CHAPTER II

CONTEXT-FREE GRAMMAR FORMS

We initiate the study of form theory by considering context-free grammar forms. As will be seen many of the reduction and normal form results for context-free grammars also hold for context-free grammar forms. These results together with some stronger ones enable us to obtain inductive proofs for closure results in the case of g-interpretations. Moreover these normal forms also enable the proof of principality of the g-grammatical families to be carried out by induction as well. These results are found in Sections II.1, II.2 and II.4. In Section II.3 the study of collections of grammar families is initiated leading to the notion of a production minimal grammar form and the effectiveness of its construction. Section II.5 is concerned with syntax analysis and pushdown acceptor forms, while Section II.6 is concerned with collections of grammatical families and some decidability results.

We use the notation $\mathcal{L}$(FIN), $\mathcal{L}$(REG), $\mathcal{L}$(LIN) and $\mathcal{L}$(CF) to denote the families of finite, regular, linear and context-free languages, respectively.

Bibliographic and historical comments are to be found in Chapter V.

II.1 The Basics of Context-Free Grammar Forms

This section introduces the two kinds of interpretation upon which most of grammar form theory is based. This is followed by a number of examples not only to give the reader some insight into these ideas but also to give some motivation for the choice of the particular interpretations that have been mainly investigated up until now. Finally the notions of the grammar and grammatical families defined by a grammar form are introduced. We also relate interpretations to grammar morphisms of a particular kind.

Throughout this book recall that we use the following useful convention (the λ-convention).

Convention

Given two languages L_1 and L_2 we say they are equal modulo λ if $L_1 - \{\lambda\} = L_2 - \{\lambda\}$. Similarly we say two language families $\mathcal{L}_1$ and $\mathcal{L}_2$ are equal modulo λ and the empty set if for every $L_1 - \{\lambda\} \neq \emptyset$ in $\mathcal{L}_1$ there is an L_2 in $\mathcal{L}_2$ such that $L_1 - \{\lambda\} = L_2 - \{\lambda\}$ and vice versa.

Essentially this means we ignore the empty set in language families and the empty word in languages.

II.1.1 Two Kinds of Interpretation

In any study of the literature of form theory the reader will notice that there are two basic definitions of interpretation which have been most studied. These we refer to as the g- and s-interpretations. The g-interpretation is the one first introduced by Ginsburg and his colleagues, while the s-interpretation was first studied in detail for EOL forms by Salomaa and his colleagues (this is also known as a strict interpretation). Since the s-interpretation is more general than the g-interpretation the results we present on context-free grammar forms will where possible be given for s-interpretations.

Before giving the definition of s-interpretation we first need the notion of a disjoint-finite-letter substitution (dfl-substitution).

Let U, V be two alphabets and μ be a (letter) substitution from U into 2^V. Then μ is a dfl-substitution if for all X, Y in U, $\mu(X) \cap \mu(Y) = \emptyset$ when $X \neq Y$.

Recall from Section I.1.1 that a context-free grammar is a couple $(G,\Rightarrow)$, where $G = (V,\Sigma,P,S)$ is a production scheme, that is $V-\Sigma$ is the nonterminal alphabet, Σ the terminal alphabet, $P \subseteq (V-\Sigma) \times V^*$

is a finite set of productions and S in $V-\Sigma$ is the sentence symbol. Moreover $\Rightarrow$ is the sequential rewrite relation over V^* induced by P (see Section I.1.1 for further details).

Letting $M, N \subseteq V^*$ then the notation $M \to N$ denotes $\{\alpha \to \beta: \alpha$ is in M and β is in $N\}$ in the following.

Without further ado we are now able to define the two notions of interpretation.

Definition

Let $G_i = (V_i, \Sigma_i, P_i, S_i)$ i = 1, 2 be two grammars. We say G_2 <u>is an s-interpretation of G_1 modulo μ</u>, denoted $G_2 \lhd_s G_1(\mu)$, where μ is a dfl-substitution on V_1^*, if conditions (i) through (iv) obtain:

(i) $\mu(A) \subseteq V_2-\Sigma_2$, for all A in $V_1-\Sigma_1$,

(ii) $\mu(a) \subseteq \Sigma_2$, for all a in Σ_1,

(iii) $P_2 \subseteq \mu(P_1)$, where $\mu(P_1) = \bigcup_{A\to\alpha \text{ in } P_1} \mu(A \to \alpha)$,

where $\mu(A \to \alpha) = \mu(A) \to \mu(\alpha)$, and

(iv) S_2 is in $\mu(S_1)$.

The definition of g-interpretation is very similar except that terminal letters can be replaced by sets of terminal words rather than just by sets of terminal letters. This notion is defined as follows.

Definition

Let $G_i = (V_i, \Sigma_i, P_i, S_i)$, i = 1, 2 be two grammars. We say G_2 <u>is a g-interpretation of G_1 modulo μ</u>, denoted $G_2 \lhd_g G_1(\mu)$, where μ is a (finite) substitution on V_1^*, if conditions (i) through (iv) obtain:

(i) μ is a dfl-substitution from $V_1-\Sigma_1$ into $V_2-\Sigma_2$,

(ii) $\mu(a) \subseteq \Sigma_2^*$, for all a in Σ_1,

(iii) $P_2 \subseteq \mu(P_1)$, where $\mu(P_1) = \bigcup_{A\to\alpha \text{ in } P_1} \mu(A) \to \mu(\alpha)$, and

(iv) S_2 is in $\mu(S_1)$.

In both cases G_1 is the <u>source</u>, <u>master</u> or <u>form grammar</u>, while G_2 is the <u>image</u> or <u>interpretation grammar</u>. Operationally we obtain s-interpretation grammars from G_1 by mapping distinct terminals into disjoint sets of terminals and distinct nonterminals into disjoint sets of nonterminals. Productions in the s-interpretation grammar are obtained as images of the productions in the form grammar G_1, while the sentence symbol is an image of S_1. We obtain g-interpretation grammars in a similar way except that terminals are mapped into sets of terminal words rather than terminal letters. One immediate result of this is that s-interpretations are necessarily length preserving while g-interpretations are not. Secondly, it means that every s-interpretation grammar is a g-interpretation grammar but the converse does not hold.

Whenever $P_2 = \mu(P_1)$ we say that G_2 is a <u>full s- or g-interpretation of G_1</u>, written $G_2 \underset{fs}{\lhd} G_1$ or $G_2 \underset{fg}{\lhd} G_1$, respectively.

We now consider some examples to give insight into s- and g-interpretations.

Convention

We often define grammars by simply listing their productions, in which case we use S, possibly subscripted, to denote the sentence symbol, early upper case Roman letters to denote nonterminals and early lower case Roman letters to denote terminals.

Example 1.1

Let G_1: $S \rightarrow a$ be the form grammar.

Then F: $S_F \rightarrow b$; $S_F \rightarrow c$ is an s-interpretation of G_1. Let $\mu(S) = \{S_F\}$ and $\mu(a) = \{b,c\}$ then $\mu(S \rightarrow a) = \{S_F \rightarrow b,\ S_F \rightarrow c\}$ hence F is also an fs-interpretation of G_1. Hence $F \underset{s}{\lhd} G_1(\mu)$, $F \underset{fs}{\lhd} G_1(\mu)$, $F \underset{g}{\lhd} G_1(\mu)$ and $F \underset{fg}{\lhd} G_1(\mu)$ by the remarks above.

Consider H: $S_H \rightarrow a^5b$; $S_H \rightarrow \lambda$; $A \rightarrow bb$. Then H is not an s-interpretation of G_1 since it cannot be obtained by using a dfl-substitution because such a substitution is length preserving.

However $H \underset{g}{\lhd} G_1(\mu)$ where μ is defined by: $\mu(S) = \{A,S_H\}$, and $\mu(a) = \{a^5b, bb, \lambda\}$ since $P_H \subseteq \mu(S \rightarrow a)$. Since $S_H \rightarrow bb$ is not in H it follows that H is not an fg-interpretation of G_1.

Finally let L be any finite language. Then L can be obtained as a g-interpretation of G_1 (in fact an fg-interpretation) by letting

$\mu(S) = \{S\}$ and $\mu(a) = L$

giving a grammar F:

$S \rightarrow x$, for all x in L.

Clearly $L(G,\Rightarrow) = L$.

Example 1.2

Let G_2: $S \rightarrow SS$; $S \rightarrow a$ be the form grammar.

Then we can obtain every context-free language as the language of some s-interpretation and hence g-interpretation of G_2. (Note that we invoke the λ-convention here.)

Consider an arbitrary context-free language L. Then it is well known that L can be generated by a context-free grammar in Chomsky Normal Form, that is its productions are either of the type $A \to BC$ or $A \to a$. Let $F = (V,\Sigma,P,Z)$ be such a grammar for L and define μ by:

$\mu(S) = V-\Sigma$ and $\mu(a) = \Sigma$.

Then $P \subseteq \mu(\{S \to SS, S \to a\})$, Z is in $\mu(S)$ and μ is trivially a dfl-substitution, hence

$F \triangleleft_s G_2(\mu)$.

However it is worth noting the, albeit obvious, fact that there are grammars which are not interpretations of G_2. Consider $H: S_H \to S_H;\ S_H \to a$ for example. On the one hand we cannot obtain nonterminals from terminals under either g- or s-interpretation, hence $S_H \to S_H$ cannot be an image of $S \to a$ and on the other hand nonterminals cannot be erased hence $S_H \to S_H$ cannot be an image of $S \to SS$.

Example 1.3

Let G_3: $S \to a;\ S \to aS;\ S \to aSS$.

As in Example 1.2 it is easy to see that every context-free language can be obtained as the language of some s-interpretation of G_3. This follows from the well known result that every context-free language has a grammar with productions only of the types $A \to a$, $A \to aB$ and $A \to aBC$. This is known as Greibach 2-standard Normal Form.

Example 1.4

Let G_4: $S \to a^i,\ 1 \le i \le 13;$

$S \to a^3Sa^2Sa^7.$

Then it will be shown later that we can obtain every context-free language via some interpretation of G_4. Essentially this follows by observing that $L(G_4,\Rightarrow) = \{a^i: i \ge 1\}$ and that G_4 is expansive. Such "two-symbol" grammars always give rise to normal form results for the context-free languages. G_2 and G_3 are special cases of a much more general "super-normal form" result.

Example 1.5

We will illustrate with this example some of the effects of relaxing the disjointness condition for nonterminals. Let G_5: $S \to AB$; $S \to a$; $A \to \lambda$; $B \to \lambda$. It is not difficult to see that as far as languages are concerned we obtain no more than G_1 under either the s- or g-interpretations (again invoking the λ-convention). This follows by observing that the production $S \to AB$ can only give rise to the empty word when used in a derivation.

However if we relax the disjointness condition for nonterminals we could define μ by:

$\mu(a) = \{a\}$ and $\mu(A) = \mu(B) = \mu(S) = \{S\}$

and obtain G_2: $S \to SS$; $S \to a$ as an interpretation of G_5. Immediately instead of obtaining only finite languages from G_5 we obtain all context-free languages! This is called a quasi-interpretation, see Section 3.2.

Example 1.6

In Examples 1.2 and 1.5 we have observed that nonterminals should not be erased or merged (and under the definition of interpretation cannot be). In this example we demonstrate why we preclude the addition of terminals or nonterminals.

Let G_6: $S \to S$; $S \to a$, then we can only obtain finite languages from G_6.

But F: $S \to aS$; $S \to a$ although looking very much like G_6 gives rise to all regular languages. This is because every regular language can be generated by a right-linear grammar and such a grammar is an interpretation of F.

Similarly G_2: $S \to SS$; $S \to a$ also looks very similar to G_6 but as we know from Example 1.2 we can obtain all context-free languages from G_2!

