(i,j,k)$ be the open-band of this space limited by the two parallel lines $\mathcal{D}(i,j,k+1)$ and $\mathcal{D}(i,j,k-1)$.

The equivalence class becomes:

$$\overline{S}(r)(x,y) = W \bigcap \left(\bigcap_{q=1}^{n} B(i_q, j_q, k_q) \right)$$

In a previous work [VC99a], we proved that the voxels set $\mathcal{E}$ centered on the lower leaning point (x_l, y_l, z_l) and defined by:

$$\mathcal{E} = \bigcup_{q=1}^{n} S(a,b,c,r)(x_l - i_q, y_l - j_q)$$

is a complete system. It is representative of the different configurations of voxels sets defined on V which generate the naive planes with normal (a,b,c). The equivalence class $\overline{\mathcal{E}}$ of that set is the intersection of the open bands $B(i,j,k)$ for (i,j) belonging to $V \ominus V$ ($\ominus$ designs the Minkowski difference between two sets) and k verifying $k = f(a,b,c,r)(x_l+i, y_l+j) - f(a,b,c,r)(x_l,y_l)$. The equivalence classes of the different configurations appearing around leaning points split the space W in polygonal areas called *Farey net associated to voxels sets defined on V*.

Example 1. Let A be the voxels set of a naive plane defined on $V = \{(0,0), (1,-1), (2,0), (2,1)\}$ and illustrated in figure 1(a). The equivalence class $\overline{A}$ of that set is the intersection of the four bands $B(0,0,0)$, $B(1,-1,0)$, $B(2,0,-1)$ and $B(2,1,-2)$ (cf. Fig. 1(b)). Each rational point in that area corresponds to the parameters of a naive plane containing that configuration of voxels set. In figure 1(c), we have the Farey net associated to the voxels set defined on V.

Now if we look for real planes $\alpha x + \beta y + z + \gamma = 0$ with $0 \le \alpha \le \beta < 1$ for which the discretization by the object boundary quantization method on the point (x,y) is the voxel (x,y,z) then the parameter γ has to verify $\gamma = \gamma' - (\alpha x + \beta y + z)$ with $0 \le \gamma' < 1$. Moreover if the dicretization on $\{(x,y)\} \oplus V$ is the set $S(a,b,c,r)(x,y)$, the parameters of that planes belong to the set:

$$\overline{S}'(x,y) = \bigcap_{q=1}^{n} \{(\alpha,\beta,\gamma' - (\alpha x + \beta y + z)) \mid 0 \le \alpha \le \beta \le 1,\ 0 \le \gamma' < 1,$$
$$(x+i_q, y+j_q, z+k_q) \in S(a,b,c,r)(x,y),\ 0 \le i_q\alpha + j_q\beta + \gamma' + k_q < 1\}$$

This second equivalence class will be used in the recognition algorithm.

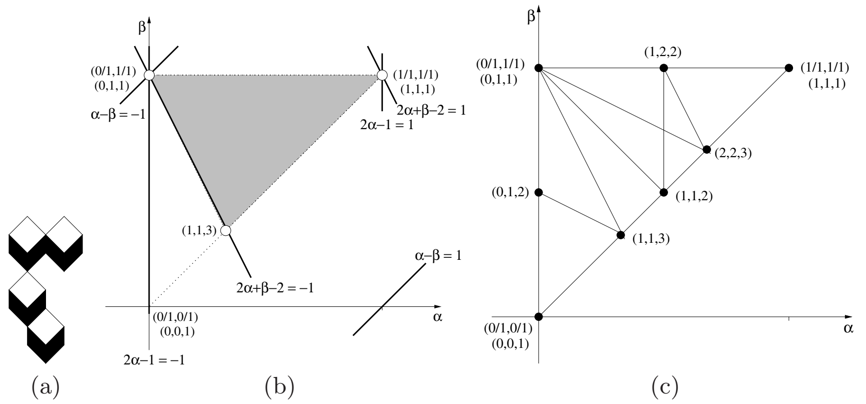


Fig. 1. (a) Set $A = \{(0,0,0),(1,-1,0),(2,0,-1),(2,1,-2)\}$; (b) Equivalence class of A; (c) Farey net associated to voxels set defined on $V = \{(0,0),(1,-1),(2,0),(2,1)\}$

4 Recognition Algorithm

Let $S = \{(x_q, y_q, z_q),\ q = 1, \cdots, n\}$ be a set of n voxels.

