Path-Width and Three-Dimensional Straight-Line Grid Drawings of Graphs *

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Abstract. We prove that every n-vertex graph G with path-width $\mathsf{pw}(G)$ has a three-dimensional straight-line grid drawing with $O(\mathsf{pw}(G)^2 \cdot n)$ volume. Thus for graphs with bounded path-width the volume is $O(n)$, and it follows that for graphs with bounded tree-width, such as series-parallel graphs, the volume is $O(n \log^2 n)$. No better bound than $O(n^2)$ was previously known for drawings of series-parallel graphs. For planar graphs we obtain three-dimensional drawings with $O(n^2)$ volume and $O(\sqrt{n})$ aspect ratio, whereas all previous constructions with $O(n^2)$ volume have $\Theta(n)$ aspect ratio.

1 Introduction

The study of straight-line graph drawing in the plane has a long history; see [37] for a recent survey. Motivated by interesting theoretical problems and potential applications in information visualisation [35], VLSI circuit design [26] and software engineering [36], there is a growing body of research in three-dimensional straight-line graph drawing.

Throughout this paper all graphs G are undirected, simple and finite with vertex set $V(G)$ and edge set $E(G)$; $n = |V(G)|$ denotes the number of vertices of G. A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *three-dimensional drawing*, represents the vertices by distinct points in 3-space with integer coordinates (called *grid-points*), and represents each edge as a line-segment between its end-vertices, such that edges only intersect at common end-vertices. If a three-dimensional drawing is contained in an axis-aligned box with side lengths $X-1$, $Y-1$ and $Z-1$, then we speak of an $X \times Y \times Z$ three-dimensional drawing with *volume* $X \cdot Y \cdot Z$ and *aspect ratio* $\max\{X,Y,Z\}/\min\{X,Y,Z\}$. This paper considers the problem of producing a three-dimensional drawing of a given graph with small volume, and with small aspect ratio as a secondary criterion.

Related Work: In contrast to the case in the plane, every graph has a three-dimensional drawing. Such a drawing can be constructed using the 'moment

* Research supported by NSERC and FCAR.

M.T. Goodrich and S.G. Kobourov (Eds.): GD 2002, LNCS 2528, pp. 42–53, 2002.

curve' algorithm in which vertex v_i, $1 \leq i \leq n$, is represented by the grid-point

$$(i, i^2, i^3) \ .$$

It is easily seen — compare with Lemma 4 to follow — that no edges cross. (Two edges *cross* if they intersect at some point other than a common end-vertex.) Cohen *et al.* [8] improved the resulting $O(n^6)$ volume bound, by proving that if p is a prime with $n < p \leq 2n$, and each vertex v_i is represented by the grid-point

$$(i, i^2 \bmod p, i^3 \bmod p)$$

then there is still no edge crossings. This construction is a generalisation of a two-dimensional technique due to Erdös [16]. Furthermore, Cohen *et al.* [8] proved that the resulting $O(n^3)$ volume bound is asymptotically optimal in the case of the complete graph K_n, and that every binary tree has a three-dimensional drawing with $O(n \log n)$ volume.

Calamoneri and Sterbini [5] proved that every 4-colourable graph has a three-dimensional drawing with $O(n^2)$ volume. Generalising this result, Pach *et al.* [30] proved that every k-colourable graph, for fixed $k \geq 2$, has a three-dimensional drawing with $O(n^2)$ volume, and that this bound is asymptotically optimal for the complete bipartite graph with equal sized bipartitions. If p is a suitably chosen prime, the main step of this algorithm represents the vertices in the ith colour class by grid-points in the set

$$\{(i, t, it) : t \equiv i^2 \pmod{p}\} \ .$$

The first linear volume bound was established by Felsner *et al.* [17], who proved that every outerplanar graph has a drawing with $O(n)$ volume. Their elegant algorithm 'wraps' a two-dimensional layered drawing around a triangular prism; see Lemma 5 for more on this method. Poranen [32] proved that series-parallel digraphs have upward three-dimensional drawings with $O(n^3)$ volume, and that this bound can be improved to $O(n^2)$ and $O(n)$ in certain special cases. Recently di Giacomo *et al.* [11] proved that series-parallel graphs with maximum degree three have three-dimensional drawings with linear volume.