In both cases we added one symbol from the grammar to one of the productions of the grammar. Again because of the sensitivity of form grammars to such operations the notion of interpretation considered here precludes the introduction of a new symbol where none previously existed. The exception to this is in the treatment of a terminal symbol, for example H: $S \to aa$ <u>is</u> a g-interpretation of G_1, since terminals can expand into terminal words but they cannot be created *ex nihilo*.

II.1.2 Grammar and Grammatical Families

We will usually write $G_1 \triangleleft_g G_2$ or $G_1 \triangleleft_s G_2$ where μ is understood. A (context-free) grammar is said to be a <u>(context-free) grammar form</u> if it is used within the framework of interpretations. We assume the existence of countable nonterminal and terminal alphabets in the following. However for simplicity we will assume nonterminal and terminal alphabets are chosen from some common countable "pool" alphabet. This means in particular that a symbol can be either a terminal or a nonterminal symbol depending on its context and hence we can avoid the renaming of symbols in some proofs.

The collection of s-interpretation grammars derived from a grammar form G is denoted $\mathcal{G}_s(G)$, referred to as the s-grammar family of G. Analogously, the collection of languages obtained from a grammar form G is denoted by $\mathcal{L}_s(G,\Rightarrow)$, referred to as the s-grammatical family of G, and defined by:

$\mathcal{L}_s(G,\Rightarrow) = \{L(G',\Rightarrow): G' \trianglelefteq_s G\}$.

We say G_1 and G_2, two grammar forms, are s-form equivalent if $\mathcal{L}_s(G_1,\Rightarrow) = \mathcal{L}_s(G_2,\Rightarrow)$ and strong s-form equivalent if $\mathcal{G}_s(G_1) = \mathcal{G}_s(G_2)$.

For any grammar form G, $G \trianglelefteq_s G$, hence $\trianglelefteq_s$ is reflexive. Clearly $\trianglelefteq_s$ is not, in general, symmetric. However since the composition of two dfl-substitutions is a dfl-substitution it follows that the composition of two s-interpretations is an s-interpretation. In other words $\trianglelefteq_s$ is transitive and therefore a pre-order.

Consider two grammar forms G_1 and G_2 with $G_1 \trianglelefteq_s G_2$, then immediately $\mathcal{G}_s(G_1) \subseteq \mathcal{G}_s(G_2)$ since s-interpretation is transitive. Conversely, if $\mathcal{G}_s(G_1) \subseteq \mathcal{G}_s(G_2)$ then G_1 is in $\mathcal{G}_s(G_2)$ and therefore $G_1 \trianglelefteq_s G_2$. It now follows immediately that $\mathcal{G}_s(G_1) = \mathcal{G}_s(G_2)$ iff $G_1 \trianglelefteq_s G_2$ and $G_2 \trianglelefteq_s G_1$. In Section III.5 we show that $\trianglelefteq_s$ is decidable, and therefore, that strong s-form equivalence is decidable.

Similarly if we base our definitions on g-interpretations we obtain $\mathcal{G}_g(G)$ the g-grammar family of G and $\mathcal{L}_g(G,\Rightarrow)$ the g-grammatical family of G. We also obtain the notions of g-form equivalence and strong g-form equivalence analogously. By the definition of g- and s-interpretations it follows that $\mathcal{G}_s(G) \subseteq \mathcal{G}_g(G)$ and $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$.

Returning to our previous examples we have already demonstrated that for:

G_1: $S \rightarrow a$ we have $\mathcal{L}_g(G_1,\Rightarrow) \supseteq \mathcal{L}(FIN)$ and $\mathcal{L}_s(G_1,\Rightarrow) \not\supseteq \mathcal{L}(FIN)$,

G_2: $S \rightarrow SS$; $S \rightarrow a$ we have $\mathcal{L}_g(G_2,\Rightarrow) \supseteq \mathcal{L}_s(G_2,\Rightarrow) \supseteq \mathcal{L}(CF)$ and hence equality holds in both cases, since we can never obtain more than the context-free languages from a grammar form,

G_3: $S \rightarrow a$; $S \rightarrow aS$; $S \rightarrow aSS$ again we have $\mathcal{L}_s(G_3,\Rightarrow) = \mathcal{L}_g(G_3,\Rightarrow) = \mathcal{L}(CF)$,

G_4: $S \rightarrow a^i$, $1 \leq i \leq 13$, $S \rightarrow a^3Sa^2Sa^7$ once more we have $\mathcal{L}_s(G_4,\Rightarrow) = \mathcal{L}_g(G_4,\Rightarrow) = \mathcal{L}(CF)$.

Note that we can derive $\mathcal{L}_g(G_3,\Rightarrow) = \mathcal{L}_g(G_4,\Rightarrow) = \mathcal{L}(CF)$ from the fact that $G_2 \underset{g}{\lhd} G_3$ and $G_2 \underset{g}{\lhd} G_4$. This is not the case however for s-interpretation. In the next section we will show that $\mathcal{L}_g(G_1,\Rightarrow) = \mathcal{L}(\text{FIN})$ and moreover that there is no grammar form G with $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(\text{FIN})$. In fact this is the only g-grammatical family which is not s-grammatical as will be shown in Section 4.4.

We close the present section by characterizing g-interpretations in terms of s-interpretations and closed terminal grammar morphisms. First observe however that whenever $G_2 \underset{s}{\lhd} G_1(\mu)$, for two grammar forms $G_i = (V_i,\Sigma_i,P_i,S_i)$, $i = 1, 2$, then an inverse of μ, μ^{-1} can be defined and this is a very fine grammar morphism (see Section I.2.4). Define μ^{-1} by, for all X_2 in V_2, $\mu^{-1}(X_2) = X_1$ in V_1 when X_2 is in $\mu(X_1)$ and $\mu^{-1}(X_2) = \emptyset$ otherwise. Since μ is a dfl-substitution on V_1 it follows from the definition of s-interpretation that μ^{-1} is a very fine grammar morphism. Define μ^{-1} by, for all X_2 in V_2, $\mu^{-1}(X_2) = X_1$ in V_1 when X_2 is in (X_1) and $\mu^{-1}(X_2) = \emptyset$ otherwise. Since μ is a dfl-substitution on V_1 it follows from the definition of s-interpretation that μ^{-1} is a very fine grammar morphism. The converse holds, namely, whenever μ: $G_2 \to G_1$ is a very fine grammar morphism then μ^{-1} is a dfl-substitution and $G_2 \underset{s}{\lhd} G_1(\mu^{-1})$.

Theorem 1.1

Let $G_i = (V_i,\Sigma_i,P_i,S_i)$, $i = 1, 2$ be two grammar forms.

(1) $G_2 \underset{s}{\lhd} G_1(\mu)$ iff μ^{-1}: $G_2 \to G_1$ is a very fine grammar morphism.

(2) $G_2 \underset{g}{\lhd} G_1(\mu)$ iff there exists some $G_0 = (V_0,\Sigma_0,P_0,S_0)$ such that $G_0 \underset{s}{\lhd} G_1(\bar{\mu})$ and a closed terminal grammar morphism ϕ: $G_0 \to G_2$.

Proof: (1) clear by the remarks above.

(2) if: Since ϕ does not affect the nonterminals of G_0, we must have $V_0-\Sigma_0 = V_2-\Sigma_2$.

Define μ on V_1 by:

$\mu(A) = \bar{\mu}(A)$, for all A in $V_1-\Sigma_1$, and

$\mu(a) = \phi(\bar{\mu}(a))$, for all a in Σ_1.

Let $P_2 = \phi(P_0)$ and $S_2 = S_0$. Clearly $G_2 \underset{g}{\lhd} G_1(\mu)$

only if: Construct G_0 as follows:

$V_0-\Sigma_0 = V_2-\Sigma_2$, $\Sigma_0 = \{[x]$: x is in $\mu(a)$, for some a in $\Sigma_1\}$,

where the [x] are new symbols. Define a homomorphism ϕ: $V_0^* \to V_2^*$ by $\phi([x]) = x$, for all [x] in Σ_0. and $\phi(A) = A$, for all A in $V_0-\Sigma_0$.

Define an s-interpretation $\bar{\mu}$ of G_1 by:

$\bar{\mu}(A) = \mu(A)$, for all A in $V_1-\Sigma_1$,

and $\bar{\mu}(a) = \{[x]: x \text{ is in } \mu(a)\}$, for all a in Σ_1.
Let $P_0 = \{A \to \alpha: A \to \gamma$ is in $\mu(B \to \beta)$ for some $B \to \beta$ in P_1,
$A \to \alpha$ is in $\bar{\mu}(B \to \beta)$ and $\phi(\alpha) = \gamma\}$
and $S_0 = S_2$.
Immediately $\phi: G_0 \to G_2$ is a closed terminal grammar morphism, hence the result. □

Since a closed terminal grammar morphism $\phi: G_0 \to G_2$ has the property that $L(G_2,\Rightarrow) = \;\; (L(G_0,\Rightarrow))$ we immediately have the following result:

Corollary 1.2

Let G be a grammar form, then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{H}(\mathcal{L}_s(G,\Rightarrow))$, where $\mathcal{H}$ denotes the homomorphic closure operator. For two grammar forms G_1 and G_2 such that $\mathcal{L}_s(G_1,\Rightarrow) = \mathcal{L}_s(G_2,\Rightarrow)$, we have $\mathcal{L}_g(G_1,\Rightarrow) = \mathcal{L}_g(G_2,\Rightarrow)$, in other words s-form equivalence implies g-form equivalence. Similarly if $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G_2,\Rightarrow)$ then $\mathcal{L}_g(G_1,\Rightarrow) \subseteq \mathcal{L}_g(G_2,\Rightarrow)$.

II.2 Isolation and Simulation

We devote this section to two technical notions which have however far-reaching consequences. We demonstrate some of these consequences by characterizing when a grammar form gives exactly the families of finite, regular and context-free languages under g-interpretations and characterizing the g-grammatical families of two-symbol forms, that is grammar forms which have only a single terminal symbol and only a single nonterminal symbol.

First observe the close relationship between the derivations in a g- or s-interpretation grammar and its form grammar.

Lemma 2.1

Let $G_i = (V_i, \Sigma_i, P_i, S_i)$, $i = 1, 2$ be two grammars such that $G_2 \triangleleft_s G_1(\mu)$ (or $G_2 \triangleleft_g G_1(\mu)$). Then for every derivation

$\alpha'_0 \Rightarrow \alpha'_1 \Rightarrow \ldots \Rightarrow \alpha'_m$ in G_2, for some α'_i in V_2^*, $0 \leq i \leq m$

and $m > 0$ there is a derivation

$\alpha_0 \Rightarrow \alpha_1 \Rightarrow \ldots \Rightarrow \alpha_m$ in G_1, for some α_i in V_1^*, such that α'_i is in $\mu(\alpha_i)$, $0 \leq i \leq m$. When $G_2 \triangleleft_s G_1(\mu)$, then $\alpha_i = \mu^{-1}(\alpha'_i)$, $0 \leq i \leq m$.

Proof: Clear. □

Hence a derivation in an interpretation grammar is always an image of a derivation in the corresponding form grammar.

The statement of Lemma 2.1 can also be expressed by saying that for each derivation tree τ_2 in G_2 there is a derivation tree τ_1 in G_1 such that τ_2 is in $\mu(\tau_1)$. In the case of s-interpretation there is exactly one derivation tree τ_1 in G_1 corresponding to each τ_2 in G_2. Clearly τ_1 and τ_2 are obtained from each other by a relabelling of their nodes. Recall that two derivation trees τ_1 and τ_2 are <u>equally shaped</u> if they are obtained from each other by a relabelling of their non-frontier nodes.

