SOME COMMON PROPERTIES FOR REGULARIZABLE GRAPHS, EDGE-CRITICAL GRAPHS AND B-GRAPHS

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1. Introduction

The purpose of this paper is to display some relations between different classes of graphs, in particular:

- the edge-critical graphs,
- the well-covered graphs ("every maximal stable set is maximum"),
- the B-graphs ("each vertex is contained in some maximum stable set"),
- the regularizable graphs,
- the quasi-regularizable graphs.

Though some of the proofs have already appeared somewhere else, we shall give them in full for the basic lemmas. No use of the Farkas lemma will be made (as for similar researches about hypergraphs) and this paper is self contained.

2. Edge-critical Graphs

Let G be a simple graph with vertex-set X. For $x \in X$, let Γx denote the set of all neighbours of x; if $A \subseteq X$. put $\Gamma A = \cup \Gamma x$ if $A \neq \emptyset$, and $\Gamma A = \emptyset$ if $A = \emptyset$. Let $\alpha(G)$ denote the stability number, i.e. the largest number of independent vertices in G. An edge e of G is <u>α-critical</u> if

$$\alpha(G-e) > \alpha(G).$$

G is <u>α-edge-critical</u> or <u>edge-critical</u> if every edge is α-critical. This concept, introduced by Zykov [24], has been studied by many authors (see Plummer [15], Erdös-Gallai [8], Hajnal [11], Berge [3], George [10], Andrasfai [1], Suranyi [22], etc...)

In this paper we shall write $G \in \mathcal{G}_1$ if G is edge-critical and has no connected component isomorphic to K_1 (isolated vertex) or K_2

(isolated edge). We shall also consider some more general classes of graphs. We write $G \in \mathcal{G}_2$ if each vertex is of degree ≥ 2 and incident to at least one critical edge. An obvious result is:

<u>Proposition 1</u>. Every graph in $\mathcal{G}_1$ is also in $\mathcal{G}_2$.
<u>Proof</u>. It suffices to show that an edge-critical connected graph different from K_1 or K_2 has no vertex of degree 1. Let a be a vertex with $d_G(a)=1$. Since G is connected and has at least three vertices, there exists a vertex x with $[a,x] \in G$. Since $G \neq K_2$, and $d_G(a)=1$, there exists a vertex $b \neq a$ adjacent to x. Since $[b,x]$ is a critical edge, there exists a set S of cardinality $\alpha(G)+1$ which contains only one edge, namely the edge $[b,x]$. Hence $a \notin S$, and $(S-\{x\}) \cup \{a\}$ is a stable set of cardinality $\alpha(G)+1$, a contradiction.

Q.E.D.

<u>Definition</u>. We shall write $G \in \mathcal{G}_3$ if for each vertex x of G there exists a maximum stable set T_x such that $x \notin T_x$, $x \not\subset T_x$.

<u>Proposition 2</u>. Every graph in $\mathcal{G}_2$ is also in $\mathcal{G}_3$.
<u>Proof</u>. Let x be a vertex of a graph G in $\mathcal{G}_2$. Thus, x is incident to a critical edge $[x,b]$, and there exists at least one edge $[x,a] \neq [x,b]$. Also, there exists a set S_{bx} of cardinality $\alpha(G)+1$ which contains only one edge, namely $[b,x]$. So $x \in S_{bx}$, $a \notin S_{bx}$. Hence, $T=S_{bx}-\{x\}$ is a maximum stable set and $x \notin T$; since $a \notin T$, we have also $T \not\supset \Gamma x$. Q.E.D.

<u>THEOREM 1</u>. In a graph $G \in \mathcal{G}_3$, every stable set S satisfies $|\Gamma S| > |S|$.
<u>Proof</u>. We shall assume that G is connected without loss of generality.

We shall show, by induction on $|S|$, that $|\Gamma S| > |S|$ for every stable set S.

First, let $S=\{x\}$ be a singleton. Then x is not an isolated vertex (because $T_x \cup \{x\}$ would be a stable set larger than T_x). Also, x is not incident to only one edge, say $[x,y]$, because $x \notin T_x$, hence $y \in T_x$, hence $\Gamma x \subset T_x$, a contradiction. Thus $|\Gamma S| > |S|$.

Assume that every stable set S with cardinality $\leq p-1$ satisfies $|\Gamma S| > |S|$, and consider a stable set S with cardinality $p > 1$. Let $a \in S$; we have

$$|\Gamma S \cap T_a| \geq |S-T_a|$$

Otherwise, $|\Gamma S \cap T_a| < |S-T_a|$, and $T_a-(\Gamma S \cap T_a) \cup (S-T_a)$ would be a stable set larger than T_a, a contradiction.

Case 1: $S\cap T_a=\emptyset$. Since $\Gamma a-T_a\neq\emptyset$,

$|\Gamma S| \geq |\Gamma S\cap T_a| + |\Gamma a-T_a| > |\Gamma S\cap T_a| \geq |S-T_a|=|S|$.