Note that three-dimensional drawings with the vertices having real coordinates have been studied by Bruß and Frick [4], Chilakamarri *et al.* [6], Chrobak *et al.* [7], Cruz and Twarog [9], Eades and Garvan [15], Garg *et al.* [18], Hong [22], Hong and Eades [23, 24], Hong *et al.* [25], Monien *et al.* [27], and Ostry [29]. Aesthetic criteria besides volume which have been considered include symmetry [22–25], aspect ratio [7, 18], angular resolution [7, 18], edge-separation [7, 18], and convexity [6, 7, 15].

Tree-Decompositions: Before stating our results we recall some definitions. A *tree-decomposition* of a graph G is a tree T together with a collection of subsets T_x (called *bags*) of $V(G)$ indexed by the vertices of T such that:

- $\displaystyle\bigcup_{x \in V(T)} T_x = V(G)$,

– for every edge $vw \in E(G)$, there is a vertex $x \in V(T)$ such that the bag T_x contains both v and w, and
– for all vertices $x, y, z \in V(T)$, if y is on the path from x to z in T, then $T_x \cap T_z \subseteq T_y$.

The *width* of a tree-decomposition is the maximum cardinality of a bag minus one. A *path-decomposition* is a tree-decomposition where the tree T is a path $T = (x_1, x_2, \ldots, x_m)$, which is simply identified by the sequence of bags $T_1, T_2, \ldots, T_m$ where each $T_i = T_{x_i}$. The *path-width* (respectively, *tree-width*) of a graph G, denoted by $\mathsf{pw}(G)$ ($\mathsf{tw}(G)$), is the minimum width of a path-decomposition (tree-decomposition) of G. A graph G is said to have *bounded path-width* (*tree-width*) if $\mathsf{pw}(G) = k$ ($\mathsf{tw}(G) = k$) for some constant k. Given a graph with bounded path-width (tree-width), the algorithm of Bodlaender [1] determines a path-decomposition (tree-decomposition) with width $\mathsf{pw}(G)$ ($\mathsf{tw}(G)$) in linear time. Note that the relationship between graph drawings and path-width or tree-width has been previously investigated by Dujmović *et al.* [13], Hliněný [21], and Peng [31], for example.

Our Results: Our main result is the following.

Theorem 1. *Every n-vertex graph G has an $O(\mathsf{pw}(G)) \times O(\mathsf{pw}(G)) \times O(n)$ three-dimensional drawing.*

Since $\mathsf{pw}(G) < n$, Theorem 1 matches the $O(n^3)$ volume bound discussed above; in fact, the drawings of K_n produced by our algorithm are identical to those produced by Cohen *et al.* [8]. We have the following corollary since every graph G has $\mathsf{pw}(G) \in O(\mathsf{tw}(G) \cdot \log n)$ [2].

Corollary 1. (a) *Every n-vertex graph with bounded path-width has a three-dimensional drawing with $O(n)$ volume.* (b) *Every n-vertex graph with bounded tree-width has a three-dimensional drawing with $O(n \log^2 n)$ volume.* □