Corollary 2.2

Let $G_i = (V_i, \Sigma_i, P_i, S_i)$, $i = 1, 2, 3$ be three grammars such that $G_1 \triangleleft_s G_3(\mu_1)$ and $G_2 \triangleleft_s G_3(\mu_2)$. Suppose τ_1 and τ_2 are derivation trees in G_1 and G_2, respectively, of a word α in $V_1^* \cap V_2^*$ and there exists a derivation tree τ in G_3 such that τ_1 is in $\mu_1(\tau)$ and τ_2 is in $\mu_2(\tau)$. Then τ_1 and τ_2 are equally shaped.

Proof: Since τ_1 is in $\mu(\tau)$, τ_1 and τ are obtained from each other by relabelling of the nodes. Similarily τ_2 and τ are obtained from each other by a relabelling of the nodes. Hence τ_1 and τ_2 are obtained from each other by a relabelling of the nodes. However by assumption τ_1 and τ_2 have the same frontier therefore τ_1 and τ_2 are equally shaped. □

Corollary 2.3

Let G_1 and G_2 be grammars such that either $G_2 \mathrel{\triangleleft_s} G_1(\mu)$ or $G_2 \mathrel{\triangleleft_g} G_1(\mu)$. Then if $L(G_2,\Rightarrow)$ is infinite then $L(G_1,\Rightarrow)$ is infinite, but not conversely. This implies that if $L(G_1,\Rightarrow)$ is finite then $L(G_2,\Rightarrow)$ is finite, but not conversely.

Proof: If $L(G_1,\Rightarrow)$ is finite then G_1 has a finite number of non-empty sentential derivations. Since μ is finite G_2 also has a finite number of non-empty sentential derivations. Hence $L(G_2,\Rightarrow)$ is finite. □

This also gives a result promised in the previous section, namely:

Corollary 2.4

Let G: $S \to a$, then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\text{FIN})$.

Proof: We have already demonstrated that $\mathcal{L}_g(G,\Rightarrow) \supseteq \mathcal{L}(\text{FIN})$ by Example 1.1. Equality holds by the previous corollary. □

We now apply Lemma 2.1 in a less trivial manner to obtain another result promised in the previous section.

Theorem 2.5

Let G be an arbitrary grammar form. Then $\mathcal{L}_s(G,\Rightarrow) \neq \mathcal{L}(\text{FIN})$.

Proof: Assume G is a grammar form for which $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(\text{FIN})$. Then $L(G,\Rightarrow)$ is finite, otherwise $\mathcal{L}_s(G,\Rightarrow) \not\subseteq \mathcal{L}(\text{FIN})$. Choose an integer $m > \max(LS(G,\Rightarrow))$. We claim $L = \{a^m\}$ is not in $\mathcal{L}_s(G,\Rightarrow)$. If it is, then there exists $G' \mathrel{\triangleleft_s} G(\mu)$ with $L = L(G',\Rightarrow)$, that is $S' \Rightarrow^+ a^m$ in G'. Now by Lemma 2.1 this implies $\mu^{-1}(S') \Rightarrow^+ \mu^{-1}(a^m)$ in G, giving a contradiction. Therefore L is not in $\mathcal{L}_s(G,\Rightarrow)$ contradicting the assumption that $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(\text{FIN})$. □

We now turn to our first technical result and "tool".

II.2.1. Isolation

Consider a grammar form $G = (V,\Sigma,P,S)$ and terminating derivations $A \Rightarrow^+ x_i$ in Σ^*, $1 \le i \le n$, for some $n \ge 1$ and some A in $V-\Sigma$. Derive an interpretation grammar $G' \lhd_s G$ and hence $G' \lhd_g G$, such that (i) $G' = (V',\Sigma,P',S)$ with $V \subseteq V'$, (ii) all productions in P whose left hand side is not A are taken into P' unchanged, and (iii) the remaining productions in P' only serve to derive the x_i from A and nothing else. Then we say that the derivations $A \Rightarrow^+ x_i$, $1 \le i \le n$ in G have been isolated in G'. In the following we formalize this notion for a derivation which is not necessarily terminating.

Let $G = (V,\Sigma,P,S)$ be a grammar form and $A \Rightarrow^+ \alpha_i$, $1 \le i \le n$, for some $n \ge 1$, be mutually distinct derivations in G, where A is in $V-\Sigma$ and the α_i are in V^*. Recall that two derivations $A \Rightarrow^+ \alpha$ and $A \Rightarrow^+ \beta$ in a grammar G are distinct if for the corresponding derivation trees neither one is a tree-prefix of the other. We construct a grammar form $G' \lhd_s G$ such that whenever $A \Rightarrow^+ \beta$ in G' with β in V^*, then the derivation can be re-arranged as $A \Rightarrow^+ \alpha_i \Rightarrow^* \beta$, for some i, $1 \le i \le n$. In other words the derivations $A \Rightarrow^+ \alpha_i$ in G become the "only possible" derivations for A in G'. Hence the $A \Rightarrow^+ \alpha_i$ have been isolated in G'. Consider the case when $n = 1$, the case for $n > 1$ we leave to the reader.

Let $A \Rightarrow^+ \alpha$ be written

(*) $A = A_0 \Rightarrow \alpha_1 \Rightarrow \dots \Rightarrow \alpha_n = \alpha$ for some $n > 0$, where the production $A_i \to \beta_{i+1}$ is used at the ith step, $0 \le i < n$. Rewrite β_i as $\gamma_{i,1}A_{i,1} \dots A_{i,n_i}\gamma_{i,n_i+1}$ where $n_i = 0$ implies $\beta_i = \gamma_{i,1}$, $1 \le i \le n$. In the case $n_i \ge 1$ then each $A_{i,j}$, is rewritten in the derivation (*), while $\gamma_{i,j}$ contains no nonterminal which is rewritten in (*). Let $V(A) = \{[A_{i,j},i,j]\colon A_{i,j} \text{ in } V-\Sigma,\ 1 \le i \le n,\ 1 \le j \le n_i\}$,

and $P(A) = \{A \to \gamma_{1,1}[A_{1,1},1,1] \dots [A_{1,n_1},1,n_1]\gamma_{1,n_1+1}\}$

$\cup \{[B,j,k] \to \gamma_{i,1}[A_{i,1},i,1] \dots [A_{i,n_i},i,n_i]\gamma_{i,n_i+1}\colon$
where B is the nonterminal rewritten at the (i-1)st step in (*) and $B = A_{j,k}$, that is B is introduced at the jth step, $j \le i - 1\}$.

Let $G' = (V \cup V(A), \Sigma, (P-\{A \to \delta\colon A \to \delta \text{ in } P\}) \cup P(A),S)$. Observe that G' has indeed isolated the derivation $A \Rightarrow^+ \alpha$ in G by a suitable renaming of the nonterminals. Moreover these new nonterminals in $V(A)$ each have a single production and these can only be applied within the derivation specified by $A \Rightarrow^{+} \alpha$.

Now define μ by:

$\mu(a) = a$, for all a in Σ,

$\mu(A) = \{A\} \cup \{[A,i,j]: 1 \leq i < n, 1 \leq j \leq m\}$,

where $m = \max(\{n_i: 1 \leq i < n\})$.

It should be clear that $P \cup P(A) \subseteq \mu(P)$ and hence $G' \lhd_s G(\mu)$ and $G' \lhd_g G(\mu)$.

This technique can be generalized to give:

Lemma 2.6

Let $G = (V,\Sigma,P,S)$ be a grammar form, A be in $V-\Sigma$ and $A \Rightarrow^{+} \alpha_i$ be mutually distinct derivations in G, where $1 \leq i \leq n$, for some $n \geq 1$. Then there exists a grammar form $G' = (V',\Sigma',P',S')$ such that

(i) $V \subseteq V'$, $\Sigma = \Sigma'$, $P - \{A \to \delta: A \to \delta \text{ in } P\} \subseteq P'$ and $S = S'$,

(ii) $G' \lhd_s G$ (and $G' \lhd_g G$), and

(iii) whenever $A \Rightarrow^{+} \beta$ in G' with β in V^* there is a derivation $A \Rightarrow^{+} \alpha_i \Rightarrow^{*} \beta$ in G', for some i, $1 \leq i \leq n$.

We now demonstrate an application of isolation.

Lemma 2.7

Let $G = (V,\Sigma,P,S)$ be a grammar form. If G is nonempty then $\mathcal{L}_g(G,\Rightarrow) \supseteq \mathcal{L}(FIN)$.

Proof: Since G is nonempty there is a nonempty word x in Σ^* such that $S \Rightarrow^{+} x$. Consider an interpretation $G' = (V',\Sigma,P',S) \lhd_s G$ which isolates this derivation. Then $L(G',\Rightarrow) = \{x\}$. Second, consider an interpretation $G'' \lhd_g G'(\mu)$, $G'' = (V'',\Sigma'',P'',S)$, where $\mu'(A) = A$, for all A in $V'-\Sigma'$ and $\mu'(a) = \{a,\lambda\}$ for all a in Σ, and P'' is chosen such that $S \Rightarrow^{+} a$ in G'', for some a in Σ is the only derivation in G''. By Corollary 2.4 slightly modified $\mathcal{L}_g(G'',\Rightarrow) = \mathcal{L}(FIN)$. Hence $\mathcal{L}(FIN) = \mathcal{L}_g(G'',\Rightarrow) \subseteq \mathcal{L}_g(G',\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$ giving the result. □

This result no longer holds under s-interpretation, for example consider G defined by the productions:

$S \to aaS$; $S \to aa$;

then $\{a\}$ is not in $\mathcal{L}_s(G,\Rightarrow)$ although G is nonempty.

We now characterize when a grammar form generates exactly the finite sets under g-interpretation.

Theorem 2.8

Let $G = (V,\Sigma,P,S)$ be a nonempty grammar form. Then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\text{FIN})$ iff G is finite.

Proof: if: Since G is finite, $\mathcal{L}_g(G,\Rightarrow) \subseteq \mathcal{L}(\text{FIN})$ by Corollary 2.3. Equality follows by Lemma 2.7.
only if: Since $L(G,\Rightarrow)$ is in $\mathcal{L}(\text{FIN})$, G is finite. □

Thus Theorem 2.5 and 2.8 provide a contrast of the effects of s- and g-interpretation with respect to the finite sets. As will be proved in Section 4.3 the only g-grammatical family which is not s-grammatical is $\mathcal{L}(\text{FIN})$.

We can consider a weaker requirement, namely when is $\mathcal{L}(\text{FIN})$ contained in an s-grammatical family. As we shall see in Section 4.3 whenever $\mathcal{L}(\text{FIN})$ is contained in an s-grammatical family $\mathcal{L}$, then $\mathcal{L}(\text{REG})$ is also contained in $\mathcal{L}$. Moreover such a containment can be characterized in terms of a simple condition on the corresponding grammar form.

II.2.2 Simulation

The second important technique can be described as follows. Let G_1 and G_2 be grammar forms such that $A \Rightarrow^+ \alpha$ in G_2 for each production $A \to \alpha$ in G_1. We say G_2 simulates G_1 or G_2 production-simulates G_1. In this case $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G_2,\Rightarrow)$.

Lemma 2.9

Let $G_i = (V_i,\Sigma_i,P_i,S)$, $i = 1, 2$ be two grammar forms such that for each production $A \to \alpha$ in P_1 there is a derivation $A \Rightarrow^+ \alpha$ in G_2 and $\Sigma_1 \subseteq \Sigma_2$. Then $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G_2,\Rightarrow)$.

Proof: We construct a grammar form $G = (V_2,\Sigma_2,P,S)$ such that $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$ and $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}_s(G_2,\Rightarrow)$.

Let $P = P_1 \cup P_2$. Clearly $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$ since G_1 is a subgrammar of G, and $\mathcal{L}_s(G_2,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$ by the same reasoning.