Case 2: $S\cap T_a\neq\emptyset$. Then $S\cap T_a$ is a stable set with cardinality $\leq p-1$, and by the induction hypothesis, $|\Gamma(S\cap T_a)| > |S\cap T_a|$. Hence

$|\Gamma S| \geq |\Gamma(S\cap T_a)|+|\Gamma S\cap T_a| > |S\cap T_a|+|S-T_a| = |S|$.

In both cases we have $|\Gamma S| > |S|$, which completes the proof.

Q.E.D.

Note that THEOREM 1 generalizes a result of Hajnal [11] who has shown that in an edge-critical graph, every stable set S satisfies $|\Gamma S| \geq |S|$.

3. Regularizable graphs and quasi-regularizable graphs

Let G be a multigraph with no loops. We denote by m(G) the number of edges in G, by $\Delta(G)$ the maximum degree of its vertices, by $\nu(G)$ the matching number, i.e. the maximum size of a matching ("set of independent edges"), and by $\tau(G)$ the transversal number $= n-\alpha(G)$. For an integer k and an edge e of G, we say that we multiply e by k if we replace e by k parallel edges; if k=0, multiplying e by k means removing e. The graph kG is the graph obtained from G by multiplying each edge by k. We say that a graph G is regularizable if by multiplying each edge by an integer ≥ 1, we get a regular multigraph (of degree $\neq 0$). We say that G is quasi-regularizable if by multiplying each edge by an integer ≥ 0, we get a regular multigraph (of degree $\neq 0$),

In this section, we shall denote:

$\mathcal{G}_4$ = class of all regularizable graphs which have no bipartite connected components;

$\mathcal{G}_5$ = class of all regularizable graphs;

$\mathcal{G}_6$ = class of all quasi-regularizable graphs.

Proposition 3. Every graph in $\mathcal{G}_4$ is also in $\mathcal{G}_5$, and every graph in $\mathcal{G}_5$ is also in $\mathcal{G}_6$. The converses are not true (for example, the graph P_4 is quasi-regularizable but not regularizable.)

A fractional transversal of G is a non-negative function t(x), defined for $x\in X$, such that for each edge [x,y]

$t(x)+t(y) \geq 1$.

$\tau^*(G)$ denotes the minimum of $\Sigma_{x\in X}t(x)$ for all fractional transversals t.

A k-transversal of G is a function t(x) on X such that:

(i) $t(x) \in \{0,1,2,\dots,k\}$,

(ii) $t(x)+t(y) \geq k$ for every edge [x,y] of G.

$\tau_k(G)$ denotes the minimum of $\Sigma_{x \in X} t(x)$ when t ranges over all the k-transversals of G. Thus $\tau_1(G)$ is the usual transversal number $\tau(G)$, i.e. the minimum cardinality of a set $T \subset X$ which meets all the edges. A partial graph H of kG is called a k-matching if $\Delta(H) \leq k$. So a 1-matching is an ordinary matching.

Denote by $\nu_k(G)$ the maximum number of edges in a k-matching. Thus $\nu_1(G)=\nu(G)$ is the usual matching number.

Lemma 1. For every graph G,

$\nu_k(G)/k \leq \tau^*(G) \leq \tau_k(G)/k$.

Checking these inequalities is easy.

Lemma 2. Let G be a simple graph. Then there exists an optimal 2-matching H of G such that each connected component of H is either a single vertex or a pair of parallel edges ("double edge") or an odd cycle. For every 2-matching of that kind there exists an optimal 2-transversal with values: 0 for a singleton of H, (0,2) or (1,1) for the two vertices belonging to a double edge of H, 1 for a vertex belonging to an odd cycle of H.

Proof. We may assume that G is connected. Let H be a maximum 2-matching. Every connected component of H which is a path or a cycle of even length can be replaced by a set of pairwise disjoint double-edges without changing m(H). No component of H is an odd path (i.e. a path of odd length) since m(H) is maximum. Thus a connected component of H is either a simple vertex or a double edge or an odd cycle. Now we shall label each vertex with 0,1 or 2, by an iterative procedure described by the following rules:

(1) label with 0 each vertex which is a singleton of H;

(2) label with 2 each vertex which is adjacent in G to a vertex previously labelled with 0;

(3) label with 0 every vertex which is adjacent in H to a vertex previously labelled with 2;

(4) every vertex which cannot be labelled by the iterative procedure described by rules (1),(2), and (3) will be labelled 1.

No odd chain, starting from a singleton of H and consisting alternately of edges of G-H and of double-edges of H, ends with a singleton because such a chain would constitute a connected component of a 2-matching H' with $m(H') > m(H)$. Similarly no odd chain of that kind can end in an odd cycle of H. No odd chain of that kind can cross itself at a vertex labelled with 0 (because there would be a better 2-matching having as connected components an odd cycle and a set of double edges).

Thus a unique label $t(x)$ is given to each vertex x by the above rules; furthermore,

$t(x)=0$ if x is a singleton of H,

$t(x)=2$ and $t(y)=0$ (or vice versa) if $[x,y]$ is a double-edge connectable to a singleton (otherwise $t(x)=t(y)=1$), $t(x)=1$ if x belongs to an odd cycle of H.