While the notion of bounded tree-width may appear to be a purely theoretic construct, graphs arising in many applications of graph drawing do have small tree-width. For example, outerplanar graphs, series-parallel graphs and Halin graphs respectively have tree-width 2, 2 and 3 (see [2, 12]). Thus Corollary 1(b) implies that these graphs have three-dimensional drawings with $O(n \log^2 n)$ volume. While linear volume is possible for outerplanar graphs [17], our result is the first known sub-quadratic volume bound for all series-parallel and Halin graphs. Another example arises in software engineering applications. Thorup [34] proved that the control-flow graphs of go-to free programs in many programming languages have tree-width bounded by a small constant; in particular, 3 for Pascal and 6 for C. Other families of graphs having bounded tree-width (for constant k) include: almost trees with parameter k, graphs with a feedback vertex set of size k, partial k-trees, bandwidth k graphs, cutwidth k graphs, planar graphs of radius k, and k-outerplanar graphs. If the size of a maximum clique is a constant

The proofs of Theorems 1 and 2 proceed in three steps. First, an ordered layering with no X-crossing is constructed from a given path-decomposition. The second step balances the number of vertices on each layer. The third step, which is essentially the converse of Lemma 1, takes an ordered layering with no X-crossing and assigns coordinates to the vertices to avoid edge crossings. The style of three-dimensional drawing produced by our algorithm, where vertices on a single layer are positioned on vertical 'rods', is illustrated in Fig. 1.

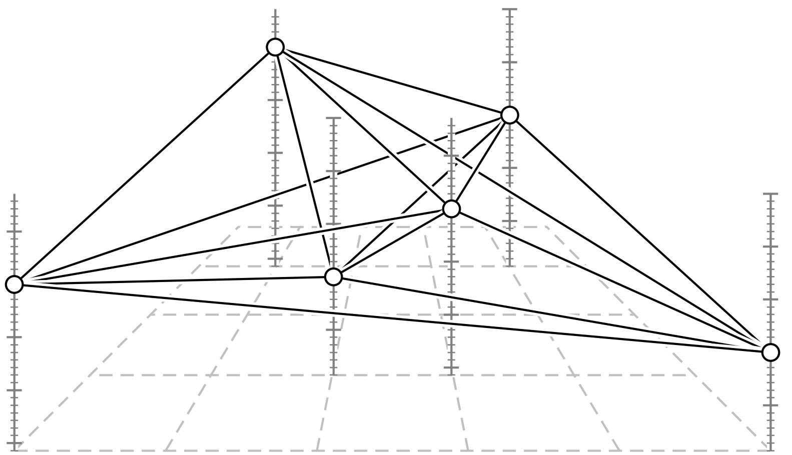

Fig. 1. A three-dimensional drawing of K_6.

Our algorithm for constructing an ordered layering makes use of the so-called normalised path-decompositions of Gupta *et al.* [20]. (The more general notion of normalised tree-decompositions was developed earlier by Gupta and Nishimura [19].) A path-decomposition $T_1, T_2, \ldots, T_m$ of width k is *normalised* if $|T_i| = k+1$ for all odd i and $|T_i| = k$ for all even i, and $T_{i-1} \cap T_{i+1} = T_i$ for all even i. The algorithm of Gupta *et al.* [20] normalises a path-decomposition while maintaining the width in linear time.

Lemma 2. *If a graph G has a normalised path-decomposition $T_1, T_2, \ldots, T_m$ of width $k-1$, then G has an ordered k-layering with no X-crossing (see Fig. 2).*

Proof. For every vertex $v \in V(G)$, let $T_{\alpha(v)}$ and $T_{\beta(v)}$ be the first and last bags containing v. Construct an ordered k-layering of G as follows. Let $T_1 = \{v_1, v_2, \ldots, v_k\}$, and position each v_i as the leftmost vertex on layer i, $1 \leq i \leq k$. Since the path-decomposition is normalised, for all bags T_j with j even, there is a unique vertex $x_j \in T_{j-1} \setminus T_j$; that is, $\beta(x_j) = j-1$. Similarly, for all bags T_j with $j > 1$ odd, there is a unique vertex $y_j \in T_j \setminus T_{j-1}$; that is, $\alpha(y_j) = j$.

