ON CENTRAL TREES OF A GRAPH *

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Abstract. The concept of central trees of a graph has attracted our attention in relation to electrical network theory. Until now, however, only a few properties of central trees have been clarified. In this paper, in connection with the critical sets of the edge set of a graph, some new theorems on central trees of the graph are presented. Also, a few examples are included to illustrate the applications of these theorems.

1. Introduction

The concept of central trees of a graph was originally introduced in 1966 by Deo [1] in relation to the reduction of the amount of labor involved in Mayeda and Seshu's method of generating all trees of a graph and subsequently considered in 1968 by Malik [2] and in 1971 by Amoia and Cottafava [3]. Also, its close relation to the formulation of a new network equation called "the 2-nd hybrid equation" (which will

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be shown in the appendix) was pointed out in 1971 by Kishi and Kajitani [4] and subsequently considered in 1979 by Kajitani [5] in a new context. Until now, however, only a few properties of central trees have been clarified [3,6,7,8].

In this paper, in connection with the critical sets of the edge set of a graph, some new theorems on central trees of the graph are given as a few extensions of the results obtained already in [6,7].

Throughout this paper, we adopt the usual set-theoretic conventions: set union, set intersection, set inclusion, proper inclusion and set difference are denoted by the familiar symbols $\cup$, $\cap$, $\subseteq$, $\subset$ and $-$, respectively. The empty set is denoted by $\emptyset$ and the cardinality of a set A is denoted by $|A|$.

2. Critical Sets

Throughout this paper, G is used to denote a nonseparable graph of rank $r[G]$ and nullity $n[G]$, and E is used to denote the edge set of G.

For any subset S of E, a graph obtained from G by deleting all edges in $E-S$ is denoted by $G \cdot S$, and a graph obtained from G by contracting all edges in $E-S$ is denoted by $G \times S$. $G \cdot S$ and $G \times S$ are called a subgraph and a contraction of G, respectively. For $R \subseteq S \subseteq E$, a graph obtained from G by deleting all edges in $E-S$ and then contracting all edges in $S-R$ is denoted by $(G \cdot S) \times R$, which is called a minor of G. Then, for $R \subseteq S \subseteq E$, we have the relations:

$$\begin{aligned}(G \cdot S) \cdot R &= G \cdot R,\\ (G \times S) \times R &= G \times R,\\ (G \cdot S) \times R &= (G \times (\overline{S} \cup R)) \cdot R,\\ (G \times S) \cdot R &= (G \cdot (\overline{S} \cup R)) \times R\end{aligned} \tag{1}$$

where $\overline{S} = E - S$. The ranks of $G \cdot S$, $G \times S$ and $(G \cdot S) \times R$ are denoted by $r[G \cdot S]$, $r[G \times S]$ and $r[(G \cdot S) \times R]$, respectively, and the nullities of $G \cdot S$, $G \times S$ and $(G \cdot S) \times R$ are denoted by $n[G \cdot S]$, $n[G \times S]$ and $n[(G \cdot S) \times R]$, respectively. Then,

(i) for $R \subseteq S \subseteq E$,

$$r[G \cdot S] = r[G \cdot R] + r[(G \cdot S) \times (S - R)], \tag{2}$$

(ii) for $R \subseteq S \subseteq E$,

$$r[(G \cdot S) \times R] + n[(G \cdot S) \times R] = |R|, \tag{3}$$

(iii) $r[G \cdot \emptyset] = 0,$ (4)

(iv) for $e \in E$,

$$r[G \cdot \{e\}] = 1, \tag{5}$$

(v) for $R \subseteq S \subseteq E$,

$$r[G \cdot R] \leqq r[G \cdot S], \tag{6}$$

(vi) for $R, S \subseteq E$,

$$r[G \cdot R] + r[G \cdot S] \geqq r[G \cdot (R \cup S)] + r[G \cdot (R \cap S)]. \tag{7}$$

