# Advances in $C$-Planarity Testing of Clustered Graphs
## (Extended Abstract)

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**Abstract.** A clustered graph $C = (G, T)$ consists of an undirected graph $G$ and a rooted tree $T$ in which the leaves of $T$ correspond to the vertices of $G = (V, E)$. Each vertex $\mu$ in $T$ corresponds to a subset of the vertices of the graph called "cluster". $c$-planarity is a natural extension of graph planarity for clustered graphs, and plays an important role in automatic graph drawing. The complexity status of $c$-planarity testing is unknown. It has been shown in [FCE95,Dah98] that $c$-planarity can be tested in linear time for $c$-connected graphs, i.e., graphs in which the cluster induced subgraphs are connected.

In this paper, we provide a polynomial time algorithm for $c$-planarity testing of "almost" $c$-connected clustered graphs, i.e., graphs for which all nodes corresponding to the non-$c$-connected clusters lie on the same path in $T$ starting at the root of $T$, or graphs in which for each non-connected cluster its super-cluster and all its siblings in $T$ are connected. The algorithm is based on the concepts for the subgraph induced planar connectivity augmentation problem presented in [GJL$^+$02]. We regard it as a first step towards general $c$-planarity testing.

## 1 Introduction

A clustered graph consists of a graph $G$ and a recursive partitioning of the vertices of $G$. Each partition is a cluster of a subset of the vertices of $G$. Clustered graphs are getting increasing attention in graph drawing [BDM02,EFN00,FCE95], [Dah98]. Formally, a *clustered graph* $C = (G, T)$ is defined as an undirected graph $G$ and a rooted tree $T$ in which the leaves of $T$ correspond to the vertices of $G = (V, E)$.

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In a cluster drawing of a clustered graph, vertices and edges are drawn as usual, and clusters are drawn as simple closed curves defining closed regions of the plane. The region of each cluster $C$ contains the vertices $W$ corresponding to $C$ and the edges of the graph induced by $W$. The borders of the regions for the clusters are pairwise disjoint. If a cluster drawing does not contain crossings between edge pairs or edge/region pairs, we call it a *c-planar* drawing. Graphs that admit such a drawing are called *c-planar*.

While the complexity status of $c$-planarity testing is unknown, the problem can be solved in linear time if the graph is $c$-connected, i.e., all cluster induced subgraphs are connected [Dah98,FCE95]. In approaching the general case, it appears natural to augment the clustered graph by additional edges in order to achieve $c$-connectivity without loosing $c$-planarity.

The results presented in this paper are the basis for a first step towards this goal. Namely, we present a polynomial time algorithm that tests $c$-planarity for "almost" $c$-connected clustered graphs, i.e., graphs for which all $c$-vertices corresponding to the non-connected clusters lie on the same path in $T$ starting at the root of $T$, or graphs in which for each non-connected cluster its super-cluster and all its siblings are connected.

The algorithm uses ideas from the linear time algorithm for subgraph induced planar connectivity augmentation presented in [GJL$^+$02]. For an undirected graph $G = (V, E)$, $W \subseteq V$, and $E_W = \{(v_1, v_2) \in E : \{v_1, v_2\} \subseteq W\}$ let $G_W = (W, E_W)$ be the subgraph of $G$ induced by $W$. If $G$ is planar, a *subgraph induced planar connectivity augmentation* for $W$ is a set $F$ of additional edges with end vertices in $W$ such that the graph $G' = (V, E \cup F)$ is planar and the graph $G'_W$ is connected.

The paper is organized as follows: After an introduction into the SPQR data structure and clustered graphs in Sect. 2, we describe in Sect. 3 a linear time algorithm for $c$-planarity testing of a clustered graph with exactly one cluster in addition to the root cluster. This algorithm can be extended to a quadratic time algorithm for $c$-planarity testing in clustered graphs with one level beyond the root level in which at most one cluster is non-connected (see Sect. 4). In Sect. 5 we present a technique to extend the previous results to graphs with arbitrarily many non-connected clusters with the restriction that for each non-connected cluster, all its siblings and its super-cluster are connected. The same technique can be applied for graphs in which all the non-connected clusters lie on the same path in $T$.