The remainder of the ordered layering is constructed by sweeping through the bags of the path-decomposition as follows. For all odd $j = 3, 5, \ldots, m$, position y_j in the same layer as the vertex x_{j-1} and immediately to the right of x_{j-1}.

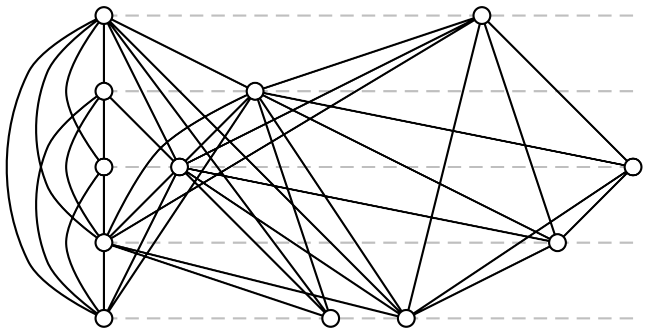

Fig. 2. An ordered 5-layering with no X-crossing produced by Lemma 2.

Clearly, x_{j-1} was the rightmost vertex in the layer before inserting y_j. Since $j-1 = \beta(x_{j-1}) < \alpha(y_j) = j$, there is no bag containing both x_{j-1} and y_j, and no edge $x_{j-1}y_j \in E(G)$. In general, two vertices in the same layer are not in a common bag and are not adjacent.

Suppose there is an X-crossing between edges vw and xy. Without loss of generality, $v <_i x$ and $y <_j w$ for some layers i and j. Thus $\beta(v) < \alpha(x)$ and $\beta(y) < \alpha(w)$. Since vw is an edge, v and w appear in some bag together; that is, $\alpha(w) \leq \beta(v)$, which implies that $\beta(y) < \alpha(x)$. This is the desired contradiction since x and y appear in some bag together. □

The second step of our algorithm is based on the algorithm of Pach *et al.* [30] for balancing the size of the colour classes in a vertex-colouring. Note that while Lemma 2 produces an ordered layering with no intra-layer edges, the remaining steps of our algorithm are valid in the more general situation that the given ordered layering possibly has intra-layer edges.

Lemma 3. *If a graph G has an ordered k-layering with no X-crossing, then for every $l > 0$, G has an ordered $\lfloor l+k \rfloor$-layering with no X-crossing and at most $\lceil \frac{n}{l} \rceil$ vertices in each layer.*

Proof. For each layer with $q > \lceil \frac{n}{l} \rceil$ vertices, replace it by $\lceil q / \lceil \frac{n}{l} \rceil \rceil$ 'sub-layers' each with exactly $\lceil \frac{n}{l} \rceil$ vertices except for at most one sub-layer with $q \bmod \lceil \frac{n}{l} \rceil$ vertices, such that the vertices in each sub-layer are consecutive in the original layer and the original order is maintained. There is no X-crossing between sub-layers of the same original layer as there is at most one edge between such sub-layers. There is no X-crossing between sub-layers from different original layers as otherwise there would be an X-crossing in the original layering. There are at most $\lfloor l \rfloor$ layers with $\lceil \frac{n}{l} \rceil$ vertices. Since there are at most k layers with less than $\lceil \frac{n}{l} \rceil$ vertices, one for each of the original layers, there is a total of at most $\lfloor l+k \rfloor$ layers. □

The third step of our algorithm is inspired by the generalisations of the moment curve algorithm by Cohen *et al.* [8] and Pach *et al.* [30]. Loosely speaking,

k then chordal, interval and circular arc graphs also have bounded tree-width. Thus Corollary 1(b) pertains to such graphs.

Since a planar graph is 4-colourable, by the results of Calamoneri and Sterbini [5] and Pach *et al.* [30] discussed above, every planar graph has a three-dimensional drawing with $O(n^2)$ volume. Of course this result also follows from the classical algorithms of de Fraysseix *et al.* [10] and Schnyder [33] for producing plane grid drawings. All of these methods produce $O(1) \times O(n) \times O(n)$ drawings, which have $\Theta(n)$ aspect ratio. Since every planar graph G has $\mathsf{pw}(G) \in O(\sqrt{n})$ [2] we have the following corollary of Theorem 1.