For any α such that $0 \leqq \alpha < \infty$, and for any subset S of E,

$$f_\alpha(S) = \alpha|S| - r[G \cdot S] \tag{8}$$

is called the deficiency of S with respect to α. A subset S_α of E is called a critical set of E with respect to α if

$$f_\alpha(S_\alpha) = \max_{S \subseteq E} f_\alpha(S). \tag{9}$$

Then, we can easily prove from (7) that if S_α^1 and S_α^2 are two critical sets of E with respect to α, then $S_\alpha^1 \cup S_\alpha^2$ and $S_\alpha^1 \cap S_\alpha^2$ are also critical sets of E with respect to α. Now, let F_α be the family of all the critical sets of E with respect to α, then we see that F_α has a unique minimal member $S_\alpha^{(0)}$ and a unique maximal member $S_\alpha^{(\infty)}$, and also we see that for any critical set S of F_α

$$S_\alpha^{(0)} \subseteq S \subseteq S_\alpha^{(\infty)} \tag{10}$$

is satisfied. Let $E_\alpha^+ = S_\alpha^{(0)}$, $E_\alpha^o = S_\alpha^{(\infty)} - S_\alpha^{(0)}$ and $E_\alpha^- = E - S_\alpha^{(\infty)}$. Here, such a unique tripartition $(E_\alpha^+, E_\alpha^o, E_\alpha^-)$ of E is called the principal partition of E with respect to α. In particular, in case of $\alpha = 1/2$, $(E_\alpha^+, E_\alpha^o, E_\alpha^-)$ is nothing but the principal partition of E defined in 1967 by Kishi and Kajitani [9,10,11]. Next, let us denote all the maximal critical sets of E with respect to all α satisfying $0 \leqq \alpha < \infty$ by $S_{\alpha_0}^{(\infty)} (= \emptyset)$, $S_{\alpha_1}^{(\infty)}$, $S_{\alpha_2}^{(\infty)}$, ... , $S_{\alpha_k}^{(\infty)}$, $S_{\alpha_{k+1}}^{(\infty)}$ $(= E)$ such that

$$\emptyset = S_{\alpha_0}^{(\infty)} \subset S_{\alpha_1}^{(\infty)} \subset S_{\alpha_2}^{(\infty)} \subset \cdots \subset S_{\alpha_k}^{(\infty)} \subset S_{\alpha_{k+1}}^{(\infty)} = E \tag{11}$$

where $0 \leqq \alpha_0 < c_1$, $c_1 \leqq \alpha_1 < c_2$, $c_2 \leqq \alpha_2 < c_3$, ... , $c_k \leqq \alpha_k < c_{k+1}$, $c_{k+1} \leqq \alpha_{k+1} < \infty$ and

$$c_i = \min_{S_{\alpha_{i-1}}^{(\infty)} \subset S \subseteq E} \frac{r[G \cdot S] - r[G \cdot S_{\alpha_{i-1}}^{(\infty)}]}{|S - S_{\alpha_{i-1}}^{(\infty)}|} \tag{12}$$

$$= \min_{S^{(\infty)}_{\alpha_{i-1}} \subset S \subseteq E} \frac{r[(G \times \overline{S^{(\infty)}_{\alpha_{i-1}}}) \cdot (\overline{S^{(\infty)}_{\alpha_{i-1}}} - \overline{S})]}{|\overline{S^{(\infty)}_{\alpha_{i-1}}} - \overline{S}|} \qquad (13)$$

Here such numbers c_i are called the critical numbers of E, and a partition $(X_0, X_1, X_2, \ldots, X_k)$ of E such that

$$X_i = S^{(\infty)}_{\alpha_i} - S^{(\infty)}_{\alpha_{i-1}} \qquad (i = 0, 1, 2, \ldots, k) \qquad (14)$$

is called the principal partition of E with respect to all α such that $0 \leqq \alpha < \infty$, which was given in 1976 by Tomizawa [12].

Here, it should be noted that all the critical sets of E with respect to all α such that $0 \leqq \alpha < \infty$ can be obtained by Tomizawa's algorithm[12].

3. Central Trees and Their Properties in Connection with Critical Sets

A tree T_s of G is called a central tree of G if

$$r[G \cdot \overline{T_s}] \leqq r[G \cdot \overline{T}] \qquad (15)$$

for every tree T of G where $\overline{T_s} = E - T_s$ and $\overline{T} = E - T$ [1].