The rule (2) shows that $t(x)$ is a 2-transversal of G. Furthermore, by lemma 1,

$$m(H)/2 \leq \nu_2(G)/2 \leq \tau^*(G) \leq \tau_2(G)/2 \leq \Sigma_{x\in X} t(x) = m(H)/2.$$

Therefore, these inequalities hold as equalities; and $t(x)$ is an optimal 2-transversal and H is an optimal 2-matching. Q.E.D.

Lemma 3. For every graph G,

$$\nu(G) = \min_k \nu_k(G)/k \leq \max_k \nu_k(G)/k = \nu_2(G)/2 = \tau^*(G)$$
$$= \tau_2(G)/2 = \min_k \tau_k(G)/k \leq \max_k \tau_k(G)/k = \tau(G).$$

Proof. - We have $\nu(G) \geq \inf \nu_k(G)/k$ because $\nu(G)=\nu_1(G)$.

- We have $\nu(G) \leq \inf \nu_k(G)/k$: let H be a maximum matching; so kH is a k-matching, and

$$\nu(G) = m(H) = m(kH)/k \leq \nu_k(G)/k.$$

Hence $\nu(G) = \min \nu_k(G)/k$.

- We have $\tau(G) \geq \sup_k \tau_k(G)/k$.

Let T be a minimal transversal set , with characteristic function $t(x)$. Then $kt(x)$ is a k-transversal; hence:

$$\tau(G) = |T| = \Sigma kt(x)/p \geq \tau_k(G)/k$$

- We have $\tau(G) \leq \sup \tau_k(G)/k$, because $\tau(G)=\tau_1(G)$. Hence, $\tau(G) = \max \tau_k(G)/k$.

- The other equalities follow from lemma 1 and lemma 2.

Q.E.D.

THEOREM 2. For a graph G, the following conditions are equivalent:

(1) G is quasi-regularizable;

(2) $t(x) \equiv 1$ is an optimal 2-transversal;

(3) G has a partial graph whose connected components are either a K_2 or an odd cycle;

(4) $|\Gamma S| \geq |S|$ for every stable set S.

<u>Proof</u>. (1) <u>implies</u> (2). If G is quasi-regularizable and has n vertices, there exists a k-regular graph $H \subseteq kG$, and by counting in two different ways the edges of the incident graph of H, we get $kn=2m(H)$. Hence:

$$n/2 = m(H)/k \leq \nu_k(G)/k \leq \tau^*(G) = \tau_2(G)/2 \leq n/2$$

(since $t(x) \equiv 1$ is a 2-transversal of G). Thus, the quasi-regularizablity implies $\tau_2(G) = n$. So the 2-transversal $t(x) \equiv 1$ is an optimal one.

(2) <u>implies</u> (1). By lemma 3 we see that $\tau^*(G) = n/2$ implies $\nu_2(G) = n$; so by lemma 2 we can cover the vertex-set of G with isolated double-edges and odd cycles; consequently G is quasi-regularizable.

(2) <u>implies</u> (3) (as above).

(3) <u>implies</u> (2) (obvious).

(2) <u>implies</u> (4).

Let S be a stable set. Put :

$t(x)=0$ if $x \in S$,

$t(x)=2$ if $x \in \Gamma S$,

$t(x)=1$ otherwise.

Clearly, $t(x)$ is a 2-transversal, and by (2), $\Sigma t(x) \geq n$. Hence

$|\Gamma S| - |S| = \Sigma_x (t(x)-1) \geq n-n = 0.$

(4) <u>implies</u> (2). Let $t(x)$ be a 2-transversal; clearly, $S=\{x \mid t(x)=0\}$ is a stable set and $\Gamma S \subseteq \{x \mid t(x)=2\}$. Hence

$$\Sigma_x t(x) = n + \Sigma_x (t(x)-1) \geq n + |\Gamma S| - |S| \geq n.$$

Hence the 2-transversal identical to 1 is an optimal one. Q.E.D.

<u>COROLLARY</u>. Let G be a connected graph of even order such that every pair of vertex-disjoint odd cycles is linked by an edge. A necessary and sufficient condition that G possess a perfect matching is that $|\Gamma S| \geq |S|$ for every stable set S.

<u>Proof</u>. By Theorem 2, $|\Gamma S| \geq |S|$ for every stable set S if and only if G can be covered with a set of K_2's and of odd cycles (vertex-disjoint). The number of odd cycles is even, and each pair of odd cycles, which is linked by an edge, can be covered by a set of K_2's. Thus G can be covered with disjoint K_2's, i.e. a perfect matching. Q.E.D

<u>REMARK</u>. This corollary has been proved by different methods by Fulkerson, McAndrew, Hoffman [9]. Theorem 2 was partly found by Tutte([23]) who proved in 1952 the following result: G has a perfect 2-matching iff for every $A \subseteq X$ the number of connected components of

G_{X-A} which are isolated vertices is $\leq |A|$.