Corollary 2. *Every n-vertex planar graph has an $O(\sqrt{n}) \times O(\sqrt{n}) \times O(n)$ three-dimensional drawing with $\Theta(\sqrt{n})$ aspect ratio.* □

This result matches the above $O(n^2)$ volume bounds with an improvement in the aspect ratio by a factor of $\Theta(\sqrt{n})$. Our final result examines the trade-off between aspect ratio and volume.

Theorem 2. *Let G be an n-vertex graph. For every r, $1 \leq r \leq n/(\mathsf{pw}(G)+1)$, G has a three-dimensional drawing with $O(n^3/r^2)$ volume and aspect ratio $2r$.*

2 Proofs

We first introduce a combinatorial structure which is the basis for a two-dimensional layered graph drawing. An *ordered k-layering* of a graph G consists of a partition $V_1, V_2, \dots, V_k$ of $V(G)$ into *layers*, and a total ordering $<_i$ of each V_i, such that for every edge vw, if $v <_i w$ then there is no vertex x with $v <_i x <_i w$. The *span* of an edge vw is $|i-j|$ if $v \in V_i$ and $w \in V_j$. An *intra-layer* edge is an edge with zero span. An *X-crossing* consists of two edges vw and xy such that for distinct layers i and j, $v <_i x$ and $y <_j w$. The next lemma highlights the intrinsic relationship between three-dimensional drawings and ordered layerings.

Lemma 1. *Let G be an n-vertex graph with an $A \times B \times C$ three-dimensional drawing. Then G has an ordered AB-layering with no X-crossing, and G has an ordered $2AB$-layering with no X-crossing and no intra-layer edges.*

Proof. Let $V_{x,y}$ be the set of vertices of G with an X-coordinate of x and a Y-coordinate of y, where without loss of generality $1 \leq x \leq A$ and $1 \leq y \leq Y$. Consider each set $V_{x,y}$ to be ordered $V_{x,y} = (v_{x,y,1}, \dots, v_{x,y,n_{x,y}})$ by the Z-coordinates of its elements. Then the ordered layering $\{V_{x,y} : 1 \leq x \leq A, 1 \leq y \leq Y\}$ has no X-crossing as otherwise there would be a crossing in the original drawing. Now, define $V'_{x,y} = \{v_{x,y,j} : j \text{ odd}\}$ and $V''_{x,y} = \{v_{x,y,j} : j \text{ even}\}$, and consider these sets to be ordered as in $V_{x,y}$. Then, as in the above, the ordered layering $\{V'_{x,y}, V''_{x,y} : 1 \leq x \leq A, 1 \leq y \leq B\}$ has no X-crossing. Moreover there is no intra-layer edges, as otherwise an edge between two vertices in $V'_{x,y}$ would have passed through a vertex in $V''_{x,y}$ (or vice versa) in the original drawing. □

Cohen *et al.* [8] allow three 'free' dimensions, whereas Pach *et al.* [30] use the assignment of vertices to colour classes to 'fix' one dimension with two dimensions free. We use an assignment of vertices to layers in an ordered layering without X-crossings to fix two dimensions with one dimension free.