[Theorem 1]

If, for a critical set S_{α_i} of E with respect to α_i such that $c_i \leqq \alpha_i < c_{i+1}$, there exists a tree T_s of G such that

(1-1) $S_{\alpha_i} \supseteq \overline{T_s} = E - T_s$, (16)

(1-2) $1 > c_i |S_{\alpha_i} - \overline{T_s}| - r[(G \cdot S_{\alpha_i}) \times (S_{\alpha_i} - \overline{T_s})]$ (17)

are satisfied, then T_s is a central tree of G.

[Proof]

Since, for a critical set S_{α_i} of E $(c_i \leqq \alpha_i < c_{i+1})$ and for any subset S of E,

$$\alpha_i |S_{\alpha_i}| - r[G \cdot S_{\alpha_i}] \geqq \alpha_i |S| - r[G \cdot S] \qquad (18)$$

is always satisfied, we have

$$\alpha_i |S_{\alpha_i}| - r[G \cdot S_{\alpha_i}] \geqq \alpha_i |\overline{T}| - r[G \cdot \overline{T}] \qquad (19)$$

for every tree T of G.

Now, suppose that there exists a tree T_s of G such that the condition (1-1) is satisfied, then we have the relations:

$$|S_{\alpha_i}| = |\overline{T_s}| + |S_{\alpha_i} - \overline{T_s}| , \qquad (20)$$

$$r[G \cdot S_{\alpha_i}] = r[G \cdot \overline{T_s}] + r[(G \cdot S_{\alpha_i}) \times (S_{\alpha_i} - \overline{T_s})] \qquad (21)$$

from which it follows that for every tree T of G we have

$$\alpha_i |S_{\alpha_i} - \overline{T_s}| - r[(G \cdot S_{\alpha_i}) \times (S_{\alpha_i} - \overline{T_s})]$$

$$\geqq r[G \cdot \overline{T_s}] - r[G \cdot \overline{T}] \qquad (22)$$

because $|\overline{T_s}| = |\overline{T}|$. Here, considering $c_i \leqq \alpha_i < c_{i+1}$, we have

$$c_i |S_{\alpha_i} - \overline{T_s}| - r[(G \cdot S_{\alpha_i}) \times (S_{\alpha_i} - \overline{T_s})]$$

$$\geqq r[G \cdot \overline{T_s}] - r[G \cdot \overline{T}] \qquad (23)$$

for every tree of G. Furthermore, suppose that the condition (1-2) is satisfied, then for every tree T of G we have

$$1 > r[G \cdot \overline{T_s}] - r[G \cdot \overline{T}] \qquad (24)$$

from which it follows that for every tree T of G

$$r[G \cdot \overline{T_s}] \geqq r[G \cdot \overline{T}] \qquad (25)$$

because both $r[G \cdot \overline{T_s}]$ and $r[G \cdot \overline{T}]$ are non-negarive integers. Hence we see that the theorem is true. (END)

[Corollary 1-1]

If, for a critical sets S_{α_i} of E with respect to α_i such that $c_i \leqq \alpha_i < c_{i+1}$, there exists a tree T_s of G such that

(1-1) $S_{\alpha_i} \supseteq \overline{T_s}$, (16)

(1-3) $1 > c_i |S_{\alpha_i} - \overline{T_s}|$ (26)

are satisfied, then T_s is a central tree of G.

[Proof]

This is obvious from the theorem 1 and the non-negative integrality of $r[(G \cdot S_{\alpha_i}) \times (S_{\alpha_i} - \overline{T_s})]$. (END)

[Example 1]

Let G be a graph shown in Fig. 1(a). Then $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and all the critical sets of E with respect to all α such that $0 \leqq \alpha < \infty$ are

$$S_{\alpha_0} = S_{\alpha_0}^{(\infty)} = \emptyset ,$$

$$S_{\alpha_1}^{1} = \{6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15\}$$

$$S_{\alpha_1}^{2} = S_{\alpha_1}^{(\infty)} = \{4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15\}$$

$$S_{\alpha_2} = S_{\alpha_2}^{(\infty)} = \{1,\ 2,\ 3\} \cup S_{\alpha_1}^{(\infty)} = E$$

where $0 \leqq \alpha_0 < c_1$, $c_1 \leqq \alpha_1 < c_2$, $c_2 \leqq \alpha_2 < \infty$, $c_1 = 1/2$ and $c_2 = 2/3$.