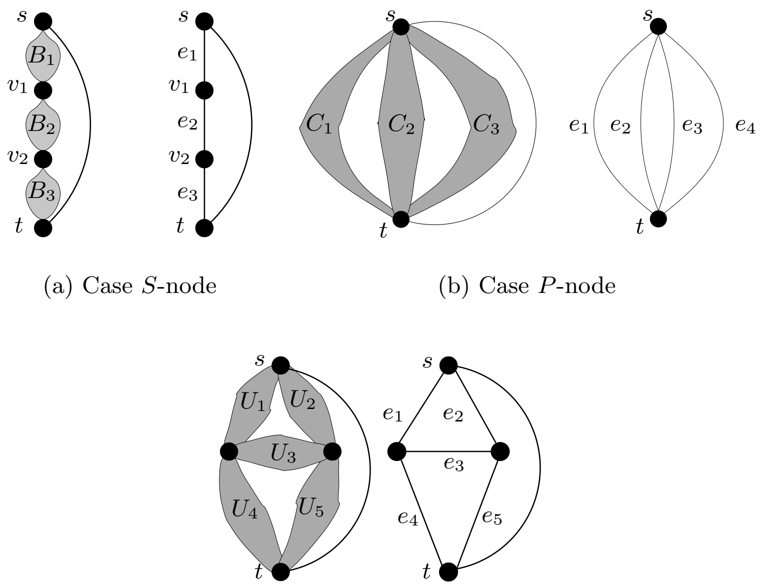
## 2 Preliminaries

### 2.1 SPQR-Trees

The data structure we use is called SPQR-tree and has been introduced by Di Battista and Tamassia [BT96]. It represents a decomposition of a planar biconnected graph according to its split pairs (pairs of vertices whose removal splits the graph or vertices connected by an edge). The construction of the SPQR-tree works recursively. At every node $\wp$ of the tree, we split the graph into the split components of the split pair associated with that node. The first split pair of the decomposition is an edge of the graph and is called the *reference edge* of the SPQR-tree. We add an edge to every split pair to make sure that they are biconnected and continue by computing the SPQR-tree for every split pair and making the resulting trees the subtrees of the node used for splitting. Every node of the SPQR-tree has two associated graphs:

- The *skeleton* of the node associated with a split pair $p$ is a simplified version of the whole graph where some split-components are replaced by single edges.
- The *pertinent graph* of a node $v$ is the subgraph of the original graph that is represented by the subtree rooted at $v$.

The two vertices of the split pair that are associated with a node $\wp$ are called the *poles* of $\wp$. There are four different node types in an SPQR-tree ($S$-,$P$-,$Q$- and $R$-nodes) that differ in the number and structure of the split components of the split pair associated with the node. The $Q$-nodes are the leaves of the tree, and there is one $Q$-node for every edge in the graph. The skeleton of a $Q$-node consists of the poles connected by two edges. The skeletons of $S$-nodes are cycles, while the skeletons of $R$-nodes are triconnected graphs. $P$-node skeletons consist of the poles connected by at least three edges. Fig. 1 shows examples for skeletons of $S$-, $P$- and $R$-nodes. Skeletons of adjacent nodes in the SPQR-tree share a pair of vertices. In each of the two skeletons, one edge connecting the two vertices is associated with a corresponding edge in the other skeleton. These two edges are called *twin edges*. The edge in a skeleton that has a twin edge in the parent node is called the *virtual edge* of the skeleton. Each edge $e$ in a skeleton represents a subgraph of the original graph. This graph together with $e$ is the *expansion graph* of $e$. All leaves of the SPQR-tree are $Q$-nodes and all inner nodes $S$-, $P$- or $R$-nodes. When we see the SPQR-tree as an unrooted tree, then it is unique for every biconnected planar graph. Another important property of these trees is that their size (including the skeletons) is linear in the size of the original graph and that they can be constructed in linear time [BT96,GM01]. As described in [BT96,GM01], SPQR-trees can be used to represent the set of all combinatorial embeddings of a biconnected planar graph. Every combinatorial embedding of the original graph defines a unique combinatorial embedding for each skeleton of a node in the SPQR-tree. Conversely, when we define an embedding for each skeleton of a node in the SPQR-tree, we define a unique embedding for the original graph. The skeleton of $S$- and $Q$-nodes are simple cycles, so they have only one embedding. But the skeletons of $R$-and $P$-nodes have at least two different embeddings. Therefore, the embeddings of the $R$- and $P$-nodes determine the embedding of the graph and we call these nodes the *decision nodes* of the SPQR-tree. The *BC-tree* of a connected graph has two types of nodes: The $c$-nodes correspond to cut-vertices of $G$ and the $b$-nodes to biconnected components (blocks). There is an edge connecting a $c$-node and a $b$-node, if the cut-vertex is contained in the block.




(a) Case $S$-node

(b) Case $P$-node

(c) Case $R$-node

**Fig. 1.** The structure of biconnected graphs and the skeleton of the root of the corresponding SPQR-tree

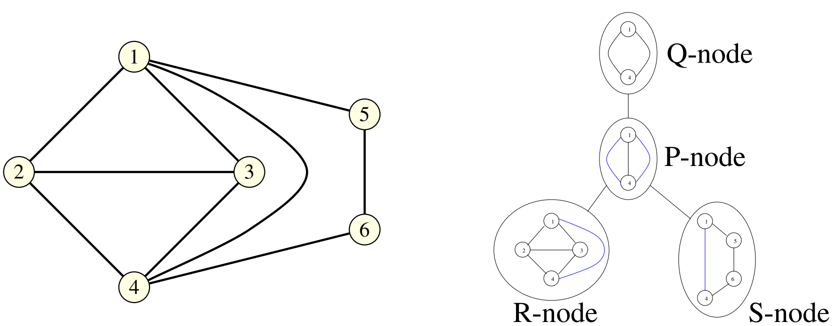



**Fig. 2.** A graph $G$ and its SPQR-tree (the $Q$-nodes of the $R$- and $S$-node are omitted)

## 2.2 Clustered Graphs

The following definitions are based on the work of Cohen, Eades and Feng [FCE95]. A *clustered graph* $C = (G, T)$ consists of an undirected graph $G$ and a rooted tree $T$ where the leaves of $T$ are the vertices of $G$. Each node $\nu$ of $T$ represents a *cluster* $V(\nu)$ of the vertices of $G$ that are leaves of the subtree rooted at $\nu$. Therefore, the tree $T$ describes an inclusion relation between clusters. $T$ is called the *inclusion tree* of $C$, and $G$ is the *underlying graph* of $C$. The root of $T$ is called *root cluster*. We let $T(\nu)$ denote the subtree of $T$ rooted at node $\nu$ and $G(\nu)$ denote the subgraph of $G$ induced by the cluster associated with node $\nu$. We define $C(\nu) = (G(\nu), T(\nu))$ to be the *sub-clustered graph* associated with node $\nu$. We define $pa(\nu)$ the parent cluster of $\nu$ in $T$ and $chl(\nu)$ the set of child clusters of $\nu$ in $T$. A *drawing* of a clustered graph $C = (G, T)$ is a representation of the clustered graph in the plane. Each vertex of $G$ is represented by a point. Each edge of $G$ is represented by a simple curve between the drawings of its endpoints. For each node $\nu$ of $T$, the cluster $V(\nu)$ is drawn as a simple closed region $R$ that contains the drawing of $G(\nu)$, such that:

- the regions for all sub-clusters of $R$ are completely in the interior of $R$;
- the regions for all other clusters are completely contained in the exterior of $R$;
- if there is an edge $e$ between two vertices of $V(\nu)$ then the drawing of $e$ is completely contained in $R$.

We say that there is an *edge-region crossing* in the drawing if the drawing of edge $e$ crosses the drawing of region $R$ more than once. A drawing of a clustered graph is *c-planar* if there are no edge crossings or edge-region crossings. If a clustered graph $C$ has a $c$-planar drawing then we say that it is *c-planar* (see Figure 3). Therefore, a $c$-planar drawing contains a planar drawing of the underlying graph.

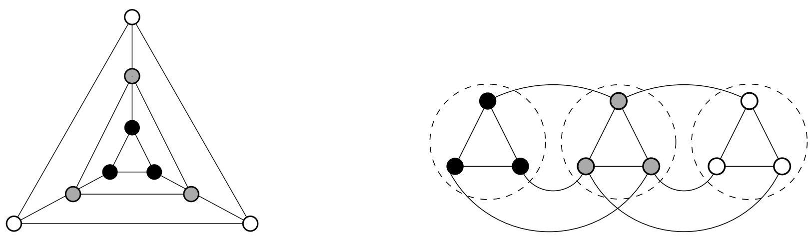


**Fig. 3.** A planar clustered graph that is not $c$-planar [FCE95] (the three disjoint clusters are represented by different types of vertices)

An edge is said to be *incident* to a cluster $V(\nu)$ if one end of the edge is a vertex of the cluster but the other endpoint is not in $V(\nu)$. An *embedding* of $C$ includes an embedding of $G$ plus the circular ordering of edges crossing the boundary of the region of each non-trivial cluster (a cluster which is not a single vertex). A clustered graph $C = (G, T)$ is *connected* if $G$ is connected. A clustered graph $C = (G, T)$ is *c-connected* if each cluster induces a connected subgraph of $G$. Suppose that $C_1 = (G_1, T_1)$ and $C_2 = (G_2, T_2)$ are two clustered graphs such that $T_1$ is a subtree of $T_2$, and for each node $\nu$ of $T_1$, $G_1(\nu)$ is a subgraph of $G_2(\nu)$. Then we say $C_1$ is a *sub-clustered graph* of $C_2$, and $C_2$ is a *super-clustered graph* of $C_1$. The following results from [FCE95] characterize $c$-planarity:

**Theorem 1.** *[FCE95] A c-connected clustered graph $C = (G, T)$ is* $c$-planar *if and only if graph $G$ is planar and there exists a planar drawing $\mathcal{D}$ of $G$, such that for each node $\nu$ of $T$, all the vertices and edges of $G - G(\nu)$ are in the outer face of the drawing of $G(\nu)$.*

**Theorem 2.** *[FCE95] A clustered graph $C = (G, T)$ is c-planar if and only if it is a sub-clustered graph of a connected and c-planar clustered graph.*

A further result from [FCE95] is a $c$-planarity testing algorithm for $c$-connected clustered graphs based on Theorem 1 with running time $O(n^2)$, where $n$ is the number of vertices of the underlying graph and each non-trivial cluster has at least two children. An improvement in time complexity is given by Dahlhaus who constructed a linear time algorithm [Dah98].

## 3 Clustered Graphs with Two Clusters

Let $C = (G, T)$ be a clustered graph with a root cluster and a cluster $\nu$. Let the graph $G$ be connected and the subgraph induced by the vertices of the cluster $\nu$ non-connected. The problem of connecting the subgraph induced by one cluster is similar to the problem of planar connectivity augmentation of an induced subgraph [GJL$^+$02].

In the following, we name the vertices of $G(\nu)$ *blue vertices*. After constructing an SPQR-tree $\mathcal{T}$ for every biconnected component of $G$ we mark for every SPQR-tree each edge in every skeleton either blue or black. An edge of a skeleton is marked blue, if its expansion graph contains blue vertices. Otherwise it is marked black.

Additionally, we assign an attribute called *permeable* to certain blue edges. Intuitively, an edge is permeable if it is possible to construct a path connecting only blue vertices through its expansion graph. Let $G(e)$ be the expansion graph of edge $e$ in skeleton $\mathcal{S}$. Since $G(e)$ is biconnected we have that in any planar embedding $G(e)$ there are exactly two faces that have $e$ on their boundary. The edge $e$ in $\mathcal{S}$ is permeable with respect to $W$, if there is an embedding $\Pi$ of $G(e)$ and a list of at least two faces $L = (f_1, \ldots, f_k)$ in $\Pi$ that satisfies the following properties:

1. The two faces $f_1$ and $f_k$ are the two faces with $e$ on their boundary.
2. For any two faces $f_i, f_{i+1}$ with $1 \leq i < k$, there is a blue vertex on the boundary between $f_i$ and $f_{i+1}$.