It remains to show that $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(G_2,\Rightarrow)$. Consider a grammar $G' \triangleleft_s G(\mu)$. It suffices to show that $L(G',\Rightarrow)$ is in $\mathcal{L}_s(G_2,\Rightarrow)$. Letting $G' = (V_2',\Sigma_2',P',S')$ it is clear that we can partition P' into $P_1' \cup P_2'$, that is those productions which are interpretations of G_1-productions and those of G_2-productions. For all productions

$A' \to \alpha'$ in P_1', $A' \to \alpha'$ is in $\mu(A \to \alpha)$ for some $A \to \alpha$ in P_1, hence there is a derivation $A \Rightarrow^+ \alpha$ in G_2. Using the technique of Lemma 2.6 construct unique isolation derivations $A' \Rightarrow^+ \alpha'$ from $A \Rightarrow^+ \alpha$ in G_2. In other words construct $G_2' \trianglelefteq_s G_2$, where $G_2' = (V_2'', \Sigma_2', P_2'', S')$ with V_2'' equal to V_2' together with all new symbols introduced by the isolating derivations and P_2'' equal to P_2' together with all the necessary new productions of the isolating derivations. Now immediately we have:

$A' \Rightarrow^+ \alpha'$ in G_2' for each production $A' \to \alpha'$ in G', and furthermore $L(G_2', \Rightarrow) = L(G', \Rightarrow)$ since the new productions in P_2'' can only be applied in such derivations. As G_2' is an s-interpretations of G_2 we obtain the result. □

One immediate consequence is the following useful corollary.

Corollary 2.10

Let $G = (V, \Sigma, P, S)$ be a grammar form and let $A \Rightarrow^* \alpha$ be in G for some A in $V-\Sigma$ and α in V^*. Then $F = (V, \Sigma, P \cup \{A \to \alpha\}, S)$ is s-form equivalent to G (and hence they are g-form equivalent).

Proof: $\mathcal{L}_s(G, \Rightarrow) \subseteq \mathcal{L}_s(F, \Rightarrow)$ is immediate. On the one hand, if $A \Rightarrow^+ \alpha$ in G then Lemma 2.9 yields the result. On the other hand, if $A \Rightarrow^0 A$ in G, it is clear that adding the production $A \to A$ to G does not affect $\mathcal{L}_s(G, \Rightarrow)$. □

As an application of the above corollary we have:

Lemma 2.11

Let $G = (V, \Sigma, P, S)$ be a reduced grammar form. Let $G_A = (V, \Sigma, P, A)$ for some A in $V-\Sigma$. Then $\mathcal{L}_g(G_A, \Rightarrow) \subseteq \mathcal{L}_g(G, \Rightarrow)$.

Proof: Since G is reduced there exists a derivation

$S \Rightarrow^* xAy \Rightarrow^+ xyz$ in Σ^*.

By a straightforward extension of Corollary 2.10 not only can we add $S \to xAy$ to P to give a g-form equivalent grammar form, but also we can add $S \to A$ to P to give a g-form equivalent grammar form. Hence assume P contains $S \to A$. We construct $G' = (V \cup \{Z\}, \Sigma, P', Z)$ such that $G' \trianglelefteq_g G(\mu)$ and $\mathcal{L}_g(G', \Rightarrow) = \mathcal{L}_g(G_A, \Rightarrow)$, where μ is defined by:

for all X in $V - \{S\}$, $\mu(X) = X$ and $\mu(S) = \{S, Z\}$.

Define P' by:

$P' = P \cup \{Z \to A\}$. Clearly $G' \trianglelefteq_g G(\mu)$ and $L(G_A, \Rightarrow) = L(G', \Rightarrow)$ and further $\mathcal{L}_g(G_A, \Rightarrow) = \mathcal{L}_g(G', \Rightarrow) \subseteq \mathcal{L}_g(G, \Rightarrow)$. □

Second, we apply Corollary 2.10 to obtain the following "back-substitution" result, a technical result which is useful in Section 4.2.

Lemma 2.12 -- The Back-Substitution Lemma

Let $G = (V,\Sigma,P,S)$ be a grammar form and A in $V-\{S\}$ be an arbitrary nonterminal with productions

$A \to \alpha_1 \mid \ldots \mid \alpha_r$ in P.

Construct $\bar{G} = (V,\Sigma,\bar{P},S)$ from G as follows: Replace each production $B \to \beta$ in P with all possible productions obtained by replacing each A in β with $\alpha_1,\ldots,\alpha_r$. The resulting set of productions is denoted by $\bar{P}$. Then G and $\bar{G}$ are s- and g-form equivalent.

Proof: $\mathcal{L}_s(\bar{G},\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$ since for each production $D \to \gamma$ in $\bar{G}$ there is a derivation $D \Rightarrow^+ \gamma$ in G. Conversely consider $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(\bar{G},\Rightarrow)$. Let $G' = (V',\Sigma',P',S')$ be an arbitrary s-interpretation of G, $G' \lhd_s G(\mu)$, then it is straightforward to construct a $\bar{G}' \lhd_s \bar{G}$ by carrying out the construction of the lemma for G' rather than G and for all A' in $\mu(A)$ rather than for A only. The resulting $\bar{G}'$ is an s-interpretation of $\bar{G}$ and by standard results $L(\bar{G}',\Rightarrow) = L(G',\Rightarrow)$. □

Third, we apply Corollary 2.10 to characterize when a grammar form gives exactly the regular languages and the context-free languages under g-interpretation. In the latter case we say such a form G is <u>g-complete</u>, i.e. $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(CF)$.

Lemma 2.13

Let $G = (V,\Sigma,P,S)$ be an infinite grammar form. Then $\mathcal{L}_g(G,\Rightarrow) \supseteq \mathcal{L}(REG)$.

Proof: Since G is infinite there exists a nonterminal A in V such that

$S \Rightarrow^+ uAv \Rightarrow^+ uwAyv \Rightarrow^+ uwxyv$ with $wy \neq \lambda$ and $x \neq \lambda$

otherwise $L(G,\Rightarrow)$ would be finite.

Construct $G_1 = (V,\Sigma,P \cup \{A \to x, A \to wAy\},S)$ then by Corollary 2.10 G_1 is g-form equivalent to G. Consider $G_1' \lhd_g G_1$ defined by $G_1' = (V,\Sigma,P_1',S)$, where P_1' consists of P, $A \to a$ and either $A \to aA$ if $w \neq \lambda$ or $A \to Aa$ if $w = \lambda$. Without loss of generality assume P_1' includes $A \to aA$.

Now define a new grammar F by $F = (V,\Sigma,P_1',A)$ then, by Lemma 2.11, $\mathcal{L}_g(F,\Rightarrow) \subseteq \mathcal{L}_g(G_1',\Rightarrow)$. Finally consider $F' = (\{A,a\},\{a\},\{A \to a;\ A \to aA\},A)$. Clearly $F' \triangleleft_g F$ and furthermore any right linear grammar is a g-interpretation of F', hence $\mathcal{L}(REG) \subseteq \mathcal{L}_g(F',\Rightarrow) \subseteq \mathcal{L}_g(F,\Rightarrow) \subseteq \mathcal{L}_g(G_1',\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$. Therefore $\mathcal{L}_g(G,\Rightarrow) \supseteq \mathcal{L}(REG)$. □

We now turn to the characterization theorem.

Theorem 2.14

Let $G = (V,\Sigma,P,S)$ be an infinite reduced grammar form. Then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(REG)$ iff G is non-self-sembedding.

Proof: if: By the previous lemma observe that $\mathcal{L}_g(G,\Rightarrow) \supseteq \mathcal{L}(REG)$. To show the converse observe that each $G' \triangleleft_g G$ is also non-self-embedding. [Assume otherwise, then because derivations in the interpretation grammar are images of derivations in the form grammar, G' is self-embedding implies G is self-embedding, a contradiction.] Hence by a standard result in language theory, $L(G',\Rightarrow)$ is regular, hence $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(REG)$.
only if: Assume G is self-embedding. Since G is reduced there exists a nonterminal A such that

$$S \Rightarrow^* uAv \Rightarrow^+ uwAyv \Rightarrow^+ uwxyv \text{ where } w \neq \lambda \text{ and } y \neq \lambda.$$

Without loss of generality, assume P contains the productions $A \to aAb$ and $A \to \lambda$ by Corollary 2.10 and previous remarks. Now $G_A = (V,\Sigma,P,A)$ has $\mathcal{L}_g(G_A,\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$. However $G_A' = (\{A,a,b\},\{a,b\},\{A \to aAb, A \to \lambda\},A)$, generates the language $L(G_A') = \{a^i b^i : i \geq 0\}$ a well known non-regular language. This yields a contradiction. □

Our next result gives necessary and sufficient conditions for a grammar form to be g-complete.

Theorem 2.15

Let $G = (V,\Sigma,P,S)$ be a reduced grammar form. Then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(CF)$ iff G is expansive.

Proof: <u>if</u>: Since G is expansive there is a nonterminal A such that $A \Rightarrow^+ u_1Au_2Au_3 \Rightarrow^+ u_1xu_2xu_3$, $x \neq \lambda$, hence by Corollary 2.10 we can assume $A \to x$ and $A \to u_1Au_2Au_3$ are in P without any loss of generality. Consider G_A and an interpretation $F \lhd_g G_A$, where F is defined by:

$F = (\{A,a\},\{a\},\{A \to a, A \to AA\},A)$,

clearly $\mathcal{L}_g(F,\Rightarrow) = \mathcal{L}(CF)$, hence $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(CF)$ by Lemma 2.11.
<u>only if</u>: Since $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(CF)$ then there is an interpretation G' with $L(G',\Rightarrow) = D_1$, the Dyck set on one letter. It is well known that G' must be expansive hence G is expansive. □

We can obtain a similar characterization theorem for the linear languages.

Theorem 2.16

Let G be a reduced grammar form. Then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(LIN)$ iff G is self-embedding and if $S \Rightarrow^+ uAxBy$ then A and B are not both self-embedding.

Proof: Left to the reader. □

It is worthwhile remarking that none of the above characterization results hold under s-interpretation, since Lemma 2.11 does not hold in this case.

We may summarize the results so far in:

Theorem 2.17

Given an arbitrary grammar form G it is decidable whether or not $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(FIN)$, $\mathcal{L}(REG)$, $\mathcal{L}(LIN)$ or $\mathcal{L}(CF)$.

We now consider two-symbol forms under g-interpretations, that is forms which have a single terminal a and a single nonterminal S. Given a two-symbol form we can characterize its language family as being one of five possible language families.

Theorem 2.18

Let $G = (\{S,a\},\{a\},P,S)$ be a two-symbol form, then $\mathcal{L}_g(G,\Rightarrow)$ is in $\{\{\emptyset\}, \mathcal{L}(FIN), \mathcal{L}(REG), \mathcal{L}(LIN), \mathcal{L}(CF)\}$.

Proof: Clearly if $L(G,\Rightarrow) = \emptyset$ or $\{\lambda\}$ then $\mathcal{L}_g(G,\Rightarrow) = \{\emptyset\}$. If G is finite then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\text{FIN})$ by Theorem 2.8. If G is infinite, clearly G is reduced, therefore $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\text{REG})$ only if G is non-self-embedding, that is all productions must have at most one S on the right hand side and either all nonterminating productions are of type $S \to a^i S$, $i \geq 0$ or all nonterminating productions are of type $S \to Sa^i$, $i \geq 0$. Similarly if G is infinite and it contains a production of type $S \to \alpha S \beta S \gamma$, $\alpha\beta\gamma$ in $\{S,a\}^*$, then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\text{CF})$. Otherwise $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\text{LIN})$. □

Before leaving this simulation section we consider two other technical results which have a simulation flavor.