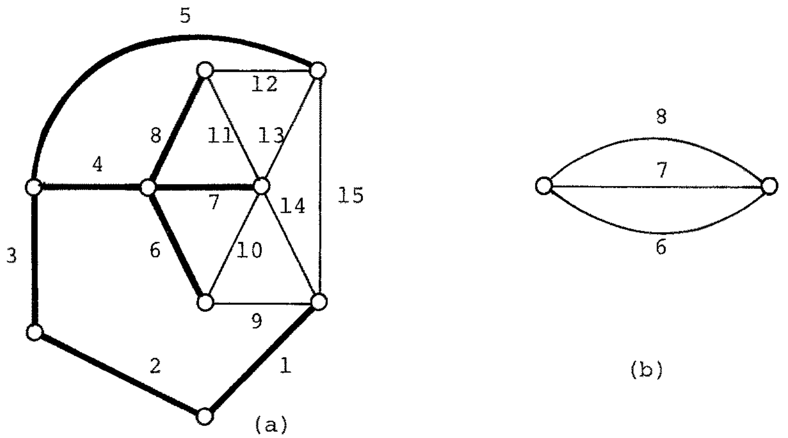

Fig. 1 Graphs for Example 1.

Now, if we choose $T_s = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ as a tree of G, then for the critical set $S_{\alpha_1}^{1}$ we have the relations:

$$S_{\alpha_1}^{1} \supseteq \overline{T}_s = \{9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15\} ,$$

$$|S_{\alpha_1}^{1} - \overline{T}_s| = |\{6,\ 7,\ 8\}| = 3,$$

$$r[(G \cdot S_{\alpha_1}^{1}) \times (S_{\alpha_1}^{1} - \overline{T}_s)] = 1$$

where $(G \cdot S_{\alpha_1}^{1}) \times (S_{\alpha_1}^{1} - \overline{T}_s)$ is shown in Fig. 1(b), and consequently we have

$$1 > c_1|S_{\alpha_1}^{1} - \overline{T}_s| - r[(G \cdot S_{\alpha_1}^{1}) \times (S_{\alpha_1}^{1} - \overline{T}_s)] = (1/2)\times 3 - 1 = 1/2.$$

Hence we see from the theorem 1 that T_s is a central tree of G.

(END)

[Example 2]

Let G be a graph shown in Fig. 2. Then $E = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14\}$ and all the critical sets of E with respect to all α such that $0 \leqq \alpha < \infty$ are

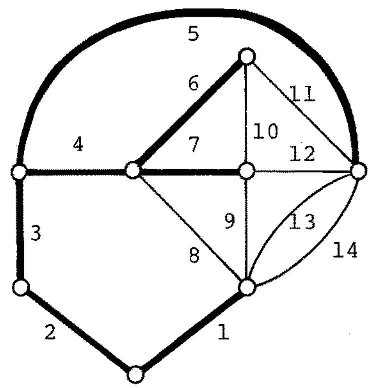

Fig. 2 A Graph for Example 2

$S_{\alpha_0} = S_{\alpha_0}^{(\infty)} = \emptyset$,

$S_{\alpha_1} = S_{\alpha_1}^{(\infty)} = \{6, 7, 8, 9, 10, 11, 12, 13, 14\}$,

$S_{\alpha_2} = S_{\alpha_2}^{(\infty)} = \{4, 5\} \cup S_{\alpha_1}^{(\infty)}$,

$S_{\alpha_3} = S_{\alpha_3}^{(\infty)} = \{1, 2, 3\} \cup S_{\alpha_2}^{(\infty)} = E$

where $0 \leqq \alpha_0 < c_1$, $c_1 \leqq \alpha_1 < c_2$, $c_2 \leqq \alpha_2 < c_3$, $c_3 \leqq \alpha_3 < \infty$, $c_1 = 4/9$, $c_2 = 1/2$ and $c_3 = 2/3$. Now, if we choose $T_s = \{1, 2, 3, 4, 5, 6, 7\}$ as a tree of G, then for the critical sets S_{α_1} we have the relations:

$S_{\alpha_1} \supseteq \overline{T_s} = \{8, 9, 10, 11, 12, 13, 14\}$,

$|S_{\alpha_1} - \overline{T_s}| = |\{6, 7\}| = 2$,

from which it follows that

$1 > c_1 |S_{\alpha_1} - \overline{T_s}| = (4/9) \times 2 = 8/9.$

Hence we see from the corollary 1-2 that T_s is a central tree of G.