A similar condition for the existence of a perfect 2-matching without K_3 was found by CORNUEJOLS and PULLEYBLANK ([7]). In [17], PULLEYBLANK shows also that G has a perfect 2-matching iff for every $A \subseteq X$,

$$|\{x \mid x \in X-A,\ \Gamma x \subseteq A\}| \leq |A|.$$

Note that in Theorem 2, (2) can be replaced by:

(2') For every $A \subseteq X$, $|\Gamma A| \geq |A|$.

<u>THEOREM 3</u>. For a simple graph G, the following conditions are equivalent:

(1) G is regularizable and has no bipartite connected component;

(2) the optimal 2-transversal is unique and is defined by $t(x) \equiv 1$;

(3) $|\Gamma S| > |S|$ for every stable set S.

<u>Proof</u>. (1) <u>implies</u> (2). Let G be a graph and let H be a regular multigraph obtained from G by edge-multiplication. Then

$$\tau_2(G)=2\tau^*(G)=2\tau^*(H)=2m(H)/\Delta(H)=2n(H)/2=n.$$

Thus, $t(x) \equiv 1$ is an optimal 2-transversal for G.

Now, assume that there exists another optimal 2-transversal $t'(x)$, and for $s=0,1,2$, put

$$A_s=\{x \mid x \in X,\ t'(x)=s\}.$$

Then $|A_0|=|A_2| \neq 0$. The set A_0 is stable (otherwise $t'(x)$ would not be 2-transversal), and $\Gamma A_0 \subseteq A_2$. We have $\Gamma A_0=A_2$ (otherwise, $t'(x)$ would not be optimal; a better 2-transversal can be obtained from $t'(x)$ by replacing a 2 by a 1. Since H is regular,

$$\Delta(H)|A_0| = m_H(A_0,A_2)$$
$$\leq \Sigma_{x \in A_2} m_H(x,A_0) \leq |A_2|\ (H)=\Delta(H)|A_0|.$$

Hence $m_H(x,A_0)=\Delta(H)$ for all $x \in A_2$, and no edge goes out of $A_0 \cup A_2$. Since G is connected, its vertex set is $A_0 \cup A_2$ and G is a bipartite graph having two vertex classes with the same cardinality. This contradicts the hypothesis.

(2) <u>implies</u> (3). Let S be a stable set, and let $H \subseteq 2G$ be an optimal 2-matching as described in Lemma 2.

Since $t(x) \equiv 1$ is an optimal 2-transversal, we have $\tau_2(G)=n$, so the connected components of H are either double edges or odd cycles. Hence

$$|\Gamma_G S| \geq |\Gamma_H S| \geq |S|.$$

If $|\Gamma_G S|=|S|$, it would follow that all the components of H meeting S are double edges. We can then define a 2-transversal $t'(x)$ by putting

$t'(x)=0$ if $x \in S$,

$\quad 2$ if $x \in \Gamma S$,

$\quad 1$ if $x \in X-(S \cup \Gamma S)$.

Since $t'(x)$ would also be an optimal 2-transversal of G, this contradicts the uniqueness of the optimal 2-transversal. Thus $|\Gamma S| > |S|$.

(3) implies (1). Now assume that $|\Gamma S| > |S|$ for every stable set S. Let H be a bipartite graph whose vertex-classes are two copies X and $\bar{X}$ of the vertex set of G, the vertices $x \in X$ and $\bar{y} \in \bar{X}$ being joined by an edge in H if and only if x and y are adjacent in G.

Let $B \subseteq X$, $B \neq \emptyset$, $B \neq X$, be a set such that the subgraph G_B has no isolated vertex. Then $\Gamma_H(B) \supseteq \bar{B}$. Now let $S \subseteq X$ be a set such that G_S has only isolated vertices. Then S is a stable set of G, and by (3),

$|\Gamma_H S| = |\Gamma_G S| > |S|$.

So, for every set $A = B \cup S$, $A \neq \emptyset$, $A \neq X$, we have

$|\Gamma_H A| > |A|$,

noting that $\Gamma_H S \cap \bar{B} = \emptyset$ if there are no edges between B and S.

First, we shall show that each edge $[a,\bar{b}]$ of H belongs to at least one perfect matching, that is , the subgraph H' of H induced by $(X \cup \bar{X})-\{a,\bar{b}\}$ has a perfect matching. For every $A \subset X-\{a\}$,

$|\Gamma_{H'} A| = |\Gamma_H A - \{b\}| \geq |\Gamma_H A| - 1 \geq |A|$.

Thus, by König's theorem, H has such a matching.

Consequently, for each edge [a,b] of G, there exists a 2-matching which saturates all the vertices and which uses the edge [a,b]. The union of all these possible 2-matchings defines a regular multigraph which is from G by edge-multiplications. Thus G is regularizable.

Q.E.D.

REMARK. It is easy to see that in Theorem 3, (3) can be replaced by

(3') For every non-empty set A, $|\Gamma A| > |A|$.

Theorem 3 was stated as above in [3],[4], but equivalent results were found independently by PULLEYBLANK [17], NEMHAUSER and TROTTER [14], BRUALDI [6]. In fact, those graphs are also called "2-bicritical" by PULLEYBLANK, with the following definition:

(4) For every x, $G_{X-\{x\}}$ is quasi-regularizable.