Lemma 2.19 -- Replacement Lemma

Let $G = (V,\Sigma,P,S)$ be a grammar form and let A in $V-\Sigma$ be an arbitrary nonterminal. Let $E = (U,\Delta,Q,A)$ be a grammar form which is s-form equivalent to the subgrammar form G_A of G, where $(U-\Delta) \cap (V-\Sigma) = \{A\}$.

Then the grammar form H obtained by removing all A-productions from P and adding all the productions in Q is s-form equivalent to G.

Proof: Without loss of generality assume G is reduced. If $A = S$, then $H = (U,\Sigma\ \Delta,(P-\{S \to \alpha\colon S \to \alpha \text{ in } P\}) \cup Q,S)$ is clearly s-form equivalent to G as desired, since the productions in H taken from P are unreachable. Therefore we now assume $A \neq S$ from hereon in.

Let the reduced subgrammar form $G_A = (V_A,\Sigma_A,P_A,A)$ and denote by G_{-A}, the grammar form defined by:

$G_{-A} = ((V-V_A) \cup \{A\} \cup \Sigma,\Sigma,P - (\{A \to \alpha\colon A \to \alpha \text{ is in } P\}),S)$.

G_{-A} is the "complement" subgrammar of G with respect to G_A. Its importance stems from the following observation:

(*) Each derivation $S \Rightarrow^* x$ in G can be uniquely rearranged as $S \Rightarrow^* \alpha \Rightarrow^* x$ in G, for some α in V^*, where $S \Rightarrow^* \alpha$ and $\alpha \Rightarrow^* x$ are derivations in G_{-A} and G_A respectively.

Consider $H = (V \cup U,\Sigma \cup \Delta,(P - \{A \to \alpha\colon A \to \alpha \text{ is in } P\}) \cup Q,S)$. We only show that $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(H,\Rightarrow)$, since the reverse inclusion follows by the same argument because G can be written as $G = (V \cup V_A,\Sigma \cup \Sigma_A,(P - \{A \to \alpha\colon A \to \alpha \text{ is in } P\}) \cup P_A,S)$. Consider an arbitrary $G' \triangleleft_s G(\mu)$, where $G' = (V',\Sigma',P',S')$ and $\mu(A) = \{A_1,\ldots,A_n\}$ say. For each A_i denote the corresponding subgrammar by G_i'. Now by assumption there is an $E_i \triangleleft_s E$ such that

$L(E_i,\Rightarrow) = L(G_i',\Rightarrow)$, for all i, $1 \le i \le n$. Let each $E_i = (U_i,\Delta_i,Q_i,A_i)$ be such that $(U_i-\Delta_i) \cap (V'-\Sigma') = \{A_i\}$, $1 \le i \le n$ and also assume that $(U_i-\Delta_i) \cap (U_j-\Delta_j) = \emptyset$ for all i, j, $1 \le i < j \le n$.

Now construct an $H' = (V'',\Sigma'',P'',S')$ as follows:

$$V'' = V'' \cup \biguplus_{i=1}^{n} U_i,\ \Sigma'' = \Sigma' \cup \biguplus_{i=1}^{n} \Delta_i \text{ and}$$

$$P'' = P' - (\{A_i \to \alpha\colon A_i \to \alpha \text{ is in } P',\ 1 \le i \le n\}) \cup \biguplus_{i=1}^{n} Q_i.$$

It should be clear that $H' \mathrel{\triangleleft_s} H$ from the construction and moreover $L(H',\Rightarrow) = L(G',\Rightarrow)$ by (*) applied to the inverse images of the derivations in G' and H'.

This completes the lemma. □

Given a grammar form $G = (V,\Sigma,P,S)$ it is straightforward to prove that G has an s-form equivalent grammar form $G' = (V',\Sigma,P',S')$ satisfying (i) $V' = V \cup \{S'\}$, where S' is a new nonterminal, (ii) $G' \mathrel{\triangleleft_s} G$ and (iii) P' is the same as P except that for each production $S \to \alpha$ in P we also take $S' \to \alpha$ into P'.

We now prove a similar but stronger result for a nonterminal which is not the sentence symbol.

Lemma 2.20 -- (A,α,B,β)-Partition Lemma

Let $G = (V,\Sigma,P,S)$ be a grammar form and let $A \to \alpha B \beta$ be an arbitrary production in P, where B is a nonterminal and $\alpha\beta$ is a word over V. Then we may assume:

(i) this is the only appearance of B on the right hand side of productions in P,

(ii) letting the reduced subgrammar form $G_B = (V_B,\Sigma_B,P_B,B)$ then $P-P_B$ has only the nonterminal B in common with V_B, and

(iii) $\alpha\beta$ has no nonterminals in common with V_B.

Proof: Without loss of generality we may assume G is reduced and by Lemma 2.19 we may also assume that the reduced grammar forms $G_B = (V_B,\Sigma_B,P_B,B)$ and $G_{-B}(V_{-B},\Sigma_{-B},P_{-B},S)$, (the sub-grammar form and complement subgrammar form of G with respect to B, respectively) have only B as their common nonterminal. Thus condition (ii) is satisfied.

Define $G_{B'} = (V_{B'},\Sigma_B,P_{B'},B') \mathrel{\triangleleft_{fs}} G_B(\mu)$, where $\mu(C) = C'$ a new nonterminal, for all C in $V_B-\Sigma_B$, $\mu(a) = a$ for all a in Σ_B and $P_B = \mu(P_B)$. This renamed version of G_B is chosen such that $(V_{B'}-\Sigma_B) \cap (V-\Sigma) = \emptyset$. By a slight generalization of Lemma 2.19, if we replace any production $C \to \delta B\gamma$ in P by $C \to \delta B'\gamma$ and add all the productions in $P_{B'}$, then the resulting grammar is s-form equivalent to G.

Carry out this replacement of B by B' for all appearances of B in the productions of P_{-B} except for the designated appearance in $A \to \alpha B\beta$. Let G' be the resulting grammar form. Clearly $G' \triangleleft_s G$, $\mathcal{L}_s(G',\Rightarrow) = \mathcal{L}_s(G,\Rightarrow)$ and moreover conditions (i) and (iii) are satisfied for G'. Letting G be the constructed G' we have the result. □

II.3 Collections of Grammar Families

In this section we are concerned solely with families of grammars defined by grammar forms under various interpretation mechanisms. Not only do we study g-interpretations and full g-interpretations, but we also introduce the notion of a quasi-interpretation. This is an interpretation in which the disjointness condition for nonterminals is dropped. We investigate compositions of these interpretations and their "inverses" in Sections 3.1 and 3.2. Finally, for a given interpretation the notion of strong form equivalence is studied. It is shown that the collection of all grammar families forms a lattice in Section 3.3, and further, for each grammar form there is a unique, up to isomorphism, (production) minimal strong form equivalent grammar form in Section 3.4.

II.3.1 Pre-orders and Closure Operators

Recall that a relation is a pre-order if it is both reflexive and transitive, for example the yield relation $\Rightarrow^*$ is a preorder. Similarly, interpretations can be viewed as relations over the class of context-free grammars, $\mathcal{G}$. For example $\lhd_g$ is such a relation and since $G \lhd_g G$ for all G and $G' \lhd_g G$, $G'' \lhd_g G'$ implies $G'' \lhd_g G$, g is a pre-order. Similarly $\lhd_s$ is a pre-order. This observation follows from the fact that (i) an identity substitution is a substitution, (ii) finite substitutions are closed under composition and (iii) dfl-substitutions are closed under composition.

A map $\theta: 2^{\mathcal{G}} \to 2^{\mathcal{G}}$ is said to be a (grammatical) operator. For example, consider the operator induced by $\lhd_g$, denoted θ_g defined by:

for all $C \subseteq \mathcal{G}$, $\theta_g(C) = \{G' \lhd_g G\text{: } G \text{ is in } C\}$.

Clearly $\mathcal{G}_g(G) = \theta_g(\{G\})$.

We say a grammatical operator θ is:

(i) inclusion preserving if for all X, Y $\subseteq \mathcal{G}$, $X \subseteq Y$ implies $\theta(X) \subseteq \theta(Y)$,

(ii) nondecreasing if for all $X \subseteq \mathcal{G}$, $X \subseteq \theta(X)$,

(iii) idempotent if for all $X \subseteq \mathcal{G}$, $\theta(\theta(X)) = \theta(X)$, that is, $\theta\theta = \theta$, and

(iv) union preserving if for all X, Y $\subseteq \mathcal{G}$, $\theta(X \cup Y) = \theta(X) \cup \theta(Y)$.

Consider θ_g. Clearly it is nondecreasing since $\lhd_g$ is reflexive; it is idempotent since $\lhd_g$ is transitive; it is union preserving by definition and hence inclusion preserving. When θ is a grammatical operator and θ fulfills (i), (ii) and (iii) above, we say θ is a closure operator. Hence θ_g is a closure operator.

We make precise the relationship between pre-orders and closure operators by way of the following definition.

Definition

Let ρ be a relation over $\mathcal{G}$, then define the induced ρ-operator, θ_ρ, by:

for all $C \subseteq \mathcal{G}$, $\theta_\rho(C) = \{G' : G' \rho G$ and G is in $C\}$.

Similarly given an operator θ define the induced θ-relation ρ_θ, by:

for all G, G' in $\mathcal{G}$, $G' \rho_\theta G$ iff G' is in $\theta(\{G\})$.

We now have:

Lemma 3.1

(i) A relation ρ over $\mathcal{G}$ is a pre-order iff the induced ρ-operator θ_ρ is a closure operator; in this case the induced θ_ρ-relation equals ρ and θ_ρ is union preserving.

(ii) An operator θ over $\mathcal{G}$ is a union-preserving closure operator iff ρ_θ is a pre-order and the induced ρ_θ-operator equals θ.

Proof: This is left to the reader. □

Since the relations $\lhd_g$, $\lhd_{fg}$, $\lhd_s$, $\lhd_{fs}$ are pre-orders, then immediately the induced operators θ_g, θ_{fg}, θ_s and θ_{fs} are closure operators. Moreover defining the inverse operators θ_g^{-1}, etc., by:

for all $C \subseteq \mathcal{G}$, $\theta_x^{-1}(C) = \{G : G' \lhd_x G$ and G' is in $C\}$,

where x is in $\{g, fg, s, fs\}$, then it follows that the inverse operators are also closure operators.

In the following we will write $\mathscr{G}_x$ and $\mathscr{G}_x^{-1}$ rather than θ_x and θ_x^{-1}, where it is assumed $\mathscr{G}_x$ and $\mathscr{G}_x^{-1}$ are extended to sets of grammars.

II.3.2 Composition of Grammatical Closure Operators

Having introduced in Section 3.1 the notion of closure operators and observed their close relationship with pre-orders, we now study in some detail compositions of closure operators of the form $\mathscr{G}_x$ and $\mathscr{G}_x^{-1}$, given by a pre-order $\lhd_x$. Composition is intended in the usual functional sense, for example $\mathscr{G}_x \mathscr{G}_y^{-1} \mathscr{G}_z^{-1}$ is defined by:

for all $C \subseteq \mathcal{G}$, $\mathscr{G}_x \mathscr{G}_y^{-1} \mathscr{G}_z^{-1}(C) = \mathscr{G}_x(\mathscr{G}_y^{-1}(\mathscr{G}_z^{-1}(C)))$.

Given a set of closure operators, $\mathcal{O}$, then we can define $\mathcal{O}^*$ to be the free monoid generated by $\mathcal{O}$ under composition, denoting the set of all finite compositions of operators from $\mathcal{O}$ where the identity closure operator $\mathcal{I}$, is defined by

for all $C \subseteq \mathcal{G}$, $\mathcal{I}(C) = C$.

Let O_1 and O_2 be two members of $\mathcal{O}^*$, we say:

$O_1 \equiv O_2$ iff for all $C \subseteq \mathcal{G}$, $O_1(C) = O_2(C)$.