We are going to establish an incremental algorithm to identify the parameters of the naive planes going through the n points of S.

The naive planes solutions are the planes $\mathcal{P}(a, b, c, r - (ax_1 + by_1 + cz_1))$ for which the parameters $\left(\frac{a}{c}, \frac{b}{c}, \frac{r}{c}\right)$ belong to the set:

$$\overline{S} = \left\{(\alpha, \beta, \gamma) \in [0,1]^2 \times [0,1[\ |\ \forall q \in \{1, \cdots, n\} \quad 0 \leq i_q\alpha + j_q\beta + \gamma + k_q < 1\right\}$$

where (i_q, j_q, k_q) is defined as to be the integer points $(x_q - x_1, y_q - y_1, z_q - z_1)$ for $q = 1, \cdots, n$.

Let $\mathcal{B}_q$ in $\mathbb{N}^4$ be the set of vectors (a, b, c, r) such that the projection in the plane $(c = 1)$ are the vertices of the convex hull of the space containing the parameters (α, β, γ) of real planes for which the dicretization on the point (i_p, j_p) is the point (i_p, j_p, k_p) for p varying from 1 to q.

We are going to construct the sets $\mathcal{B}_q$ for $q = 1, \cdots, n$. The following algorithm gives at step q the set $\mathcal{B}_q$ or the empty set if there is no solution. In the first case, the solution with the minimal periodicity can correspond to a vertex of the convex-hull, the median point between the projection of two vertices of $\mathcal{B}_q$ or the median point of the area limited by the projection of the vectors of $\mathcal{B}_q$.

As the discretization of all real planes of the working space goes through the origin (here the origin is taken at point (x_1, y_1, z_1)), the algorithm starts with

$$\mathcal{B}_1 = \{(0,0,1,0),(0,1,1,0),(1,1,1,0),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$$

composed by the six vectors (a, b, c, r) such that the projection in the plane $(c = 1)$ are the vertices of the convex-hull limiting the solution space of (α, β, γ).

Algorithm at step q, $q \geq 2$:
We introduce the point (i_q, j_q, k_q).
Let L_q and L_q^+ the functions from $\mathbb{N}^4$ to $\mathbb{Z}$ defined by:

$$L_q(a, b, c, r) = ai_q + bj_q + ck_q + r$$
$$L_q^+(a, b, c, r) = L_q(a, b, c, r) - c$$

The naive plane of parameters (a, b, c, r) goes through the voxel (i_q, j_q, k_q) if and only if $0 \leq L_q(a, b, r, c) < c$. Consequently the vectors (a, b, c, r) of $\mathcal{B}_q$ verify $L_p(a, b, c, r) \geq 0$ and $L_p^+(a, b, c, r) \leq 0$ for $p = 1, \cdots, q$.

Initialization: $\mathcal{B}_q = \emptyset$.

For all the vectors V_i in $\mathcal{B}_{q-1}$, $i = 1, \cdots, \#(\mathcal{B}_{q-1})$ do:

1. Process

   Step 1 : If $L_q(V_i) \geq 0$ and $L_q^+(V_i) \leq 0$ then the projection of V_i is still on the convex hull of the domain solution. We insert V_i in $\mathcal{B}_q$. More particularly, if $L_q(V_i) = 0$ (resp. $L_q^+(V_i) = 0$) we can say that the voxel (i_q, j_q, k_q) (resp. $(i_q, j_q, k_q - 1)$) is a lower leaning point of the naive plane of parameters V_i.

   Step 2 : If $L_q(V_i) < 0$ (resp. $L_q^+(V_i) > 0$), we are going to search the point P such as $L_q(P) = 0$ (resp. $L_q^+(P) = 0$). To do that, we use an algorithm based on the notion of median point [Far16,Gra92].