We call a skeleton $\mathcal{S}$ of a node $\wp$ of $\mathcal{T}$ permeable if the pertinent graph of $\wp$ and the virtual edge of $\mathcal{S}$ have the two properties stated above. Thus $\mathcal{S}$ is permeable if the twin edge of its virtual edge is permeable.

**Theorem 3.** *Let $C = (G, T)$ be a clustered graph such that the following conditions hold:*

- *$G$ is series-parallel*

– *$C$ contains only one non-trivial non-root cluster $\nu$ and $W$ is its corresponding vertex set.*

*Let $\mathcal{T}$ be set of the SPQR-trees of every biconnected component of $G$. Let $\mathcal{C} := \{\mathcal{V} \in 2^W | \mathcal{V}$ is a circle in the expansion graph of a P-node and contains both pole vertices$\}$ be the set of the vertex sets of all circles in the expansion graphs of P-nodes that contain both pole vertices. Let the subgraph $G_W$ induced by $W$ in $G$ allow a planar connectivity augmentation for $W$. If for every P-node the following property $(*)$ holds, then $C$ is c-planar:*

$(*)$ *In every P-node, there is at the most one circle $\mathcal{V}$ of $\mathcal{C}$ such that*
1. *the union of the expansion graphs of two children $\wp_1$ and $\wp_2$ contains $\mathcal{V}$ and*
2. *for $i \in \{1, 2\}$ the cut of $G - G_W$ with the expansion graph of child $\wp_i$ is nonempty.*

*Proof.* We calculate the $BC$-tree of $G$ and for every block $B$ the $SPQR$-tree of its biconnected component. The $SPQR$-tree of a series-parallel graph does not contain $R$-nodes.

Hence $G_W$ has a planar connectivity augmentation, we assume that $G_W$ is connected. As we introduce a minimum cardinality edge set applying the planar connectivity augmentation, we do not loose c-planarity. According to Theorem 1 we need to show that a planar embedding exists such that the subgraph $G - G_W$ is embedded into the outside of $G_W$. It follows that we need to show that $G - G_W$ is not embedded partially inside of $G_W$ (if it is embedded completely in an inner face we choose a face $f$ that has an edge $e = (v, w)$ with $v \in W$ and $w \notin W$ as outer face). Consider now the $P$-, $S$- and $Q$-nodes. For $P$-nodes we have the following cases

– the $P$-node is black, that is it contains only black edges,
– the P-node is blue and there exists exactly one blue edge in a $P$-node. The expansion graph of this blue edge contains the subgraph $G_W$ or
– the P-node is blue and permeable.

A blue P-node that is not permeable such that case 2 does not hold cannot exist for the following reasoning. If there is at least one blue edge in the $P$-node and another blue vertex or blue edge in another node, at least one pole vertex has to be blue due to the connectivity of $G_W$. If there are at least two blue edges in the $P$-node, the pole vertex must be blue.

Similar reasoning holds for $S$- and $Q$-nodes of the SPQR-trees and for the cut vertices of graph $G$. The latter are blue if at least two blocks which they belong to are blue.

We have to show that there does not exist a planar embedding such that the subgraph $G - G_W$ is embedded in the outside of $G_W$ if and only if there exists a P-node $\mathcal{P}$ with more than one circle of vertices of $W$ in the corresponding expansion graph fulfilling conditions 1. and 2. Note, that $G(\nu)$ is equal to $G_W$ and that $\mathcal{P}$ is in this case permeable as both pole vertices belong to $G(\nu)$.

If there exists a P-node $\mathcal{P}$ with more than one circle of vertices of $W$ in the corresponding expansion graph fulfilling conditions 1. and 2., then we order the edges that correspond to the union of expansion graphs fulfilling conditions 1. and 2. consecutively in $\mathcal{P}$. We want to find an embedding according to Theorem 1. As we have only two clusters (a root cluster and cluster $\nu$) this is equal to the fact, that $G(\nu)$ can be embedded into the outside of $G - G(\nu)$. As $G(\nu)$ is connected (there exists a planar connectivity augmentation) there has to be an embedding $\Pi$ of G with a sequence of faces $f_1, \ldots, f_k$ such that there is at least one vertex of $G - G(\nu)$ in the boundary between two consecutive faces $f_i$ and $f_{i+1}$ and the boundaries of all those faces contain all vertices of $G - G(\nu)$. This is equal to the fact that $G - G_W$ has a planar connectivity augmentation for $V - W$ in $G$ as described in [GJL$^+$02]. The faces containing vertices of $G - G(\nu)$ in their boundaries in $\mathcal{P}$ are contained in the sequence of faces $f_1, \ldots, f_k$. As the pole vertices are contained in $G(\nu)$ and there exists at least two circles of vertices of $G(\nu)$ that are contained in at least three expansions graphs, the sequence of faces $f_1, \ldots, f_k$ cannot be consecutive and therefore there cannot exist a planar connectivity augmentation of $G - G_W$ for $V - W$ in $G$. Therefore, we cannot find an embedding according to Theorem 1 and therefore $C$ is not c-planar.