We will address the following problem:

Is $\mathcal{O}^*/\equiv$ finite?

and if it is, then provide representatives of each equivalence class.

To demonstrate that this is not always trivial we will consider the closure operators defined by four new pre-orders, which are either restrictions or generalizations of g-interpretations.

Let $G = (V,\Sigma,P,S)$ and $G' = (V',\Sigma',P',S')$ be two grammars. We say G' is a <u>very full g-interpretation</u> of G modulo μ, if (a) $G' \triangleleft_g G(\mu)$, (b) $\mu(X) \neq \emptyset$, for all X in V and (c) $P' = \mu(P)$. We denote this by $G' \triangleleft_{vg} G(\mu)$. Notice that a vg-interpretation is a further restriction of a full g-interpretation. If μ is a finite letter substitution on $V - \Sigma$ and a substitution on Σ, we say G' is a quasi-interpretation of G modulo μ, if (a) $\mu(A) \subseteq V' - \Sigma'$, for all A in $V - \Sigma$, (b) $\mu(a) \subseteq \Sigma'^*$, for all a in Σ, (c) $P' \subseteq \mu(P)$ and (d) S' is in $\mu(S)$. This is denoted by $G' \triangleleft_q G(\mu)$. We also define a <u>very full</u> quasi-interpretation, denoted $G' \triangleleft_{vq} G(\mu)$, analogously to the very full g-interpretation. A quasi-interpretation is similar to a g-interpretation except that different nonterminals can give rise to the same nonterminal. It is convenient to define a special interpretation, the <u>two-symbol interpretation</u> G' of a grammar G, denoted $G' \triangleleft_2 G(\mu)$, if G' is a two-symbol form over $\{S',a'\}$, $\mu(A) = S'$, for all A in $V - \Sigma$, $\mu(a) = a'$ for all a in Σ, and $P' = \mu(P)$. Note that when $G' \triangleleft_2 G(\mu)$ we also have $G' \triangleleft_{vq} G(\mu)$.

The q-interpretation, by its very nature, is much coarser than the g-interpretation in a sense we will now make precise.

<u>Notation</u>

For notational convenience in the remainder of this section we will use $\acute{x}$ synonymously with $\triangleleft_x$ and $\grave{x}$ synonymously with $\triangleright_x$.

<u>Lemma 3.2</u>

$\acute{vq}\ \acute{g} = \acute{g}\ \acute{vq} = \acute{q} = \acute{g}\ \acute{2}$.

Proof: Since $\acute{vg}$, $\acute{vq}$, $\acute{g}$ and $\acute{2}$ are all restrictions of $\acute{q}$, we have $\acute{vq}\ \acute{g}$, $\acute{g}\ \acute{vq}$, $\acute{g}\ \acute{2} \subseteq \acute{q}$. Consider the converse inclusions. Let $G = (V,\Sigma,P,S)$, $G' = (V',\Sigma',P',S')$ and $G' \underset{q}{\lhd} G(\mu)$.

(a) $\acute{q} \subseteq \acute{vq}\ \acute{g}$. Construct $G" = (V",\Sigma',P",S")$, where $V" = \{[A,B]$: A in $V - \Sigma$ and B in $\mu(A)\} \cup \Sigma'$ and $S" = [S,S']$. Define $\mu"$ by $\mu"(A) = \{[A,B]$: $[A,B]$: B in $\mu(A)\}$, for all A in $V - \Sigma$ and $\mu"(a) = \mu(a)$, for all a in Σ and define μ' by $\mu'(a) = \{a\}$ for all a in Σ' and $\mu'([A,B]) = \{B\}$, for all $[A,B]$ in $V"$. Finally, choose $P" \subseteq \mu"(P)$ such that $\mu'(P") = P'$, in other words $P" = \mu'^{-1}(P')$. Since $\mu"$ is a dfl-substitution on $V - \Sigma$, $G \underset{g}{\lhd} G(\mu")$, and since μ' is not a dfl-substitution on $V' - \Sigma'$, but $P' = \mu'(P")$ we have $G' \underset{vq}{\lhd} G"(\mu')$. Hence the result.

(b) $\acute{q} \subseteq \acute{g}\ \acute{2}$. Construct $G" = (\{A,a\},\{a\},P",A)$ such that $G" \underset{2}{\lhd} G$, essentially by identifying all the terminals with a and all the non-terminals with A. Now define μ' by $\mu'(a) = \bigcup_{b \text{ in } \Sigma} \mu(b)$ and $\mu'(A) = \bigcup_{B \text{ in } V-\Sigma} \mu(B)$. Immediately $P' \subseteq \mu'(P")$ therefore $G' \underset{g}{\lhd} G"(\mu')$ giving the required result.

(c) $\acute{q} \subseteq \acute{g}\ \acute{vq}$. Since $\acute{2} \subseteq \acute{vq}$ the inclusion follows from (b). □

Corollary 3.3

Let G be a grammar form, then $\mathcal{L}_q(G,\Rightarrow)$ is in $\{\{\emptyset\}, \mathcal{L}(\text{FIN}), \mathcal{L}(\text{REG}), \mathcal{L}(\text{LIN}), \mathcal{L}(\text{CF})\}$.

Proof: By Lemma 3.2 and Theorem 2.18. □

Thus under q-interpretations only 5 language families are obtainable. Incidentally, we have also shown that $\mathcal{G}_{vq}$ and $\mathcal{G}_g$ commute. Since $\acute{vg} \subseteq \acute{g}$, $\acute{vg} \subseteq \acute{q}$ and $\acute{vq} \subseteq \acute{q}$ we have immediately that

$$\acute{vg}\ \acute{g} = \acute{g} = \acute{g}\ \acute{vg},\quad \acute{vg}\acute{q} = \acute{q} = \acute{q}\ \acute{vg} \text{ and } \acute{vq}\ \acute{q} = \acute{q} = \acute{q}\ \acute{vq}.$$

Similarly since $\acute{vg} \subseteq \acute{vq}$ and $\acute{vg}\acute{vq} = \acute{vq} = \acute{vq}\acute{vg}$ each composition of two pre-orders and two operators commutes. Moreover each composition can always be reduced to one pre-order or one operator. Thus we have shown

Theorem 3.3

Let $\mathcal{O} = \{\mathcal{G}_x$: x = vg,vq,g,q$\}$ then $\mathcal{O}^*/\equiv$ is finite and is equal to $\{[\mathcal{G}]_\equiv\} \cup \{[\mathcal{G}_x]_\equiv$: x = vg, vq,g,q$\}$. Moreover $\mathcal{O}$ is a commutative semigroup under composition (see Table 3.1).

	$\acute{v}g$	$\acute{v}q$	$\acute{g}$	$\acute{q}$
$\acute{v}g$	$\acute{v}g$	$\acute{v}q$	$\acute{g}$	$\acute{q}$
$\acute{v}q$	$\acute{v}q$	$\acute{v}q$	$\acute{q}$	$\acute{q}$
$\acute{g}$	$\acute{g}$	$\acute{q}$	$\acute{g}$	$\acute{q}$
$\acute{q}$	$\acute{q}$	$\acute{q}$	$\acute{q}$	$\acute{q}$

Table 3.1: Multiplication Table for $\{\mathcal{G}_x : x = vg, vq, g, q\}$.

By symmetry note that we have also shown that if $\mathcal{O} = \{\mathcal{G}_x^{-1} : vg, vq, g, q\}$ then $\mathcal{O}^*/\equiv$ is finite. Continuing our investigation we now turn to compositions of the form $\mathcal{G}_x \mathcal{G}_y^{-1}$ and $\mathcal{G}_x^{-1} \mathcal{G}_y$, alternatively, $\acute{x}\grave{y}$ and $\grave{x}\acute{y}$.

We demonstrate in the following that even when $\mathcal{O} = \{\mathcal{G}_x, \mathcal{G}_x^{-1} : x = vg, vq, g, q\}$, $\mathcal{O}^*/\equiv$ is finite. However we will discover that for some $\acute{x}$ and $\grave{y}$, $\acute{x}\grave{y} \neq \grave{y}\acute{x}$, in other words $\acute{x}$ and $\grave{y}$ do not commute. In this case, it is not even clear that either $\acute{x}\grave{y}$ or $\grave{y}\acute{x}$ is a pre-order. For consider $\acute{x}\grave{y}$, then $\acute{x}\grave{y}$ is a pre-order iff $\grave{y}\acute{x} \subseteq \acute{x}\grave{y}$. For if $\grave{y}\acute{x} \subseteq \acute{x}\grave{y}$, it follows that $\acute{x}\grave{y}\acute{x}\grave{y} \subseteq \acute{x}\acute{x}\grave{y}\grave{y} = \acute{x}\grave{y}$ and clearly $\acute{x}\grave{y} \subseteq \acute{x}\grave{y}\acute{x}\grave{y}$, hence $\acute{x}\grave{y}$ is a pre-order. On the other hand if $\acute{x}\grave{y}$ is a pre-order, then $\acute{x}\grave{y}\acute{x}\grave{y} = \acute{x}\grave{y}$ and, since $\grave{y}\acute{x} \subseteq \acute{x}\grave{y}\acute{x}\grave{y}$ then $\grave{y}\acute{x} \subseteq \acute{x}\grave{y}$. Therefore for $\acute{x}$ and $\grave{y}$ to commute, in other words $\acute{x}\grave{y} = \grave{y}\acute{x}$, it is both necessary and sufficient that $\acute{x}\grave{y}$ and $\grave{y}\acute{x}$ be pre-orders. Conversely, if $\acute{x}$ and $\grave{y}$ do not commute then at least one of $\acute{x}\grave{y}$ and $\grave{y}\acute{x}$ is not a pre-order.

We first obtain:

Theorem 3.4

$\acute{x}\grave{y} = \mathcal{G} \times \mathcal{G}$, for x, y in {g,q}.

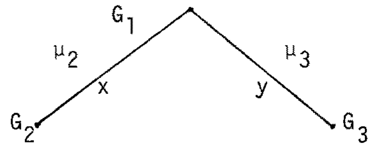
Proof: Let $G_i = (V_i, \Sigma_i, P_i, S)$, i = 1, 2 be two arbitrary grammars and c_1, c_2 be two new terminal symbols not in $V_1 \cup V_2$. Note that we have assumed S is common to G_1 and G_2. Since we are dealing with interpretations this is no loss of generality. Construct
$G = (V_1 \cup V_2 \cup \{c_1, c_2\}, \Sigma_1 \cup \Sigma_2 \cup \{c_1, c_2\}, P, S)$, where
$P = \{A \to \alpha c_i : A \to \alpha \text{ is in } P_i,\ i = 1, 2\}$.
Defining μ_i by $\mu_i(X) = X$, for all X in $V_1 \cup V_2$, $\mu_i(c_i) = \lambda$ and $\mu_i(c_j) = \emptyset$, $i \neq j$, i, j = 1, 2, then $G_i \triangleleft_g G(\mu_i)$, i = 1, 2. Therefore $\mathcal{G} \times \mathcal{G} \subseteq \acute{x}\grave{y}$, for all x, y in {g,q}, since $\acute{g} \subseteq \acute{q}$, giving the desired result. □

We say $G' \underset{sub}{\blacktriangleleft} G$ if G' is a subgrammar of G. Clearly $\acute{g} = s\acute{u}b\ \acute{v}g$ and $\acute{q} = s\acute{u}b\ \acute{v}q$, from the definitions. We use this simple observation to simplify some of the following proofs.