   For `each vectors` V_j, $j > i$, `belonging to` $\mathcal{B}_{q-1}$ `and verifying` $L_q(V_j) > 0$ `(resp.` $L_q^+(V_j) < 0$`)` **do**
       $P_1 = V_i$ `and` $P_2 = V_j$
       While $L_q(P_1) + L_q(P_2) \neq 0$ `(resp.` $L_q^+(P_1) + L_q^+(P_2) \neq 0$`)` **do**
           if $L_q(P_1) + L_q(P_2)$ `and` $L_q(V_i)$ `(resp.` $L_q^+(P_1) + L_q^+(P_2)$ `and` $L_q^+(V_i)$`) have the same sign` **then**
               $P_1 = P_1 + P_2$ `and` $P_2 = V_j$
           else
               $P_1 = V_i$ `and` $P_2 = P_1 + P_2$
       End While
   End For
   `The solution is given by` $P = P_1 + P_2$.

   The point (i_q, j_q, k_q) (resp. $(i_q, j_q, k_q - 1)$) is a lower leaning point of the naive plane of parameters P.
   We insert the point P in $\mathcal{B}_q$.

2. Validation of $\mathcal{B}_q$

   (a) For each vector $V = (a, b, c, r)$ in $\mathcal{B}_q$ we verify if the projection in the plane $(c = 1)$ is a vertex of the convex hull. If V can be written as a combination of vectors of $\mathcal{B}_q$ then the projection of V is on the convex hull but it is not a vertex. So we suppress that point from the list.

(b) If $\#(\mathcal{B}_q) \leq 2$, there is no solution and we suppress all the vertices from the list.
(c) If $\#(\mathcal{B}_q) = 3$, we verify that the points $(a/c, b/c)$ corresponding to the vectors (a, b, c, r) of $\mathcal{B}_q$ are not alined otherwise we suppress the vertices from the list.

Example 2. We are going to illustrate this algorithm on an example. We want to know if the set of voxels in figure 2 belong to a naive plane of the studied 48th part of space. We start with a first point defined as the origin of the voxels

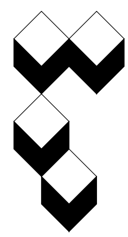


Fig. 2. Set of voxels to recognize

set (cf. Fig 3(a)). The parameters set of naive planes including the origin are the rational points (α, β, γ) contained in the domain limited by the projection $(a/c, b/c, r/c)$ of the vectors (a, b, c, r) of $\mathcal{B}_1$. The set $\mathcal{B}_1$ is composed by the six vectors:

$$\mathcal{B}_1 = \{(0,0,1,0),(0,1,1,0),(1,1,1,0),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$$

As it was previously mentioned, it is equivalent to say that parameters (α, β) belong to the area limited by the points $(a/c, b/c)$ (cf. Fig 3(b)). We introduce the second point $(1, -1, 0)$ (cf. Fig 4(a)). Let us compute the value $L_2(V)$ and $L_2^+(V)$ on the different vectors V from $\mathcal{B}_1$:

(a,b,c,r)	$L_2(a,b,c,r) = a-b+r$	$L_2^+(a,b,c,r) = a-b+r-c$
$(0,0,1,0)$	0	-1
$(0,1,1,0)$	-1	-2
$(1,1,1,0)$	0	-1
$(0,0,1,1)$	1	0
$(0,1,1,1)$	0	-1
$(1,1,1,1)$	1	0

The vectors $(0,0,1,0)$, $(1,1,1,0)$, $(0,0,1,1)$, $(0,1,1,1)$ and $(1,1,1,1)$ verify the property indiced by step 1 of the algorithm. Ve insert these vectors in $\mathcal{B}_2$. As $L_2(0,1,1,0) < 0$, we apply the algorithm presented in step 2. We introduce the new vectors $(0,1,2,1)$ and $(1,2,2,1)$. But $(0,1,2,1)$ is the sum of vectors