Clearly (3) implies (4) because every stable S' of $G'=G_{X-\{x\}}$ satisfies $|\Gamma_{G'} S'| \geq |\Gamma_G S'|-1 \geq |S'|$, and (4) follows from Theorem 2. Conversely, if (3) is false, there exists in G a stable S with $|\Gamma_G S| \leq |S|$. So for $x \in \Gamma_G S$, $G'=G_{X-\{x\}}$ satisfies $|\Gamma_{G'} S| < |S|$, and (4) is false.

Proposition 4. Every graph in $\mathcal{G}_3$ is also in $\mathcal{G}_4$.

This follows from Theorem 1 and Theorem 3.

Lemma Let G=(X,Y;E) be a bipartite connected graph. Then G is regularizable if and only if

$|\Gamma S| > |S| \quad (S \subseteq X,\ S \neq X)$,

$|\Gamma T| > |T| \quad (T \subseteq Y,\ T \neq Y)$,

$|X| = |Y| \neq \emptyset$.

Proof. This follows immediately from the theorem of Konig, which gives a necessary and sufficient condition that for each edge [x,y], the subgraph induced by $(X-\{x\}) \cup (Y-\{y\})$ has a perfect matching, that is G has a perfect matching containig [x,y]. (The union of all these perfect matchings gives a regular multigraph, which shows that G is regularizable). Q.E.D.

THEOREM 4. A graph G is regularizable iff $|\Gamma S| \geq |S|$ for every stable set S, and $|\Gamma S| = |S| \Longrightarrow \Gamma(\Gamma S) = S$.

This follows immediately from this lemma and Theorem 3.

In [5], we have also shown that the line-graph of an r-uniform hypergraph with no vertex of degree 1 and no edge meeting less than r other edges is regularizable. An impotant class to be considered is the claw-free graphs, i.e. the graphs which have no induced subgraphs isomorphic to $K_{1,3}$ (for instance, the line-graphs). M. Las Vergnas [13] and D. Sumner [21] have shown independently that a connected claw-free graph with an even number of vertices has a perfect matching (and therefore is quasi-regularizable). A claw-free graph is not always regularizable, as we can see with the following graph: take an even cycle, whose vertices are colored alternately with red and blue, and add a few (at least one) triangular chords connecting two blue vertices at distance 2. Such a graph is called a C-graph; Jaeger and Payan have shown:

THEOREM 5([12]). A connected claw-free graph G is regularizable if and only if G has no pendant vertices and is not a C-graph.

Proof. It suffices to show that a connected claw-free graph G with no pendant vertices and which is not regularizable is a C-graph.

Thus, G is not bipartite (otherwise, a vertex in one vertex-class is adjacent to exactly two vertices in the other class, so G is an even cycle, so G is regularizable, a contradiction).

So, by Theorem 4, there exists a stable set S with $|\Gamma S| \leq |S|$. Each vertex in S has at least 2 neighbours in S (otherwise there is a pendant vertex), and each vertex in ΓS has less than 3 neighbours in S (otherwise there is an induced $K_{1,3}$). So the number of edges $m(S,\Gamma S)$ between S and ΓS satisfies:

$$2|S| \leq \Sigma_{x\in S}\, d(x) = m(S,\Gamma S) \leq 2|\Gamma S| \leq 2|S|.$$

Hence $|S|=|\Gamma S|$ and every vertex in S has exactly two neighbours in ΓS, and every vertex in ΓS has exactly two neighbours in ΓS. Furthermore, a vertex in ΓS has no neighbour in $X-(S\cup\Gamma S)$, and since G is connected, $X=S\cup\Gamma S$.

Since G is not bipartite, $G_{\Gamma S}$ has at least one edge $e=[x,y]$; the vertices x and y have a common neighbour z in S (since G is claw-free). So G is a C-graph. Q.E.D.

The same proof shows that a claw-free graph with no pendant vertex is quasi-regularizable. So, a claw-free graph with no pendant vertex is quasi-regularizable.

4. Well-covered graph

A graph G is <u>well-covered</u>, or $G\in\mathcal{G}_7$, if G has no isolated vertices and if every maximal stable set is also a maximum stable set. This class is independant of $\mathcal{G}_1$. For instance, the graph P_4 (elementary chain with four vertices) is well-covered but is not in $\mathcal{G}_1$; the graph C_9 is in $\mathcal{G}_1$ but is not well-covered. Nevertheless, we shall see that the two classes have similar properties.

A graph G is a <u>B-graph</u>, or $G\in\mathcal{G}_8$, if G has no isolated vertices and if each vertex belongs to some maximum stable set.

<u>Proposition 5</u>. Every well-covered graph is a B-graph.

(Trivial).

<u>Proposition 6</u>. Every graph in $\mathcal{G}_2$ is a B-graph.

(Trivial).

The converse is not true: the graph C_6 is a B-graph, but is not in $\mathcal{G}_2$. The characterizations of well-covered graphs and of B-graph are difficult problems as quoted by Plummer [16]; Ravindra [19] has shown that a 2-connected graph G with no odd cycles of length ≥ 5 is a B-graph if and only if G is isomorphic to K_3, or to K_4, or is a bipartite graph with a perfect matching.