If $C$ is not c-planar, then there is an embedding $\Pi$ in which the vertices of $G - G(\nu)$ cannot be embedded into the outside of $G(\nu)$ according to Theorem 1. As $G(\nu)$ is connected (there exists a planar connectivity augmentation) there does not exist a sequence of consecutive faces $f_1, \ldots, f_k$ which are consecutive so that there is at least one vertex of $V - W$ on the boundary between two faces and all vertices of $V - W$ are included in the union of the boundaries of $f_1, \ldots, f_k$. Therefore there exists a sequence of faces $f_1, \ldots, f_k$ in which their boundaries contain all vertices of $G - G(\nu)$ and there is a minimum number of consecutive faces $f_i$ and $f_{i+1}$ so that on the boundary between $f_i$ and $f_{i+1}$ is no vertex of $G - G(\nu)$. This is equal to the fact, that on the boundary between $f_i$ and $f_{i+1}$ are vertices of $G(\nu)$. Let $\mathcal{F}$ be the set of all the faces $f_i$ and $f_{i+1}$. As the S- and Q-nodes have skeletons with only one embedding and a skeleton represents the whole corresponding biconnected component in $G$, we have to consider the P-nodes. As the pole vertices are contained in $G(\nu)$ and $\mathcal{F}$ is a minimum cardinality face set, there has to be $l$ paths of vertices of $G(\nu)$ from one pole vertex to the other with $l = \frac{|\mathcal{F}|}{2} + 2$ and $l \geq 3$. Combining the paths to circles (the first and the last vertex of the paths are the pole vertices), we get more than one circle fulfilling condition 1. and 2. □

Note, that (∗) in the previous theorem can be replaced by: There exists a planar connectivity augmentation for $V - W$ in $G$, if the subgraph $G_W$ is connected using the planar connectivity augmentation. Figure 4 gives an example for a clustered graph with underlying series-parallel graph which is not $c$-planar.

The previous theorem can be easily extended to $c$-connected clustered graph $C = (G, T)$ with series-parallel underlying graph $G$.

**Theorem 4.** *Let $C = (G, T)$ be a clustered graph where $G$ is a connected planar graph and $W$ is the vertex set corresponding to the only non-trivial non-root cluster in $T$. We further assume that the subgraph $G_W$ induced by $W$ is connected.*

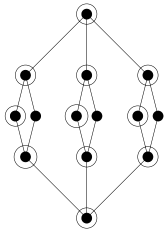


**Fig. 4.** Isolations in the expansion graph of P-nodes: The vertices with a circle around are vertices of the connected cluster, the others belong to the root cluster

*The clustered graph C is c-planar if and only if there is an embedding of G that contains no cycle of blue vertices that separates the black vertices.*

Therefore $c$-planarity can be destroyed in the expansion graphs of $R$-nodes and in the expansion graphs of $P$-nodes if the clustered graph is $c$-connected.

**Theorem 5.** *A connected clustered graph $C = (G, T)$ where $G$ is planar with one non-connected cluster $\nu$ is c-planar if and only if*

1. *there exists a planar connectivity augmentation of the subgraph induced by the non-connected cluster $\nu$ and*
2. *there is an embedding of $G$ that contains no circle of vertices of $G(\nu)$ that separates vertices of $G - G(\nu)$.*

*Proof.* The proof follows from the previous theorems. □

Note, that item 2. in the previous theorem can be replaced by: There exists a planar connectivity augmentation for the vertices of $G - G(\nu)$ in $G$.

As a result, we are able to deal with $c$-planarity of a special subclass of clustered graphs using planar connectivity augmentation and SPQR-trees.

According to Theorem 5, we know how to test planar connectivity augmentation of a subgraph of a planar graph. We now show how to test whether there exists an embedding of $G$ that contains no circle in $G(\nu)$ that separates $G - G(\nu)$.

Consider a clustered graph $C = (G, T)$ that has only one non-root cluster $\nu$ that is non-connected. Therefore the subgraph $G(\nu)$ has more than one connected component. As we choose a minimum cardinality planar augmenting edge set $\mathcal{M}$ and take the pole vertices belonging to $G(\nu)$ into account to augment $C$ to a $c$-connected clustered graph and if such an augmentation exists, $c$-planarity of $C$ is maintained if $C$ is $c$-planar. Therefore to test whether $C$ is $c$-planar is equal to the following:

Let $C_i$, $i = 1, \ldots, l$, $l \geq 2$ be the connected components of $G(\nu)$.

I. There exists a drawing such that the drawing of $G - C_i$ can be drawn outside of the drawing $C_i$ for all $i = 1, \ldots, l$ and
II. there exists a planar connectivity augmentation between all $C_i$, $i \in \mathbb{N}$ and therefore for $G(\nu)$.

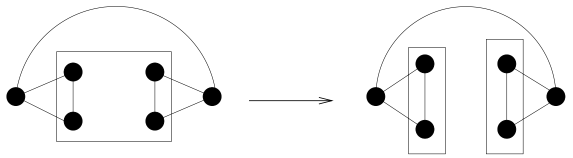


**Fig. 5.** A clustered graph $C$ with a non-connected cluster. By splitting the non-connected cluster, $C$ is extended to a $c$-connected clustered graph.

We test I. on an auxiliary modified clustered graph $\tilde{C}$ of $C$ as follows. Cluster $\nu$ is split into $l$ dummy clusters so that each connected component $C_i$ $i = 1, \ldots, l$ corresponds to a dummy cluster. By construction $\tilde{C}$ is a $c$-connected clustered graph and can be tested according Feng, Eades and Cohen and according Dahlhaus for $c$-planarity.