(END)

[Theorem 2]

If, for a critical set S_{α_i} of E with respect to α_i such that $c_i \leqq \alpha_i < c_{i+1}$, there exists a tree T_s of G such that

(2-1) $S_{\alpha_i} \subseteq \overline{T_s} = E - T_s$, (27)

(2-2) $1 > (1 - \alpha_i)\, |\overline{T_s} - S_{\alpha_i}| - n[(G \cdot \overline{T_s}) \times (\overline{T_s} - S_{\alpha_i})]$ (28)

are satisfied, then T_s is a central tree of G.

[Proof]

As in the proof of the theorem 1, for a critical set S_{α_i} of E and for every tree T of G, there holds

$$\alpha_i |S_{\alpha_i}| - r[G \cdot S_{\alpha_i}] \geqq \alpha_i |\overline{T}| - r[G \cdot \overline{T}]. \tag{19}$$

Now, suppose that there exists a tree T_s of G such that the condition (2-1) is satisfied, then we have the relations:

$$|\overline{T}_s| = |S_{\alpha_i}| + |\overline{T}_s - S_{\alpha_i}| , \tag{29}$$

$$r[G \cdot \overline{T}_s] = r[G \cdot S_{\alpha_i}] + r[(G \cdot \overline{T}_s) \times (\overline{T}_s - S_{\alpha_i})]$$

from which it follows that for every tree T of G we have

$$- \alpha_i |\overline{T}_s - S_{\alpha_i}| + r[(G \cdot \overline{T}_s) \times (\overline{T}_s - S_{\alpha_i})]$$

$$\geqq r[G \cdot \overline{T}_s] - r[G \cdot \overline{T}] \tag{30}$$

because $|\overline{T}_s| = |\overline{T}|$. Since

$$|\overline{T}_s - S_{\alpha_i}| = r[(G \cdot \overline{T}_s) \times (\overline{T}_s - S_{\alpha_i})] + n[(G \cdot \overline{T}_s) \times (\overline{T}_s - S_{\alpha_i})] \tag{31}$$

is satisfied, we have

$$(1 - \alpha_i) |\overline{T}_s - S_{\alpha_i}| - n[(G \cdot \overline{T}_s) \times (\overline{T}_s - S_{\alpha_i})]$$

$$\geqq r[(G \cdot \overline{T}_s] - r[G \cdot \overline{T}] \tag{32}$$

for every tree T of G. Furthermore, suppose that the condition (2-2) is satisfied, then for every tree T of G we have

$$1 > r[G \cdot \overline{T}_s] - r[G \cdot \overline{T}] \tag{33}$$

from which it follows that for every tree T of G

$$r[G \cdot \overline{T}_s] \leqq r[G \cdot \overline{T}] \tag{34}$$

because both $r[G \cdot \overline{T}_s]$ and $r[G \cdot \overline{T}]$ are non-negative integers. Hence we see that the theorem is true. (END)

[Corollary 2-1]

If, for a critical set S_{α_i} of E with respect to α_i such that $c_i \leqq \alpha_i < c_{i+1}$, there exists a tree T_s of G such that

$$\text{(2-1)} \quad S_{\alpha_i} \subseteq \overline{T}_s , \tag{27}$$

$$\text{(2-2)} \quad 1 > (1 - \alpha_i) |\overline{T}_s - S_{\alpha_i}| \tag{35}$$

are satisfied, then T_s is a central tree of G.

[Proof] This is obvious from the theorem 2. (END)

[Example 3]