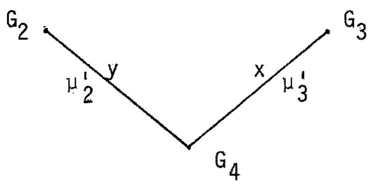
Theorem 3.5

$\acute{x}\check{y} \subseteq \check{y}\acute{x}$ for all x, y in {vg,vq,g,q}

Proof: Throughout we have $G_i = (V_i,\Sigma_i,P_i,S_i)$, i = 1, 2, 3 with $G_2 \underset{x}{\blacktriangleleft} G_1(\mu_2)$ and $G_3 \underset{y}{\blacktriangleleft} G_1(\mu_3)$. We wish to show there exists $G_4 = (V_4,\Sigma_4,P_4,S_4)$ such that $G_4 \underset{y}{\blacktriangleleft} G_2$ and $G_4 \underset{x}{\blacktriangleleft} G_3$. Diagrammatically we have



and we wish to construct G_4 such that



(a) $\acute{v}g\ \grave{v}q \subseteq v\grave{q}\ \acute{v}g$. Define a relation $\psi \subseteq (V_2 - \Sigma_2) \times (V_3 - \Sigma_3)$ by:

$A_2\ \psi\ A_3$ iff there exists an A in $V_1 - \Sigma_1$ with A_i in $\mu_i(A)$, i = 2, 3.

Define μ_i', i = 2, 3 by $\mu_i'(a) = \lambda$, for all a in Σ_i, i = 2, 3, and, for all A_2 in $V_2 - \Sigma_2$.

$\mu_2'(A_2)$ = either $\{[B_2,A_3]\colon B_2\ \psi\ A_3$ and $A_2\ \psi\ A_3\}$ if there is an A in $V_3 - \Sigma_3$ such that $A_2\ \psi\ A$ or A_2 otherwise,

and for all A_3 in $V_3 - \Sigma_3$:

$\mu_3'(A_3)$ = either $\{[A_2,A_3]\colon A_2\ \psi\ A_3\}$ if there is an A in $V_2 - \Sigma_2$ such that $A\ \psi\ A_3$ or A_3 otherwise.

We now claim that $\mu_2'\mu_2(X) = \mu_3'\mu_3(X)$ for all X in V_1 which appear in at least one production of G_1. Since μ_i and μ_i' are very full interpretations for i = 2, 3, if the claim is true we let $P_4 = \mu_2'\mu_2(P_1) = \mu_3'\mu_3(P_1)$. Note that $\mu_i'(A_i) = A_i$, only when A_i does not appear in any production of P_i, hence if there is an A in $V_1 - \Sigma_1$ such that A_i is in $\mu_i(A)$, then A does not appear in any production in

P_1 either and further A_i does not appear in any production in P_4, $i = 2, 3$.

For all a in Σ_1, $\mu_2'\mu_2(a) = \mu_3'\mu_3(a) = \lambda$, hence $\Sigma_4 = \emptyset$. Consider $[A_2,A_3]$ in $\mu_3'\mu_3(A)$, then by definition $A_2 \psi A_3$ and A_2 is in $\mu_2(A)$, hence $[A_2,A_3]$ is in $\mu_2'\mu_2(A)$. Conversely, if $[A_2,A_3]$ is in $\mu_2'\mu_2(A)$ then by definition there exists B_2 in $\mu_2(A)$ and A_3 in $\mu_3(A)$, therefore $[A_2,A_3]$ is in $\mu_2'(B_2)$ only if $A_2 \psi A_3$, in which case $[A_2,A_3]$ is in $\mu_3'\mu_3(A)$.

Finally let $V_4 = \mu_2'(V_2 - \Sigma_2) \cup \mu_3'(V_3 - \Sigma_3)$ and $P_4 = \mu_2'\mu_2(P_1) = \mu_3'\mu_3(P_1)$, then $G_4 \underset{vq}{\triangleleft} G_2(\mu_2')$ and $G_4 \underset{vg}{\triangleleft} G_3(\mu_3')$ by careful scrutiny of μ_2' and μ_3'.

(b) $\acute{v}g\ \grave{v}g \subseteq \grave{v}g\ \acute{v}g$. Modify the proof in (a) replacing the definition of μ_2' on nonterminals by:

For all A_2 in $V_2 - \Sigma_2$,

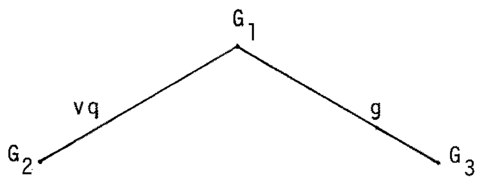
$\mu_2'(A_2)$ = either $\{[A_2,A_3]\colon A_2 \psi A_3\}$ if there is an A in $V_3 - \Sigma_3$ such that $A_2 \psi A$ or A_2 otherwise.

(c) $\acute{v}q\ \grave{v}q \subseteq \grave{v}q\ \acute{v}q$. Define $\mu_i'(a) = \lambda$, for all a in Σ_i, and $\mu_i'(A) = S_4$, for all A in $V_i - \Sigma_i$, $i = 2, 3$. Clearly $\mu_2'\mu_2(X) = \mu_3'\mu_3(X)$ for all X in V_1, hence letting $G_4 = (\{S_4\},\emptyset,\mu_2'\mu_2(P_1),S_4)$ we have $G_4 \underset{vq}{\triangleleft} G_i(\mu_i')$, $i = 2, 3$.

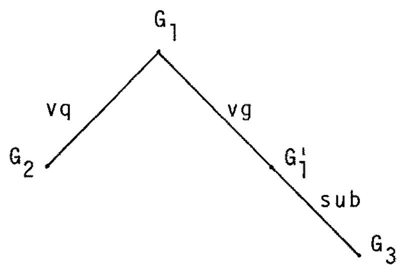
(d) $\acute{v}q\ s\grave{u}b \subseteq s\grave{u}b\ \acute{v}q$. Since $G_3 \underset{sub}{\triangleleft} G_1(\mu_3)$, μ_2 is well defined on $V_3 \subseteq V_1$, hence letting $G_4 = (V_2,\Sigma_2,\mu_2(P_3),S_2)$ implies $G_4 \underset{vq}{\triangleleft} G_3$ and $G_4 \underset{sub}{\triangleleft} G_2$, since $\mu_2(P_3) \subseteq \mu_2(P_1) = P_2$.

(e) $\acute{v}g\ s\grave{u}b \subseteq s\grave{u}b\ \acute{v}g$. As in (d).

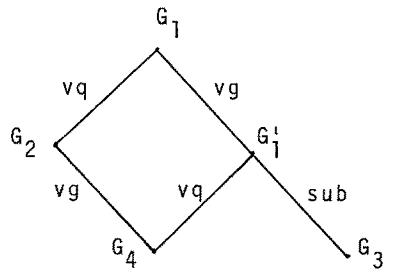
To complete the theorem we use the observation that $\acute{q} = s\acute{u}b\ \acute{v}q$ and $\acute{g} = s\acute{u}b\ \acute{v}q$. For example, given



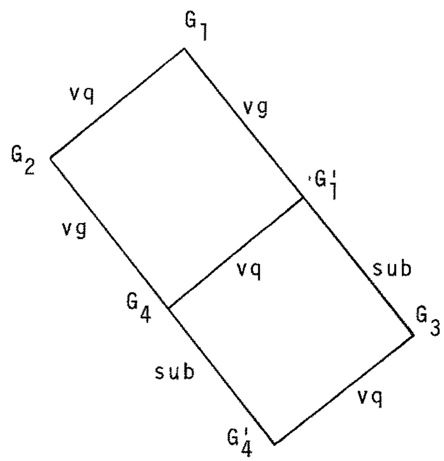
then we know there exists G_1' such that



Now by (c) above we obtain:



and by (d) we can complete this to give



and therefore $G_4' \underset{g}{\lhd} G_2$ and $G_4' \underset{vq}{\lhd} G_3$ as desired giving $\acute{vq}\ \grave{g} \subseteq \grave{g}\ \acute{vq}$.
We can carry out the diagram chasing for all other cases bar those $\acute{x}\grave{y}$, where x, y are in {g,q}. However in any of these cases let $G_4 = (\{S\},\emptyset,\emptyset,S)$ whence $\acute{x}\grave{y} \subseteq \grave{y}\acute{x}$. □

We summarize the results so far in Table 3.2.

	$\grave{vg}$	$\grave{vq}$	$\grave{g}$	$\grave{q}$
$\acute{vg}$	$\subset$	$\subset$	$\subset$	$\subset$
$\acute{vq}$	$\subset$	$\subset$	$\subset$	$\subset$
$\acute{g}$	$\subset$	$\subset$	$\bar{u}$	$\bar{u}$
$\acute{q}$	$\subset$	$\subset$	$\bar{u}$	$\bar{u}$

Table 3.2

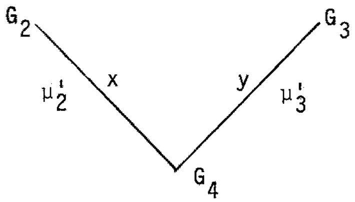
$\subset$ indicates $\acute{x}\grave{y} \subseteq \grave{y}\acute{x}$

$\bar{u}$ indicates $\acute{x}\grave{y} = \grave{y}\acute{x} = \mathcal{C}\times\mathcal{C}$, the universal relation

We have used symmetry to fill out this table and the fact that since $\bar{\bar{u}} = \acute{x}\grave{y}$, for all x, y in {g,q} and $\acute{x}\grave{y} \subseteq \grave{y}\acute{x}$, then $\acute{x}\grave{y} = \grave{y}\acute{x} = \bar{\bar{u}}$.

We turn now to the consideration of the relations $\grave{x}\acute{y}$, for all x, y in {vg,vq,g,q}.

For consistency, we will assume that we are given



in each of the following theorems for which the inclusion $\grave{x}\acute{y} \subseteq \acute{y}\grave{x}$ is to be proved.

Theorem 3.6

$\grave{vg}\ \acute{y} \not\subseteq \acute{y}\ \grave{vg}$, for y = vg and vq.

Proof: We first prove a stronger result, namely $\grave{vg}\acute{vg} \not\subseteq \acute{vq}\grave{vg}$.
Let G_2, G_3, G_4 be defined by the productions:

G_2: $S \to BaC$; $S \to B\bar{C}$
G_3: $S \to BaC$; $S \to \bar{B}C$
G_4: $S \to BC$; $S \to \bar{B}C$; $S \to B\bar{C}$; $S \to \bar{B}\bar{C}$.

Clearly $G_4 \underset{vg}{\blacktriangleleft} G_i$, i = 2, 3.
Now assume there exists G_1 such that $G_2 \underset{vq}{\blacktriangleleft} G_1(\mu_2)$ and $G_3 \underset{vg}{\blacktriangleleft} G_1(\mu_3)$.
Without loss of generality we may assume G_1 is reduced. Clearly V_1 consists of at least 4 nonterminals and one terminal, since assuming otherwise implies

either $\mu_3(D)$ contains at least two nonterminals from $V_3 - \Sigma_3$ for some D in $V_1 - \Sigma_1$, in which case since G_1 is reduced, both must appear in the productions of G_1 and hence $\mu_3(P_1) \neq P_3$, whatever the choice, a contradiction,

or G_1 has no terminals, in which case P_3 has no terminals, a contradiction.

Further V_1 contains exactly 4 nonterminals, since G_1 is reduced and nonterminals are preserved under vg-interpretation, hence let them be denoted S, B, $\bar{B}$, C where μ_3 is the identity on $V_1 - \Sigma_1$, without loss of generality.

$\mu_2(P_1)$, by definition. It is clear that $G_2 \underset{g}{\lhd} G_1(\mu_2)$ if $y = g$ and $G_2 \underset{q}{\lhd} G_1(\mu_2)$ otherwise. □

Our final result is:

Theorem 3.8

$\grave{vq}\ \acute{y} \subseteq \acute{y}\ \grave{vq}$ for $y = vq$, g and q.