**Definition 1.** *Let $C$ be a clustered graph with a connected root cluster and a non-connected cluster $\nu$. Let $C_{sub}$ be the clustered graph created by splitting $\nu$ into one dummy cluster for each connected component of the subgraph induced by $\nu$ that contain at least two vertices (see Fig. 5). A connected component that contains only one vertex is treated as a trivial cluster. We call $C_{sub}$ the c-split clustered graph of $C$.*

Note, that $C_{sub}$ is a $c$-connected clustered graph.

**Theorem 6.** *Let $C$ be a clustered graph with a connected root cluster and a non-connected cluster $\nu$. Let $C_{sub}$ be its c-split clustered graph. If $C_{sub}$ is not c-planar then $C$ is not c-planar.*

*Proof.* If $C_{sub}$ is not $c$-planar, there exists an edge-crossing or a cluster-crossing in at least one cluster $\mu$. The subgraphs induced by the dummy clusters of $C_{sub}$ are connected components of the corresponding non-connected cluster $\nu$ of $C$. Thus there exists an edge-crossing or a cluster-crossing in $\nu$ and $C$ is not $c$-planar. □

Note that $C$ must not be $c$-planar if its $c$-split clustered graph $C_{sub}$ is $c$-planar (see Fig. 6). In the case that $C_{sub}$ is $c$-planar and $C$ is not there will not exist a planar connectivity augmentation in $C$ for the non-connected cluster $\nu$.

Thus after a positive result after the application of the $c$-planarity test by Dahlhaus we apply the planar connectivity augmentation algorithm on $G$ and the vertices of $\nu$ as subset $W$ (if we have pole vertices that belong to $G(\nu)$, we use them for connectivity). Together with Theorem 5 we get the following theorem.

**Theorem 7.** *Let $C = (G, T)$ be a connected clustered graph and $\nu$ its only non-trivial non-root cluster that is non-connected. $C$ can be tested for c-planarity in linear time with respect to the number of vertices of $C$ and in the positive case embedded in linear time.*

**Input**: A clustered graph $C = (G, T)$ that contains a connected root cluster and a non-connected cluster $\nu$.
**Result** : `true` if and only if there is a $c$-planar connectivity augmentation for $\nu$; in the positive case an embedding $\Pi$ and the minimum cardinality augmenting edge set.
Compute $C_{sub}$ by splitting the non-connected cluster for each connected component of the subgraph $G(\nu)$;
Apply the linear time $c$-planarity test on $C_{sub}$;
**if** the test return false **then**
  `return false`;
Apply the subgraph induced planarity augmentation algorithm for $C$ and $\nu$;
**if** a planar connectivity augmentation exists **then**
  Compute $\Pi$;
  `return true`;
**else**
  `return false`;

**Algorithm 1.** The algorithm for clustered graphs $C = (G, T)$ that contain a connected root cluster and a non-connected cluster $\nu$. It computes an embedding $\Pi$ and the minimum cardinality augmenting edge set.

Next we consider the case that $G$ is non-connected and there is only one cluster $\nu$ that is not the root cluster and is non-connected. For all connected components of $G$ we apply our algorithm for clustered graphs with one non-connected cluster. Then we choose for each connected component a face as outer face that contains at least one blue vertex $v_1$ and one non-blue vertex $v_2$. We connect the blue vertices in the outer face so that the edges of a minimum cardinality augmenting edge set is inserted (as described in [GJL$^+$02]). In the positive case a $c$-planar embedding with a minimum cardinality augmenting edge set will be computed. Thus we have that the clustered graph has a $c$-planar connectivity augmentation for the non-connected cluster $\nu$ which leads us to the following theorem.

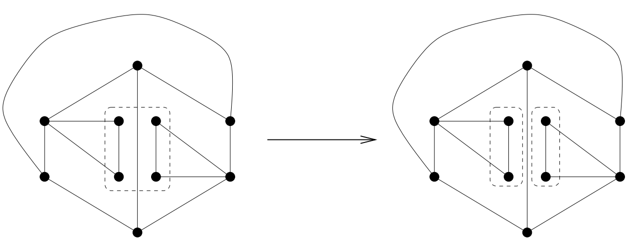


**Fig. 6.** An example where $C$ is not $c$-planar but its $c$-split clustered graph $C_{sub}$ is $c$-planar

**Theorem 8.** *Let $C = (G, T)$ be a not necessarily connected clustered graph and $\nu$ its only non-trivial non-root cluster that is non-connected. $C$ can be tested for c-planarity in linear time concerning the number of vertices of $C$ and in the positive case embedded in $O(n)$.*

## 4 One-Level Clustered Graphs

As shown in the last section it is possible to test $c$-planarity of a clustered graph with one non-connected cluster and a root cluster in $O(n)$ time, where $n$ is the number of vertices of $G$. Next, we consider a clustered graph $C = (G, T)$ with a cluster tree $T$ with only one level below the root cluster. So now we allow more than one non-root cluster. Further let only one child cluster of the root cluster be non-connected. We assume, that every non-trivial cluster has at least two vertices.

We construct a $c$-split clustered graph $C_{sub}$ of $C$ as described in Sect. 3. If $G$ is not connected, we apply this technique to every connected component of $G$.