Let G be a graph shown in Fig. 3. Then E = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16} and all the critical sets of E with respect to all α such that $0 \leqq \alpha < \infty$ are

$$S_{\alpha_0} = S_{\alpha_0}^{(\infty)} = \phi ,$$

$$S_{\alpha_1} = S_{\alpha_1}^{(\infty)} = \{14, 15, 16\} ,$$

$$S_{\alpha_2}^{1} = \{12, 13\} \cup S_{\alpha_1}^{(\infty)} ,$$

$$S_{\alpha_2}^{2} = \{10, 11, 12, 13\} \cup S_{\alpha_1}^{(\infty)} ,$$

$$S_{\alpha_2}^{3} = \{8, 9, 12, 13\} \cup S_{\alpha_1}^{(\infty)} ,$$

$$S_{\alpha_2}^{4} = S_{\alpha_2}^{(\infty)} = \{8, 9, 10, 11, 12, 13\} \cup S_{\alpha_1}^{(\infty)} ,$$

$$S_{\alpha_3} = S_{\alpha_3}^{(\infty)} = \{1, 2, 3, 4, 5, 6, 7\} \cup S_{\alpha_2}^{(\infty)} = E$$

where $0 \leqq \alpha_0 < c_1$, $c_1 \leqq \alpha_1 < c_2$, $c_2 \leqq \alpha_2 < c_3$, $c_3 \leqq \alpha_3 < \infty$, $c_1 = 1/3$, $c_2 = 1/2$ and $c_3 = 4/7$.

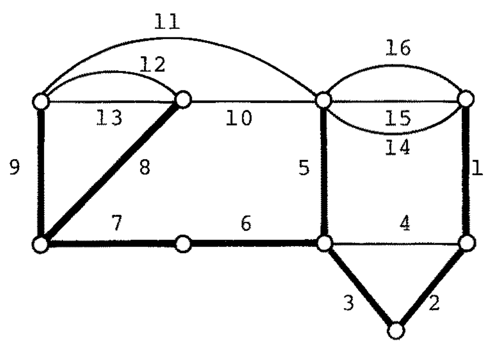

Fig. 3 A Graph for Example 3.

Now, if we choose $T_s^{(1)} = \{1, 2, 3, 5, 6, 7, 8, 9\}$ as a tree of G, then for the critical set $S_{\alpha_2}^{2}$ we have the relations

$$S_{\alpha_2}^{2} \subsetneq \overline{T_s^{(1)}} = \{4, 10, 11, 12, 13, 14, 15, 16\} ,$$

$$|\overline{T_s^{(1)}} - S_{\alpha_2}^{2}| = |\{4\}| = 1$$

from which it follows that

$$1 > (1-\alpha_2)|\overline{T_s^{(1)}} - S_{\alpha_2}^2| = (1-\alpha_2)\times 1 = 1-\alpha_2$$

Thus, $\alpha_2 > 0$. Here, since there exists α_2 such that $\alpha_2 > 0$ and $c_2 = 1/2 \leqq \alpha_2 < c_3 = 4/7$, we see from the corollary 2-1 that $T_s^{(1)}$ is a central tree of G.

On the other hand, if we choose $T_s^{(2)} = \{1, 2, 4, 5, 6, 7, 10, 11\}$ as a tree of G, then for the critical set $S_{\alpha_2}^3$ we have the relations:

$$S_{\alpha_2}^3 \subseteq \overline{T_s^{(2)}} = \{3, 8, 9, 12, 13, 14, 15, 16\},$$

$$|\overline{T_s^{(2)}} - S_{\alpha_2}^3| = |\{3\}| = 1$$

from which it follows that

$$1 > (1-\alpha_2)|\overline{T_s^{(2)}} - S_{\alpha_2}^3| = 1-\alpha_2.$$

Accordingly, we get $\alpha_2 > 0$. Since there exists α_2 such that $\alpha_2 > 0$ and $c_2 = 1/2 \leqq \alpha_2 < c_3 = 4/7$, we also see from the corollary 2-1 that $T_s^{(2)}$ is a central tree of G.

(END)

Now, considering that the condition (2-1) is equivalent to

$$(2'\text{-}1)\quad T_s \subseteq \overline{S_{\alpha_i}} = E - S_{\alpha_i} \tag{36}$$

we have the relations

$$\overline{T_s} - S_{\alpha_i} = \overline{S_{\alpha_i}} - T_s, \tag{37}$$

$$(G\cdot\overline{T_s})\times(\overline{T_s} - S_{\alpha_i}) = (G\times\overline{S_{\alpha_i}})\cdot(\overline{S_{\alpha_i}} - T_s) \tag{38}$$

from which it follows that the theorem 2 and its corollary 2-1 can be rewritten as follows:

[Theorem 2']