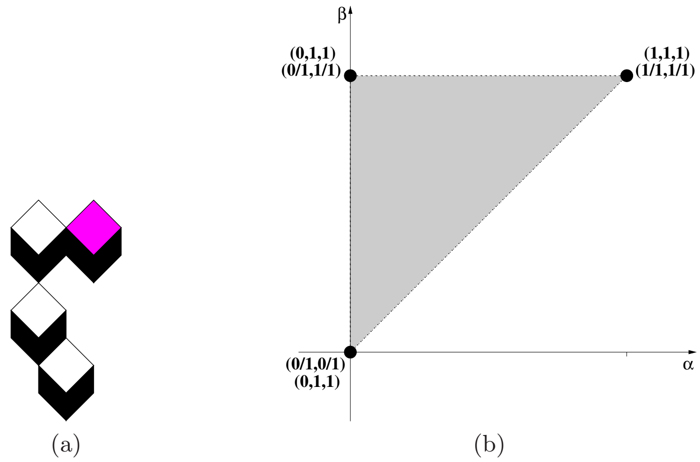


Fig. 3. (a) Set $\{(0,0,0)\}$; (b) Equivalence class

$(0,0,1,0)$ and $(0,1,1,1)$. Similarly, the vector $(1,2,2,1)$ is the sum of vectors $(0,1,1,1)$ and $(1,1,1,0)$. Consequently, the vectors $(0,1,2,1)$ and $(1,2,2,1)$ are not present in $\mathcal{B}_2$.
Finally, we have:

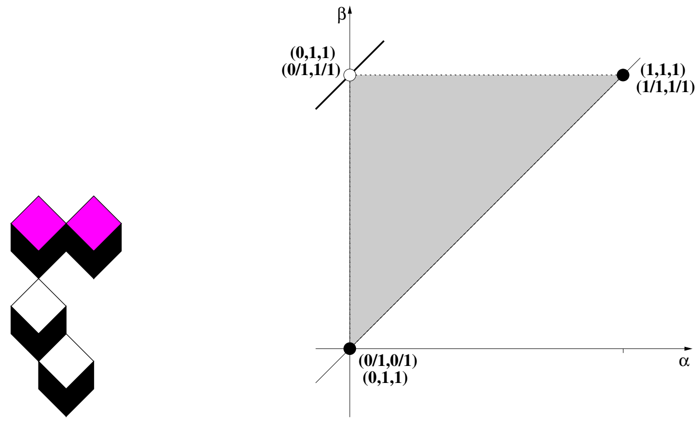
$$\mathcal{B}_2 = \{(0,0,1,0),(1,1,1,0),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$$

Every naive planes for which the projection $(a/c, b/c)$ of the normal (a,b,c) is contained in the area limited by the points $(a'/c', b'/c')$ with (a',b',c',r') belonging to $\mathcal{B}_2$ are solutions (cf. Fig 4(b)). We introduce the third point $(2,0,-1)$ (cf. Fig 5(a)). We compute the value $L_3(V)$ and $L_3^+(V)$ on the different vectors V of $\mathcal{B}_2$:

(a,b,c,r)	$L_3(a,b,c,r) = 2a - c + r$	$L_3^+(a,b,c,r) = 2a - 2c + r$
$(0,0,1,0)$	-1	-2
$(1,1,1,0)$	1	0
$(0,0,1,1)$	0	-1
$(0,1,1,1)$	0	-1
$(1,1,1,1)$	2	1

The vectors $(1,1,1,0)$, $(0,0,1,1)$ and $(0,1,1,1)$ verify the property indiced by step 1 of the algorithm. We insert these vectors in $\mathcal{B}_3$. As $L_3(0,0,1,0) < 0$ and $L_3^+(1,1,1,1) > 0$, we apply for these vectors the algorithm presented in step 2. We make appear the new vectors $(1,1,2,0)$, $(1,1,3,1)$, $(2,2,3,2)$, $(1,1,2,2)$ and $(1,2,2,2)$. But we have: $(1,1,3,1) = (1,1,2,2) + (1,1,1,0)$ and $(2,2,3,2) = (1,1,2,2) + (1,1,1,0)$. These two vectors are not inserted.
Finally, we have:

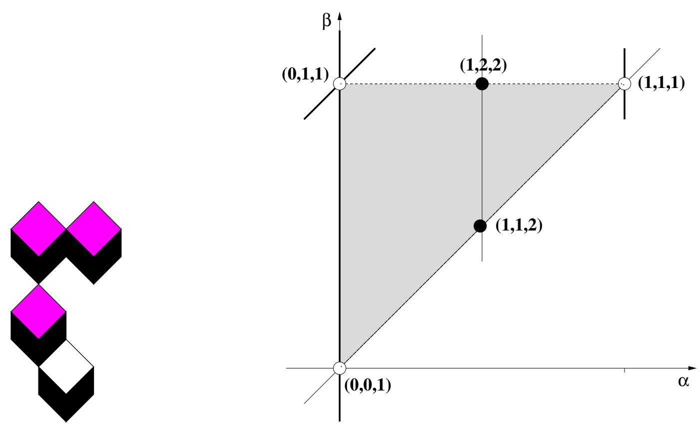
$$\mathcal{B}_3 = \{(1,1,1,0),(0,0,1,1),(0,1,1,1),(1,1,2,0),(1,1,2,2),(1,2,2,2)\}$$



(a) (b)

Fig. 4. (a) Set $\{(0,0,0),(1,-1,0)\}$; (b) Equivalence class

Every naive planes for which the projection $(a/c, b/c)$ of the normal (a,b,c) is contained in the area limited by the points $(a'/c', b'/c')$ with (a', b', c', r') belonging to $\mathcal{B}_3$ are solutions (cf. Fig 5(b)). More particularly, the naive planes with normal $(1,1,2)$ or $(1,2,2)$ contains the initial configuration of 3 voxels. Finally,



(a) (b)

Fig. 5. (a) Set $\{(0,0,0),(1,-1,0),(2,0,-1)\}$; (b) Equivalence class

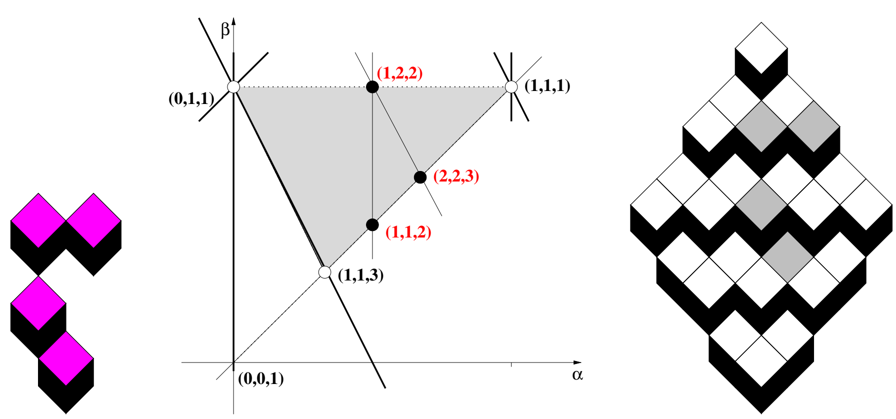
we introduce the last point $(2,1,-2)$ (cf. Fig 6(a)). We compute the value $L_4(V)$ and $L_4^+(V)$ on the different vectors V of $\mathcal{B}_3$:

(a,b,c,r)	$L_4(a,b,c,r) = 2a+b-2c+r$	$L_4^+(a,b,c,r) = 2a+b-3c+r$
$(1,1,1,0)$	1	0
$(0,0,1,1)$	-1	-2
$(0,1,1,1)$	0	-1
$(1,1,2,0)$	-1	-3
$(1,1,2,2)$	1	-1
$(1,2,2,2)$	2	0

The vectors $(1,1,1,0)$, $(0,1,1,1)$, $(1,1,2,2)$ and $(1,2,2,2)$ verify the property indiced by step 1 of the algorithm. We insert these vectors in $\mathcal{B}_4$. As the value L_4 is negative for the vectors $(0,0,1,1)$ and $(1,1,2,0)$, we applied for these vectors the algorithm presented in step 2. We make appear the new vectors $(1,1,2,1)$, $(1,1,3,3)$, $(2,2,3,0)$, $(1,2,4,4)$ and $(3,4,6,2)$. But we have: $(1,2,4,4) = (1,1,3,3) + (0,1,1,1)$ and $(3,4,6,2) = (1,1,2,1) + (0,1,1,1) + (2,2,3,0)$.These two vectors are not inserted.
Finally, we have:

$$\mathcal{B}_4 = \{(1,1,1,0),(0,1,1,1),(1,1,2,2),(1,2,2,2),(1,1,2,1),(1,1,3,3),(2,2,3,0)\}$$