For a graph G with n vertices, a transversal set is the complement of a stable set, and therefore, the cardinality of a minimum transversal set is $\tau(G) = n - \alpha(G)$. A vertex x is <u>τ-critical</u> if $\tau(G-x) < \tau(G)$, that is if there exists a maximum stable set S_x which does not contain x. A graph G whose vertices are all τ-critical is said to be <u>τ-vertex-critical</u>, or to be in class $\mathcal{G}_9$.

<u>Proposition 7</u>. Every B-graph is τ-vertex-critical.

(Trivial).

<u>Proposition 8</u>. Every graph in $\mathcal{G}_3$ is τ-vertex-critical.

(Trivial).

The converse is not true. The graph in FIGURE 1 is in $\mathcal{G}_9$ but not in $\mathcal{G}_8$ or in $\mathcal{G}_3$.

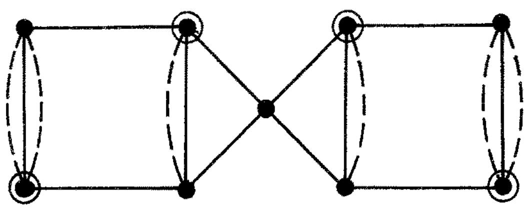

FIGURE 1. $\alpha = 4$

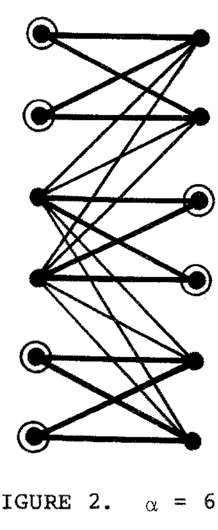

FIGURE 2. $\alpha = 6$

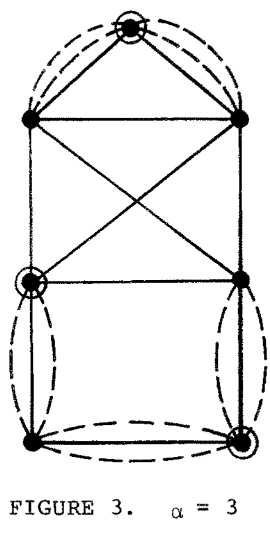

FIGURE 3. $\alpha = 3$

A τ-vertex-critical graph is not necessarily regularizable, as we can see with the graph in FIGURE 2; a graph regularizable is not necessarily τ-vertex-critical, as we can see with the graph represented in FIGURE 3.

However, we shall show that every τ-vertex-critical graph is quasi-regularizable. We first need a lemma.

Lemma. In a graph G, a stable set S is maximum if and only if every stable set T disjoint from S can be matched into S.
Proof. 1. Let S be a maximum stable set, and let T be a disjoint stable set.

Let $H=(T,S;E)$ be the bipartite graph defined by the edges having one end point in T and the other in S.
For $B \subseteq T$, we have $|B| \leq |\Gamma_H B|$ (otherwise, $|B| > |\Gamma_H B|$, and $B \cup (S-\Gamma_H B)$ would be a stable set with cardinality $> |S|$, a contradiction).

Thus, by the theorem of König, there exists a matching between T and S saturating all the vertices in T.

2. Now, assume that every stable set T can be matched into S; let B be a maximum stable set, $B \neq S$; then $T=B-S$ can be matched into S, and therefore into $S-B$. So, $|B-S| \leq |S-B|$.

Hence $|S| \geq |B|$, and S is a maximum stable set. Q.E.D.

THEOREM 6. Every τ-vertex-critical graph is quasi-regularizable.
Proof. Let G be a τ-vertex-critical graph; so for every vertex a, there exists a maximum stable set T_a with $a \notin T_a$.

Now, we show that $|\Gamma S| \geq |S|$ for every stable set S by induction on $|S|$.
- if $|S|=1$, this is trivial.
- if $|S|= p > 1$, consider a vertex $a \in S$, and a maximum stable set T_a which does not contain a. By the lemma, $S-T_a$ can be matched into T_a, and therefore into T_a-S; also, $S \cap T_a$ can be matched into $X-(S \cap T_a)$, by the induction hypothesis (because $|S-T_a| < |S|= p$). Thus S can be matched into X-S and $|\Gamma S| \geq |S|$. Q.E.D.

Remark that the converse is not true; the graph in FIGURE 3 is quasi-regularizable but is not τ-vertex-critical (because all the maximum stable sets contain a).

The results of this section can be summarized by the diagram shown in FIGURE 4.

We see that $G \in \mathcal{G}_4 \not\Rightarrow G \in \mathcal{G}_9$, because the graph in FIGURE 3 is regularizable and not τ-vertex-critical (because of the point a). Also, $G\in\mathcal{G}_9 \not\Rightarrow G\in\mathcal{G}_5$, because the graph in FIGURE 2 belongs to $\mathcal{G}_7$, $\mathcal{G}_8$, $\mathcal{G}_9$, but not to $\mathcal{G}_5$.