Lemma 4. *If a graph G has an ordered k-layering $\{(V_i, <_i) : 1 \leq i \leq k\}$ with no X-crossing then G has a $k \times 2k \times 2k \cdot n'$ three-dimensional drawing, where n' is the maximum number of vertices in a layer.*

Proof. Let p be the smallest prime such that $p > k$. Then $p \leq 2k$ by Bertrand's postulate. For each i, $1 \leq i \leq k$, represent the vertices in V_i by the grid-points

$$\{(i, i^2 \bmod p, t) : 1 \leq t \leq p \cdot |V_i|,\ t \equiv i^3 \pmod{p}\} ,$$

such that the Z-coordinates respect the given linear ordering $<_i$. Draw each edge as a line-segment between its end-vertices. Suppose two edges e and e' cross such that their end-vertices are at distinct points $(i_\alpha, i_\alpha^2 \bmod p, t_\alpha)$, $1 \leq \alpha \leq 4$. Then these points are coplanar, and if M is the matrix

$$M = \begin{pmatrix} 1 & i_1 & i_1^2 \bmod p & t_1 \\ 1 & i_2 & i_2^2 \bmod p & t_2 \\ 1 & i_3 & i_3^2 \bmod p & t_3 \\ 1 & i_4 & i_4^2 \bmod p & t_4 \end{pmatrix}$$

then the determinant $\det(M) = 0$. We proceed by considering the number of distinct layers $N = |\{i_1, i_2, i_3, i_4\}|$.

• $N = 1$: By the definition of an ordered layering e and e' do not cross.

• $N = 2$: If either edge is intra-layer then e and e' do not cross. Otherwise neither edge is intra-layer, and since there are no X-crossings in the ordered layering, e and e' do not cross.

• $N = 3$: Without loss of generality $i_1 = i_2$. It follows that $\det(M) = (t_2 - t_1) \cdot \det(M')$, where

$$M' = \begin{pmatrix} 1 & i_2 & i_2^2 \bmod p \\ 1 & i_3 & i_3^2 \bmod p \\ 1 & i_4 & i_4^2 \bmod p \end{pmatrix} .$$

Since $t_1 \neq t_2$, $\det(M') = 0$. However, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is non-zero since i_2, i_3 and i_4 are distinct and p is a prime, a contradiction.

• $N = 4$: Let M' be the matrix obtained from M by taking each entry modulo p. Then $\det(M') = 0$. Since $t_\alpha \equiv i_\alpha^3 \pmod{p}$, $1 \leq \alpha \leq 4$,

$$M' \equiv \begin{pmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{pmatrix} \pmod{p} .$$

Since each $i_\alpha < p$, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is non-zero since $i_\alpha \neq i_\beta$ and p is a prime. This contradiction proves there are no edge crossings. The produced drawing is at most $k \times 2k \times 2k \cdot n'$. □

We now prove the theorems.

Proof of Theorem 1. By Lemma 2, G has an ordered k-layering with no X-crossing, where $k = \mathsf{pw}(G) + 1$. By Lemma 3 with $l = k$, G has an ordered $(2k)$-layering with no X-crossing and at most $\lceil \frac{n}{k} \rceil$ vertices on each layer. By Lemma 4, G has a $2k \times 4k \times 4k \cdot \lceil \frac{n}{k} \rceil$ three-dimensional drawing, which is at most $2(\mathsf{pw}(G) + 1) \times 4(\mathsf{pw}(G) + 1) \times 4(n + \mathsf{pw}(G) + 1)$. The result follows since $1 \leq \mathsf{pw}(G) < n$. □

Proof of Theorem 2. By Lemma 2, G has an ordered k-layering with no X-crossing, where $k = \mathsf{pw}(G) + 1$. By Lemma 3 with $l = \frac{n}{r}$, G has an ordered $\lfloor \frac{n}{r} + k \rfloor$-layering with no X-crossing and at most r vertices in each layer. By assumption $r \leq n/(\mathsf{pw}(G) + 1)$. Thus $k \leq \frac{n}{r}$ and the number of layers is at most $\frac{2n}{r}$. By Lemma 4, G has a $\frac{2n}{r} \times \frac{4n}{r} \times 4n$ three-dimensional drawing, which has volume $32n^3/r^2$ and aspect ratio $2r$. □

3 Commentary

Consider the following open problems concerning straight-line grid drawings.