If, for a critical set S_{α_i} of E with respect to α_i such that $c_i \leqq \alpha_i < c_{i+1}$, there exists a tree T_s of G such that

$$(2'\text{-}1)\quad T_s \subseteq \overline{S_{\alpha_i}}, \tag{36}$$

$$(2'\text{-}2)\quad 1 > (1-\alpha_i)|\overline{S_{\alpha_i}} - T_s| - n[(G\times\overline{S_{\alpha_i}})\cdot(\overline{S_{\alpha_i}} - T_s)] \tag{39}$$

are satisfied, then T_s is a central tree of G. (END)

[Corollary 2'-1]

If, for a critical set S_{α_i} of E with respect to α_i such that $c_i \leqq \alpha_i < c_{i+1}$, there exists a tree T_s of G such that

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(2'-1) $T_s \subseteq \overline{S}_{\alpha_i}$, (36)

(2'-3) $1 > (1 - \alpha_i)|\overline{S}_{\alpha_i} - T_s|$ (40)

are satisfied, then T_s is a central tree of G. (END)

Also, as a special case of the theorem 1 and 2, the following known theorem and corollary can be derived:

[Theorem 3]

If, for a critical set S_{α_i} of E with respect to α_i such that $c_i \leqq \alpha_i < c_{i+1}$, there exists a tree T_s of G such that

(3-1) $T_s = \overline{S}_{\alpha_i}$ (41)

is satisfied, then T_s is a central tree of G. (END)

[Corollary 3-1]

If there exists a tree T_s of G such that for a critical set $S_{1/2}$ of E with respect to 1/2 there holds

(3-2) $T_s = \overline{S}_{1/2}$, (42)

then T_s is a central tree of G. (END)

This corollary was given and proved in 1977 by Kawamoto, Kajitani and Shinoda [6]. In 1980, as an extension of the corollary, the theorem 3 was proved in an elegant way by Shinoda, Kitano and Ishida [7]. Indeed it was the proof technique of the theorem 3 shown in [7] that suggested the present investigation.

4. Conclusions

In this paper, in connection with the critical sets of the edge set of a nonseparable graph, some new theorems on central trees of the graph have been given as a few extensions of the results obtained already in [6, 7].

Since all the critical sets of the edge set of a nonseparable graph can be easily obtained by Tomizawa's algorithm [12], the theorems and their corollaries presented in this paper may be very useful.

[12] N. Tomizawa: Strongly irreducible matroids and principal partitions of a matroid into strongly irreducible minors, Trans. Inst. Elec. comm. Eng. Japan, Vol. J59-A, pp.83-91, 1976.

Appendix The 2-nd hybrid equation and a central tree

Let N(G) be an electrical network whose underlying graph is G and whose edge-immittance matrix is a non-singular diagonal matrix. Each edge κ in N(G) is represented by either (a) or (b) of Fig. A where

s : complex variable in the Laplace transformation;

$v_\kappa(s)$: voltage of edge κ ;

$i_\kappa(s)$: current of edge κ ;

$e_\kappa(s)$: voltage of voltage source in edge κ ;

$j_\kappa(s)$: current of current source in edge κ ;

$z_\kappa(s)$: edge-impedance of edge κ ; and

$y_\kappa(s)$: edge-admittance of edge κ .

Among $v_\kappa(s)$, $e_\kappa(s)$, $i_\kappa(s)$, $j_\kappa(s)$, $z_\kappa(s)$ and $y_\kappa(s)$ there holds either

$$v_\kappa(s) = z_\kappa(s) \cdot (i_\kappa(s) + j_\kappa(s)) - e_\kappa(s) \tag{A-1}$$

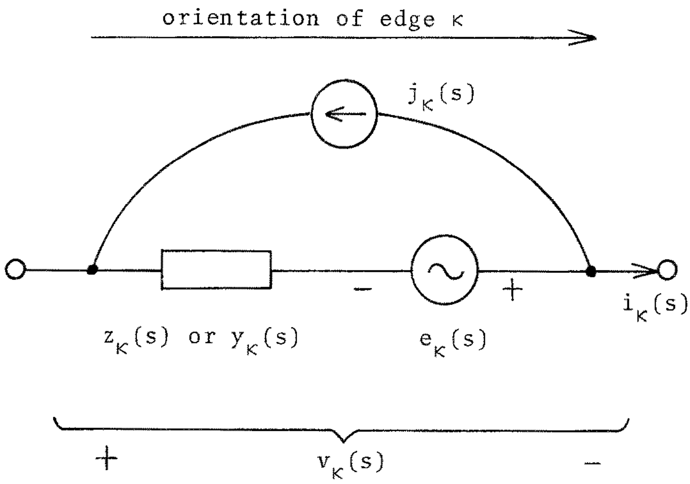
or

$$i_\kappa(s) = y_\kappa(s) \cdot (v_\kappa(s) + e_\kappa(s)) - j_\kappa(s). \tag{A-2}$$