First, we test if the $c$-split clustered graph $C_{sub}$ of $C$ is $c$-planar. We call $G_{mod}$ by the $c$-planarity test of Feng, Eades and Cohen modified graph $G$. Then we apply the planar connectivity augmentation algorithm described in [GJL+02] for the vertices belonging to $\nu$ in $G_{mod}$. If one exists, then an embedding $\Pi$ and a minimum cardinality augmenting edge set is computed and `true` is returned. Otherwise, $C$ is not $c$-planar. If we get $\Pi$, we can apply the techniques used in the $c$-planarity embedding algorithm of Feng, Eades for the connected clusters using $\Pi$ to obtain a $c$-planar embedding of $C$. For the case that the root cluster is non-connected, we apply further the planar connectivity augmentation algorithm for the root cluster.

**Theorem 9.** *Let $C = (G, T)$ be a connected clustered graph with a cluster tree $T$ with only one level below the root cluster and $\nu$ its only non-root non-connected cluster. Let $C_{sub}$ be the c-split clustered graph of $C$. Let $G_{mod}$ be the modified graph $G$ obtained by the c-planarity test by Feng, Eades and Cohen of $C_{sub} = (G, T_{sub})$. $C$ is c-planar if and only if $C_{sub}$ is c-planar and there exists a planar connectivity augmentation of the vertices belonging to the subgraph $G(\nu)$ in $G_{mod}$.*

*Proof.*

- "$\Leftarrow$" We have that $C_{sub}$ is $c$-planar and there exists a planar connectivity augmentation of $G(\nu)$ in $G_{mod}$. Thus $G$ is planar and there exists a planar drawing of $G$, such that for every node $\mu$ of $T_{sub}$, all the vertices and edges of $G - G(\mu)$ are in the external face of the drawing of $G(\mu)$. Note that this holds for the connected clusters of $C$ and for the dummy clusters in $C_{sub}$ constructed of $\nu$. Therefore, $G_{mod}$ allows only those $c$-planar embeddings that respect the connected clusters of $C$ and takes the connected components of $G(\nu)$ into account. Hence the planar connectivity augmentation within $G_{mod}$ has introduced a minimum cardinality augmenting edge set, connecting $G(\nu)$ such that the boundary of its external face in any planar drawing of $G(\nu)$ consists of a connected not simple cycle. Furthermore, the minimum cardinality augmenting edge set connects the connected components of the original $G(\nu)$, so that there exists at the most one edge between two connected components. Therefore $G_{mod} - G(\nu)$ is embedded in the outer face of $G(\nu)$ after planar connectivity augmentation. Hence, $C$ is $c$-planar.

- **"$\Rightarrow$"** We have that $C$ is $c$-planar. Thus $C_{sub}$ is $c$-planar. Hence the vertices of $G(\nu)$ can be embedded so that there is a sequence of faces $f_1, \ldots, f_k$ with the following property: for all $1 \leq i < k$, there is at least one vertex of $W$ on the boundary between $f_i$ and $f_{i+1}$ and the boundaries of the faces $f_i$ $(1 \leq i \leq k)$ contain all vertices of $W$. Therefore $G$ has a planar connectivity augmentation in respect to the vertices of $G(\nu)$.

□

**Theorem 10.** *Let $C = (G, T)$ be a connected clustered graph with a cluster tree $T$ with only one level below the root cluster and $\nu$ its only non-root non-connected cluster. $C$ can be tested for c-planarity in $O(n^2)$ time in respect to the number $n$ of vertices of $C$.*

*Proof.* We can create the $c$-split clustered graph $C_{sub}$ in $O(n)$ time where $n$ is the number of vertices of $G$. The $c$-planarity testing can be done in $O(n^2)$ time and the planar connectivity augmentation of the subgraph induced by the non-connected clustered graph in $O(n)$ time. As a result the algorithm can be implemented in $O(n^2)$ time where $n$ is the number of vertices of $G$. □

## 5 Multi-level Clustered Graphs

We extend the algorithm of the previous section to clustered graphs with more than one level in the tree $T$. Consider a clustered graph $C = (G, T)$ with at least two non-connected clusters where $G$ is connected. Then if for every non-connected cluster $\nu$ in the cluster tree $T$ the parent cluster and all siblings of $\nu$ are connected, we show that it is possible to connect the non-connected clusters using the planar connectivity augmentation [GJL+02].

To do so, we compute the $c$-split clustered graph $C_{sub}$ of $C$. Then, we traverse $T$ towards the root starting at the leaves in order to do the followings: For every non-connected cluster $\nu$ of $C$ that has connected siblings $\mu$ and a connected parent $pa(\nu)$, we test whether the subgraph $G(pa(\nu))$ is planar, test whether the edges that are incident to $pa(\nu)$ can be drawn into the outside of the drawing of $G(pa(\nu))$ (see Figure 7) and test whether there exists a planar connectivity augmentation of the vertices of $\nu$. If a planar connectivity augmentation exists, then an embedding $\Pi$ and a minimum cardinality augmenting edge set is computed and `true` is returned. Otherwise, $C$ is not $c$-planar. If we get $\Pi$, we can apply the techniques used in the $c$-planarity embedding algorithm of Feng and Eades for the connected clusters using $\Pi$ to obtain a $c$-planar embedding of $C$.

**Input**: A clustered graph $C = (G, T)$ that contains a connected root cluster, a non-connected child cluster $\nu$ and an arbitrarily number of connected child clusters.

**Result** : `true` if and only if there is a $c$-planar connectivity augmentation for $\nu$; in the positive case an embedding $\Pi$ and the minimum cardinality augmenting edge set will be computed.