1. A graph with degree bounded by some constant k is $(k+1)$-colourable, and thus by the theorem of Pach *et al.* [30], has a three-dimensional drawing with $O(n^2)$ volume. Pach *et al.* [30] ask whether every graph with bounded degree has a three-dimensional drawing with $o(n^2)$ volume?
2. As discussed in Section 1 every planar graph has a three-dimensional drawing with $O(n^2)$ volume. Felsner *et al.* [17] ask whether every planar graph has a three-dimensional drawing with $O(n)$ volume? Even a volume bound of $o(n^2)$ would be interesting.

As a final observation, we show that a generalisation of the 'wrapping' algorithm of Felsner *et al.* [17] can be applied in conjunction with our algorithm, which may be helpful in solving the above open. Note that Felsner *et al.* [17] prove the case $s = 1$ (with improved constants in the volume).

Lemma 5. *Let a graph G have an ordered k-layering $\{(V_i, <_i) : 1 \leq i \leq k\}$ with no X-crossing. If the maximum edge span is s, then G has an $O(s) \times O(s) \times O(n)$ three-dimensional drawing.*

Proof. Let $t = 2s + 1$. Construct an ordered t-layering of G by merging the layers $\{V_i : i \equiv j \pmod{t}\}$ for each j, $0 \leq j \leq t - 1$, with vertices in V_α appearing before vertices in V_β in the new layer j, for all $\alpha, \beta \equiv j \pmod{t}$ with

$\alpha < \beta$. The given ordering of each V_i is preserved in the new layers. It remains to prove that there is no X-crossing. Consider two edges vw and xy. Let i_1 and i_2, $1 \le i_1 < i_2 \le k$, be the minimum and maximum layers containing v, w, x or y in the ordered k-layering.

Firstly consider the case that $i_2 - i_1 > 2s$. Then without loss of generality v is on layer i_2 and y is on layer i_1. Thus w is on a greater layer than x, and even if x (or y) appear on the same layer as v (or w) in the new t-layering, x (or y) will be to the left of v (or w). Thus these edges do not form an X-crossing in the ordered t-layering. Otherwise $i_2 - i_1 \le 2s$. Thus any two of v, w, x or y will appear on the same layer in the t-layering if and only if they are on the same layer in the given ordered k-layering (since $t > 2s$). Hence the only way for these four vertices to appear on exactly two layers in the ordered t-layering is if they were on exactly two layers in the given k-layering, in which case, by assumption vw and xy do not form an X-crossing.

Therefore there are no X-crossings. By Lemma 3 with $l = t$, G has an ordered $2t$-layering with no X-crossing and at most $\lceil \frac{n}{t} \rceil$ vertices in each layer. Since $t = 2s + 1$, by Lemma 4, G has a $2(2s + 1) \times 4(2s + 1) \times 4(2s + 1)\lceil \frac{n}{2s+1} \rceil$ three-dimensional drawing, which is $2(2s+1) \times 4(2s+1) \times 4(n+2s)$. The result follows since $s \le n$. □

Lemma 5 also shows that small path-width is not necessary for a graph to have a three-dimensional drawing with small volume. The $\sqrt{n} \times \sqrt{n}$ plane grid graph has path-width $\Theta(\sqrt{n})$, but has an ordered layering with maximum edge span 1. Therefore it has a three-dimensional drawing with $O(n)$ volume by Lemma 5.

Note Added in Proof

The results in this paper have recently been extended. In particular, Wood [38] has proved that every graph G from a proper minor-closed family has a $O(1) \times O(1) \times O(n)$ three-dimensional drawing if and only if G has $O(1)$ queue-number, and Dujmović and Wood [14] have proved that graphs of bounded tree-width have three-dimensional drawings with $O(n)$ volume.

Acknowledgements

The authors are grateful for stimulating discussions with Prosenjit Bose, Hubert de Fraysseix, Stefan Langerman, Giuseppe Liotta, and Patrice Ossona de Mendez.

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