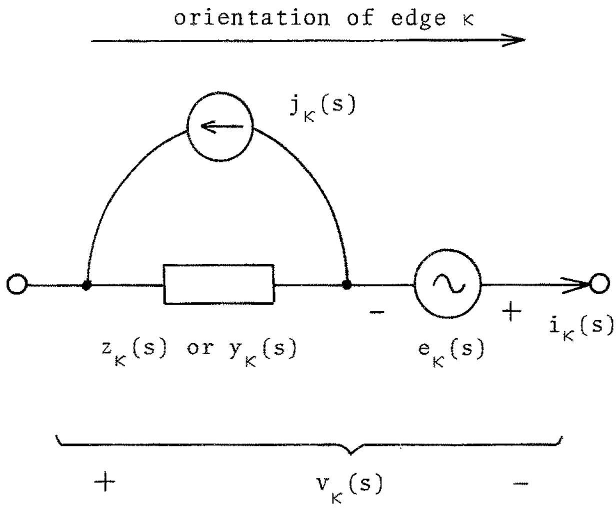
Here, (A-1) or (A-2) are called the v-i relations of edge κ .

For a tree t of G, t* is a tree of G which is at the maximal distance from the tree t. $\bar{t}$ and $\bar{t}^*$ are the cotrees of t and t*, respectively. Since each edge in $\bar{t} \cap \bar{t}^*$, together with some (or all) edges in $\bar{t} \cap t^*$, defines the fundamental tieset with respect to t*, it follows from Kirchhoff's voltage law that the voltages of the edges in $\bar{t} \cap \bar{t}^*$ can be uniquely expressed as the linear combinations of the voltages of the edges in $\bar{t} \cap t^*$.

Now, applying the v-i relations to the edges in $\bar{t}$, we see that the currents of the edges in $\bar{t}$ can be uniquely expressed as the linear combinations of the voltages of the edges in $\bar{t} \cap t^*$. Also, since each edge in t defines the fundamental cutset with respect to t, it follows from Kirchhoff's current law that the currents of the edges in t can be uniquely expressed as the linear combinations of the voltages of the edges in $\bar{t} \cap t^*$.



(a)



(b)

Fig. A An edge κ in N(G).

Moreover, applying the v-i relations to the edges in t, we see that the voltages of the edges in t can be uniquely expressed as the linear combinations of the voltages of the edges in $\bar{t} \cap t^*$. Namely, we see from the above that the voltages and the currents of all edges of N(G) can be uniquely expressed as the linear combinations of the voltages of the edges in $\bar{t} \cap t^*$.

Here, substituting the voltages of all edges of N(G) expressed as the linear combinations of the voltages of the edges in $\bar{t} \cap t^*$ into a system of Kirchhoff's voltage equations based on the fundamental tie-sets in G which are defined by the edges in $\bar{t} \cap t^*$ with respect to t, we obtain a system of equations whose variables are the voltages of the edges in $\bar{t} \cap t^*$. Such a system of equations is called the <u>2-nd hybrid equation</u> of N(G), since the elements in the coefficient matrix of the 2-nd hybrid equation are expressed in quadratic polynomials of edge-immittances.

The order of the 2-nd hybrid equations is $d(t) = |\bar{t} \cap t^*|$. d(t) varies under the choice of t. Since d(t) is the distance between t and t*, and since t is called a central tree of G if $d(t) \leq d(t')$ for every tree t' of G, we see that the 2-nd hybrid equation of minimum order can be obtained by choosing a central tree of G as t.

The above was originally pointed out in 1971 by Kishi and Kijitani [4] and subsequently considered in 1979 by Kajitani in a new context [5].