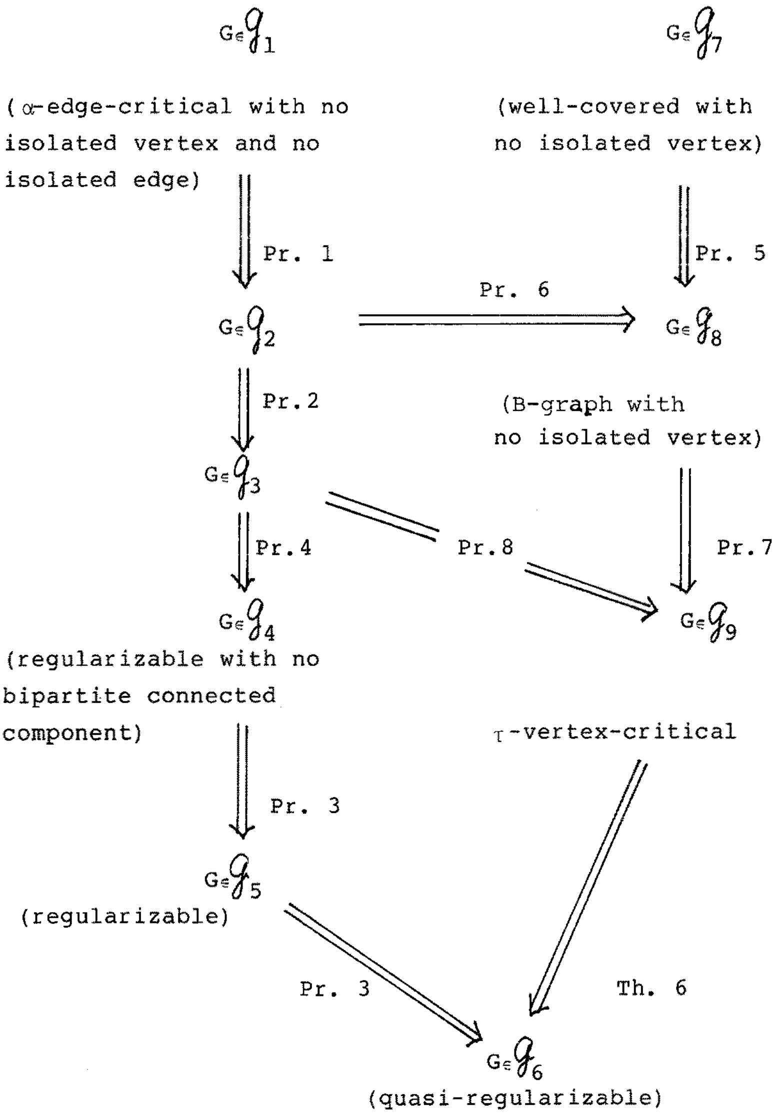

FIGURE 4

5. Case of bipartite graphs

If G is bipartite, we have $G\notin\mathcal{G}_1$, $G\notin\mathcal{G}_2$, $G\notin\mathcal{G}_3$, $G\in\mathcal{G}_4$, and the other properties considered in the preceding sections are easier to characterize.

We have:

Proposition 9. For a bipartite graph $G=(X,Y;E)$, the following properties are equivalent:

- G is quasi-regularizable,
- G has a perfect matching,
- G is a B-graph,
- G is τ-vertex-critical.

Proof. If G is quasi-regularizable, then G has a perfect matching by Theorem 2 (characterization 3).

If G has a perfect matching, then X is a minimum transversal set (by the König theorem), hence X and Y are both maximum stable sets; so G is a B-graph.

If G is a B-graph, then G is τ-vertex-critical, by Proposition 5.

If G is τ-vertex critical, then G is quasi-regularizable, by Theorem 6. Q.E.D.

Proposition 10. (Ravindra [18]). A tree T is well-covered if and only if its pendant edges constitute a perfect matching.

Proof. 1. An edge is pendant if it is incident to a vertex of degree 1. Let G be a graph (not necessarily a tree) whose pendant edges constitute a perfect matching M; then the pendant vertices constitute a stable set of cardinality n/2, and $\alpha(G)=n-\tau(G)\leq n-|M|=n/2$. So $\alpha(G)=n/2$. If a stable set S_0 has less than n/2 elements, there exists an edge $e\in M$ which does not meet S_0, so S_0 plus the pendant vertex attached to e is also a stable set. This shows that every maximal stable set is also maximum, i.e. $G\in\mathcal{G}_7$.

2. Now, let T be a well-covered tree. So, $T\in\mathcal{G}_8$, and since T is bipartite, T has a perfect matching M (by Proposition 7 and Theorem 6).

Hence, $\alpha(T)=n-\tau(T)=n-|M|=n/2$, and a maximal stable set has exactly one point in each edge of M.

Now, let $e=[a,b]\in M$, and assume that e is not a pendant edge. Then there exists two edges $[b,b']$ and $[a,a']$ with $a'\neq b'$, $[a',b']\notin T$ (since T has no cycle). Therefore, the maximal stable set which contains a' and b' does not meet $\{a,b\}$ and cannot be maximum: a contradiction. Thus, every edge in M is pendant (and every pendant edge is in M, because the matching M is perfect). So T has the required property.