Every naive planes for which the projection $(a/c, b/c)$ of the normal (a,b,c) is contained in the area limited by the points $(a'/c', b'/c')$ with (a', b', c', r') belonging to $\mathcal{B}_4$ are solutions (cf. Fig 6(b)). More particularly, the naive planes with normal $(1,1,2)$ or $(1,2,2)$ or $(2,2,3)$ contains the set of 4 voxels. We can verify in figure 6(c) that this configuration of voxels set is contained in the naive plane with normal $(1,1,2)$.



(a) (b) (c)

Fig. 6. (a) Set $\{(0,0,0),(1,-1,0),(2,0,-1),(2,1,-2)\}$; (b) Equivalence class; (c) Part of the naive plane with normal $(1,1,2)$l

5 Polyhedrization of a Voxels Object

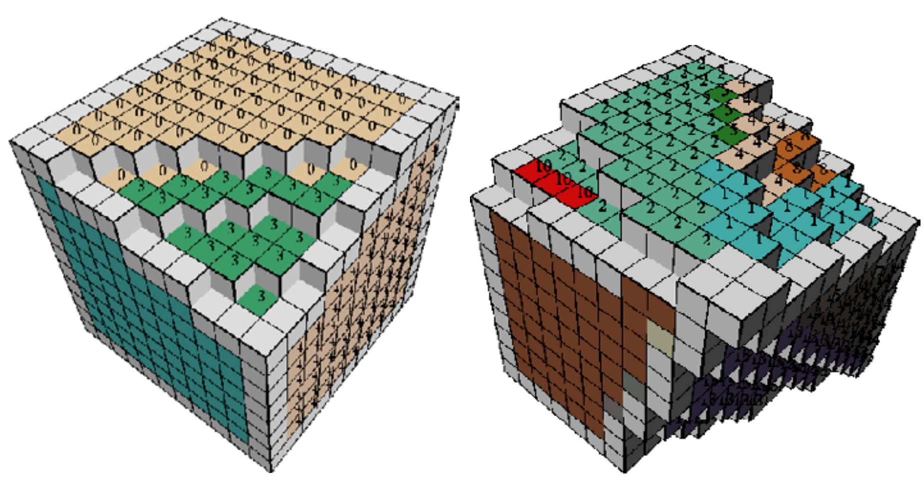
The polyhedrization of a voxelized object has been studied using mainly approximation approaches [BF94]. Here we propose an algorithm which fully works on the discrete representation of digital naive planes. The polyhedrization can be made in different ways in order to obtain straight or smooth angles between adjacent planes.

The present algorithm is an iterative process based on the recognition algorithm.

Let V_0 be a voxel of the object such that his surfel with normal N belongs to the boundary of the object. We considere this voxel as the origin of the plane. By applying the algorithm of recognition, we verify that V_0 is center of a tricube such as all the surfels of the voxels with normal N belong to the boundary of the object. If V_0 is not center of a tricube we color its surfel in white. Otherwise, we verify that its neighbours are centers of tricubes of the same naive plane. If a neighbour belongs to that plane we mark the surfel as belonging to that plane and we can analyze in a same way its neighbours. Otherwise, this voxel will be treated as the origin of a new plane.

We analyze in the same way all surfels of the object boundary until they are colored in white or marked.

The obtained results are shown on a chanfrein cube and a cube cut by three naive planes (cf. Fig 7).



(a) (b)

Fig. 7. Chanfrein cube and polytope: Voxels with the same gray level belong to a same naive plane; white voxels belong to the boundaries between adjacent planes

6 Conclusion

A generic algorithm for coplanar voxels recognition has been presented. This algorithm analyses any configuration of voxels set either connected or not connected. It is fully discrete working in the dual space issued from Farey net representation of the normal equation of a digital naive plane. This algorithm has been used for polyhedrization of the boundary of voxelized objects. As perspective, a lot of work has to be achieved precisely in computation of the exact area evaluation of such a boundary using the Pick theorem well known in 2D [Sta86,CM91].

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