Compute $C_{sub}$ by splitting the non-connected cluster for each connected component of the subgraph $G(\nu)$;
Apply the $c$-planarity test by Cohen, Eades and Feng on $C_{sub}$;
**if** the test return false **then**
  `return false`;
Apply on $G_{mod}$ (see Theorem 9) in respect to the vertices of $G(\nu)$ the planar connectivity augmentation algorithm;
**if** a planar connectivity augmentation exists **then**
  Compute $\Pi$;
  `return true`;
**else**
  `return false`;

**Algorithm 2.** The algorithm for clustered graphs $C = (G, T)$ that contain a connected root cluster, a non-connected child cluster $\nu$ and an arbitrarily number of connected child clusters. It computes an embedding $\Pi$ and the minimum cardinality augmenting edge set if $C$ is $c$-planar.

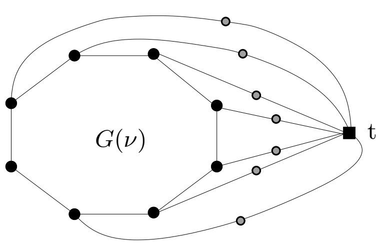



**Fig. 7.** Constructing an auxiliary graph $G_{mod}$ from the connected subgraph $G(\nu)$ where the incident edges of $\nu$ are connected with a dummy vertex $t$ [FCE95]

**Theorem 11.** *Let $C = (G, T)$ be a connected clustered graph where its non-connected clusters $\nu$ have a connected parent cluster and only connected sibling clusters. Let $C_{sub} = (G, T_{sub})$ be its c-split clustered graph. Let $G_{mod}$ be the graph constructed by the c-planarity test of Feng, Eades and Cohen applied to $pa(\nu)$ of $C_{sub}$ and where the subgraphs $G(chl(pa(\nu)))$ are replaced with their wheel graphs. If $C_{sub}$ is c-planar and there exists for every non-connected cluster $\nu$ a planar connectivity augmentation for the vertices of the subgraph $G(\nu)$ in $G_{mod}$ then $C$ is c-planar.*

*Proof.* The proof is by construction. We extend the $c$-planarity test by Cohen, Eades and Feng as follows: For every connected parent cluster $pa(\nu)$ with a non-connected child cluster $\nu$, we construct the graph $G_{mod}$ as stated in the theorem (see Fig. 7). Then we apply the linear time planarity test based on PQ-trees [BL76,CEL67] to $G_{mod}$ and in the positive case the planar connectivity augmentation for the subgraph induced by the non-connected cluster [GJL$^+$02].

As we do this recursively for every connected parent node that has a non-connected cluster by taking the wheel graphs as constructed in the $c$-planarity test into account, we can test $c$-planarity in $O(n^2)$ time where $n$ is the number of vertices of $G$. □

**Theorem 12.** *Let $C = (G, T)$ be a connected clustered graph where its non-connected clusters $\nu$ have a connected parent cluster and only connected sibling clusters. $C$ can be tested for c-planarity and in the positive case embedded in $O(n^2)$ time with respect to the number $n$ of vertices of $C$.*

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**Input**: A clustered graph $C = (G, T)$ that contains a connected root cluster and non-connected clusters with connected parent cluster and sibling clusters.

**Result** : `true` if and only if there is a $c$-planar connectivity augmentation for $\nu$; in the positive case an embedding $\Pi$ and the minimum cardinality augmenting edge set will be computed.

Compute $C_{sub}$ by splitting the non-connected cluster for each connected component of the subgraph $G(\nu)$;

Change the $c$-planarity test of Cohen, Eades and Feng as follows and apply it to $C_{sub}$;

**for** for every connected parent cluster of a non-connected cluster **do**
- Construct $G_{mod}$ as described in Theorem 11;
- Test planarity of $G_{mod}$;
- **if** the planarity test returns false **then**
  - return false;
- Apply the subgraph induced planar connectivity augmentation algorithm for the vertices of $G(\nu)$ in $G_{mod}$;

**if** a planar connectivity augmentation exists **then**
- Compute $\Pi$;
- `return true`;

**else**
- `return false`;

**Algorithm 3.** The algorithm for clustered graphs that contain a connected root cluster and non-connected clusters with a connected parent cluster and sibling clusters. It computes an embedding $\Pi$ and the minimum cardinality augmenting edge set if $C$ is $c$-planar.

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We note that this technique can be applied to connected clustered graphs $C = (G, T)$ where the non-connected clusters lie on the same path from the root to the leaves. This is done again by computing the $c$-split clustered graph $C_{sub}$ for every connected component $C_i$ of $C$, testing it for $c$-planarity. In this step, every connected component is modified with wheel graphs. Then we get graph $G_{mod}$.

We now traverse the path of non-connected clusters in the original cluster tree $T$ from the leaves to the root and apply to the vertices of these clusters the planar connectivity augmentation algorithm [GJL$^+$02] in $G_{mod}$. This can be done in $O(n^2)$ time where $n$ is the number of vertices in the underlying graph $G$. In the positive case, an embedding can be computed in $O(n^2)$ time.

Finally, we consider two non-connected clusters that are siblings in an arbitrary clustered graph where all other clusters are connected and $G$ is planar. If the two clusters are contained in two different connected components (if additionally the root is non-connected) or if they are contained in two different biconnected components or in two different subtrees of a BC-tree, we can apply the one-cluster-method for each connected or biconnected component independently. This can be extended to an arbitrarily number of non-connected clusters that are siblings under the condition that they are in different connected or biconnected components.

Combining the techniques used in this paper, there is a large class of clustered graphs that can now be tested for $c$-planarity.

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