Q.E.D.

Lemma. Let G be a well-covered graph having a perfect matching M such that no alternating chain constitute two disjoint odd cycles linked by an odd chain. Then for each edge $[a,b]\in M$, the set $\{a,b\}\cup\Gamma a\cup\Gamma b$ induces on G a complete-bipartite graph.

Proof. By a theorem of Sterboul [20], if there exists a perfect matching of the described kind, then $\tau(G)=\nu(G)$. Hence $\alpha(G)= n-\tau(G)= n-\nu(G)= n/2$.

So, a maximum stable set has exactly one point in each edge of M.

Let $[a,b]\in M$. If a' is a neighbour of a and b' a neighbour of b, then $a'\neq b'$ (otherwise a maximum stable set containing $\{a'\}$ cannot meet [a,b]). Also, [a',b'] is an edge of G (otherwise a maximum stable set containing $\{a',b'\}$ does not meet [a,b]).

So $\{a,b\}\cup\Gamma a\cup\Gamma b$ induces on G a complete bipartite graph.

Q.E.D.

Proposition 11. Let G be a connected regularizable bipartite graph; then G is well-covered if and only if G is isomorphic to a complete bipartite graph $K_{r,r}$.

Proof. Clearly $K_{r,r}$ is well-covered.

Conversely, Let $G=(X,Y;E)$ be a well-covered bipartite graph. If G is not isomorphic to a complete-bipartite graph $K_{r,s}$, then there exists a $x\in X$ and a $y\in Y$ whose distance $d(x,y)$ is larger than 1, so there exists a set $\{x,b,a,y\}$ which induces a P_4. The edge [a,b] belongs to some perfect matching M (since a regular bipartite multigraph has the edge-coloring-property). Applying the lemma with the edge [a,b], we get a contradiction.

Thus G is isomorphic to $K_{r,s}$, and since G is regularizable, $|\Gamma S| \geq |S|$ for all stable set S, hence r=s. Q.E.D.

REFERENCES

[1] B. Andrásfai, On critical graphs, Théorie des Graphes (Rome I.C.C.), Paris, (1967), 9-19.
[2] C. Berge, Graphes et Hypergraphes (Dunod, Paris, 1970).
[3] C. Berge, Une propriete des graphes k-stables critiques, Combinatorial Structures (Gordon and Breach, New York, 1970) 7-11.
[4] C. Berge, Regularizable Graphs, Proc. I.S.I. Conference on Graph Theory, Calcutta, 1976.
[5] C. Berge, Regularizable Graphs, Annals of Discrete Math. 3 (1978) 11-19.
[6] R. A. Brualdi, Combinatorial Properties of symmetric non-negative matricies, Coll. Th. Combinat. Rome, 2 (1976) 99-120.
[7] G. Cornuejols, W. Pulleyblank, A matching problem with side conditions, Discrete Math. 29 (1980) 135-139.

[8] P. Erdős, T. Gallai, On the minimal number of vertices representing the edges of a graph, Publ. Math. Inst. Hung. Acad. Sci. 6 (1961) 181-203.
[9] D. R. Fulkerson, A. J. Hoffman, M. H. McAndrew, Some properties of graphs with multiple edges, Can. J. Math. (1965) 166-177.
[10] A. George, On line-critical graphs, Thesis, Vanderbilt Univ., Nashville, TN (1971).
[11] A. Hajnal, A theorem on k-saturated graphs, Can. J. Math. 17 (1965) 720-772.
[12] F. Jaeger, C. Payan, A class of regularizable graphs, Annals of Discrete Math. 3 (1978) 125-127.
[13] M. Las Vergnas, A note on matchings, Acts Coll. Brussells, CER, 17, 1975, 255-260.
[14] G. L. Nemhauser, L. E. Trotter, Vertex packings: structural properties and algorithms, Math. Programming 8 (1975) 232-248.
[15] M. D. Plummer, On a family of line critical graphs, Monatsh. Math. 71 (1967) 40-48.
[16] M. D. Plummer, Some covering concepts in Graphs, J. Comb. Theory B (1970) 46-48.
[17] W. R. Pulleyblank, Minimum node covers and 2-bicritical graphs, Math. Programming 17 (1979) 91-103.
[18] G. Ravindra, Well coverd Graphs, I.I.T. Madras (1976).
[19] G. Ravindra, B-Graphs, Symposium on Graph Theory, I.S.I. (1976).
[20] F. Sterboul, A characterization of the graphs in which the transversal number equals the matching number, J. Comb. Theory B 27 (1979) 228-229.
[21] D. P. Sumner, Graphs with 1-factors, Proc. Am. Math. Soc. 42, 1974, 8-12.
[22] L. Suranyi, On line-critical graphs, Infinite and finite sets, (North Holland, Amsterdam, 1975) 1411-1444.
[23] W. Tutte, The factors of graphs, Canad. J. Math. 4 (1952) 314-328.
[24] A. A. Zykov, On some properties of linear complexes, Math. USSR Sb. 24 (1949) 163-188.