Proof: We will first prove $\grave{vq}\ \acute{vq} \subseteq \acute{vq}\ \grave{vq}$.

(a) Let $W_i = \{x\colon x \text{ in } \Sigma_i^*,\ A \to \alpha x \beta \text{ is in } P_i,\ \alpha\beta \text{ in } V_i^*\}$, $i = 2, 3$ and let $G_1 = (V_1, \Sigma_1, P_1, S_1)$ be defined as follows:

$V_1 - \Sigma_1 = \{[A_2, A_3]\colon A_i \text{ in } V_i - \Sigma_i,\ i = 2, 3\}$,

$\Sigma_1 = \{[x_2, x_3]\colon x_i \text{ in } W_i,\ i = 2, 3\}$,

$P_1 = \{[A,B] \to [x_0,y_0][A_1,B_1] \ldots [A_n,B_n][x_n,y_n]\colon$
$A \to x_0A_1 \ldots x_n \text{ is in } P_2,\ B \to y_0B_1 \ldots B_n \text{ is in } P_3\}$,

and

$S_1 = [S_2, S_3]$.

Let μ_2 and μ_3 be defined by:

$\mu_i([X_2,X_3] = X_i$, for $i = 2, 3$, for all $[X_2,X_3]$ in V_1.

Now for each $p\colon C \to z_0C_1 \ldots C_nz_n$ in P_4 there are productions $q\colon A \to x_0A_1 \ldots A_nx_n$ and $r\colon B \to y_0B_1 \ldots B_ny_n$ in P_2 and P_3 respectively, such that p is in $\mu_2'(q) \cap \mu_3'(r)$. Conversely, given q in P_2, there is a p in P_4 and an r in P_3 satisfying p is in $\mu_2'(q) \cap \mu_2'(r)$, and similarly for each r in P_3. Hence P_1 as defined contains encoded versions of both P_2 and P_3 which are recovered by the substitutions μ_2 and μ_3, respectively. Therefore

$G_i \underset{vq}{\lhd} G_1(\mu_i)$, $i = 2, 3$.

(b) $\grave{vq}\ \acute{g} \subseteq \acute{g}\ \grave{vq}$. Proceed as in (a) except that P_1 should also include

$\{[A_0,B_0] \to [x_0,y_0][A_1,B_1] \ldots [A_n,B_n][x_n,y_n]\colon$
$B_0 \to y_0B_1 \ldots B_ny_n$ is in P_3, A_i is in $V_2 - \Sigma_2$, and x_i is in W_2, $0 \leq i \leq n\}$.

Clearly $G_3 \underset{vq}{\lhd} G_1(\mu_3)$ with μ_3 as in (a). We need to show that $G_2 \underset{g}{\lhd} G_1$. We define μ_2 as follows:

For each A in $V_2 - \Sigma_2$ which appears in a production in P_2, select just one B in $V_3 - \Sigma_3$ with $\mu_2'(A) \cap \mu_3'(B) \neq \emptyset$; if A does not appear in any production choose B arbitrarily from $V_3 - \Sigma_3$. Let $\mu_2([A,B]) = A$ if B has been selected for A and $\mu_2([A,B]) = \emptyset$ otherwise, and let $\mu_2([x,y]) = \{x\}$, for all $[x,y]$ in Σ_1. As in (a), for each q in P_2 there is a p in P_4 with p in $\mu_2'(q)$ and an r in P_3 with p in $\mu_3'(r)$, Consider P_1. Since $V_1 - \Sigma_1$ consists of four nonterminals, then P_1 contains only productions of two types, namely,

$S \to xByCz$

and

$S \to u\bar{B}vCw$

where u, v, w, x, y, z are terminal words.
Turning to G_2 we now obtain a contradiction. Clearly S is in $\mu_2(S)$ and $\{C,\bar{C}\} \subseteq \mu_2(C)$. Since $G_2 \underset{vq}{\triangleleft} G_1(\mu_2)$ then $S \to BaC$ is in $\mu_2(P_1)$ and further, there is a production p: $S \to xXyCz$ in P_1 such that

$S \to BaC$ is in $\mu_2(p)$.

However this implies that $S \to Ba\bar{C}$ is also in $\mu_2(p)$ and hence in P_2, a contradiction. Therefore G_1 does not exist. Hence $\grave{v}g\ \acute{v}g \subseteq \acute{v}q\ \grave{v}g$.

To complete the theorem observe that:

$\acute{v}g\ \grave{v}g \subseteq \acute{v}q\ \grave{v}g$ therefore $\grave{v}g\ \acute{v}g \not\subseteq \acute{v}g\ \grave{v}g$, and

$\grave{v}g\ \acute{v}g \subseteq \grave{v}g\ \acute{v}q$ and therefore $\grave{v}g\ \acute{v}q \not\subseteq \acute{v}q\ \grave{v}g$. □

Having established this non-inclusion result, it then follows that we cannot make use of diagram extension to obtain results for $\acute{v}g\ \grave{g}$ and $\grave{v}g\ \acute{q}$. Therefore we treat these cases separately.

Theorem 3.7

$\grave{v}g\ \acute{y} \subseteq \acute{y}\ \grave{v}g$ for y = g and q.

Proof: Without loss of generality we will assume G_4 has no terminal symbols. Define a homomorphism h by $h(A) = A$, for all nonterminals A in $V_2 \cup V_3$ and $h(a) = \lambda$, for all terminals a in $\Sigma_2 \cup \Sigma_3$. Since μ_2' is a very full interpretation we can assume A is in $\mu_2'(A)$, for all A in $V - \Sigma_2$. Hence for all p in P_2 there exists q in P_4 such that $h(p) = q$.

Construct $G_1 = (V_1,\Sigma_1,P_1,S_1)$ from G_3 as follows:
$V_1 - \Sigma_1 = V_3 - \Sigma_3$, $\Sigma_1 = \Sigma_3 \cup \{c\}$, where c is a new symbol not in V_3,
$P_1 = \{A \to \alpha_1 c\alpha_2 \ldots \alpha_n c\colon A \to \alpha_1 \ldots \alpha_n$ is P_3, $n \geq 1$, α_1 is in Σ_3^*, α_i is in $(V_3 - \Sigma_3)\Sigma_3^*$, $1 < i \leq n\}$, and $S_1 = S_3$.
Clearly $G_3 \underset{vg}{\triangleleft} G_1(\mu_3)$ where $\mu_3(X) = X$, for all X in V_3 and $\mu_3(c) = \lambda$.
Define μ_2 by:

$\mu_2(A) = \mu_3'(A)$, for all A in $V_1 - \Sigma_1 = V_3 - \Sigma_3$, and

$\mu_2(a) = \{x\colon x$ is in Σ_2^* and $B \to \alpha x\beta$ is in P_2, for some α,β in $V_2^*\}$, for all a in Σ_1.

Then $P_2 \subseteq \mu_2(P_1)$ since λ is in $\mu_2(a)$ for all a and for all p in P_2 there exists q in P_4 with $h(p) = q$, which in turn implies there is r in P_3 with q in $\mu_3'(r)$ and r' in P_1 with $h(r') = h(r)$, hence p is in therefore q can be recovered from P_1. Hence q is in $\mu_2(P_1)$, and $G_2 \underset{g}{\triangleleft} G_1(\mu_2)$ by definition of μ_2.

(c) As in (b) defining μ_2 by:

$\mu_2([X,Y]) = X$ for all $[X,Y]$ in V_1,

clearly by the above arguments $G_2 \underset{q}{\triangleleft} G_1(\mu_2)$. □

We are now in a position to extend Table 3.2 considerably, completing our investigation of the compositions of two pre-orders. The result is seen in Table 3.3. By our earlier remarks this means that the pairs $(\grave{v}g, \acute{v}g)$, $(\grave{v}g, \acute{v}q)$ and $(\grave{v}q, \acute{v}g)$ are not commuting pairs. Similarly, we know that $\acute{v}g\ \grave{v}g$ is not a pre-order

	$\acute{v}g$	$\acute{v}q$	$\acute{g}$	$\acute{q}$
$\grave{v}g$	$\not\subset$	$\not\subset$	=	=
$\grave{v}q$	$\not\subset$	=	=	=
$\grave{g}$	=	=	$\bar{\bar{u}}$	$\bar{\bar{u}}$
$\grave{q}$	=	=	$\bar{\bar{u}}$	$\bar{\bar{u}}$

where $\not\subset$ means $\grave{x}\acute{y} \not\subseteq \acute{y}\grave{x}$ but $\acute{y}\grave{x} \subseteq \grave{x}\acute{y}$.

Table 3.3

and neither are $\acute{v}q\ \grave{v}g$ and $\acute{v}g\ \grave{v}q$. In terms of operators this means that $\mathcal{G}^{-1}_{vg}$, $\mathcal{G}_{vg}$ do not commute, and similarly for the others.

Let us turn to the composition of three pre-orders. Since by Table 3.1 we can always replace or simplify $\acute{x}\acute{y}$ by a single pre-order, and similarly with $\grave{x}\grave{y}$ (by symmetry), we only need consider compositions of the forms

$\acute{x}\grave{y}\acute{z}$ or $\grave{x}\acute{y}\grave{z}$

and by symmetry we only need consider one or the other.

Consider $\acute{x}\grave{y}\acute{z}$. If $\acute{x}$ and $\grave{y}$ commute (see Table 3.3) we can replace $\acute{x}\grave{y}\acute{z}$ by $\grave{y}\acute{x}\acute{z}$ and this in turn, using Table 3.1, by $\grave{y}\acute{w}$. Similarly if $\grave{y}$ and $\acute{z}$ commute we obtain $\acute{v}\grave{y}$. This only leaves those combinations for which neither $\acute{x}$, $\grave{y}$ nor $\grave{y}$, $\acute{z}$ commute. These are

(i) $\acute{v}g\ \grave{v}g\ \acute{v}g$; (ii) $\acute{v}g\ \grave{v}g\ \acute{v}q$;

(iii) $\acute{v}g\ \grave{v}q\ \acute{v}g$; (iv) $\acute{v}q\ \grave{v}g\ \acute{v}g$; (v) $\acute{v}q\ \grave{v}g\ \acute{v}q$.

Now using the general transformation $\acute{x}\grave{y}\acute{z} \subseteq \grave{y}\acute{x}\acute{z}$ which is valid by Table 3.2, we obtain $\acute{x}\grave{y}\acute{z} \subseteq \grave{y}\acute{w}$ and if $w = z$ then $\acute{x}\grave{y}\acute{z} = \grave{y}\acute{z}$. This is the case for (i), (ii), (iii) and (v), giving (i) $\grave{v}g\ \acute{v}g$ (ii) $\grave{v}g\ \acute{v}q$ (iii) $\grave{v}q\ \acute{v}g$ and (v) $\grave{v}g\ \acute{v}q$. However with (iv) we obtain:

$\acute{v}q\ \grave{v}g\ \acute{v}g \subseteq \grave{v}g\ \acute{v}q$.

Thus it remains to prove that the reverse inclusion holds even in this case.

Let $G = (V,\Sigma,P,S)$ be a grammar, then we say X in V is an <u>extra symbol</u> if it does not appear in any production in P. We say G has extra symbols if V contains at least one extra symbol. To complete the investigation of compositions of three pre-orders we need:

Theorem 3.9

(i) Let $G = (V,\Sigma,P,S)$ be a grammar form, then there exists a grammar form G_1 with no extra symbols such that $G \triangleleft_{vg} G_1$.

(ii) Let $G_i = (V_i,\emptyset,P_i,S_i)$, $i = 1, 2$ be two grammar forms with no extra symbols and $G_1 \triangleleft_{vq} G_2(\mu)$. Then there exists a grammar form G such that $G \triangleleft_{vg} G_i$, $i = 1, 2$.

Proof: (i) Let $G_1 = (V_1,\Sigma_1,P,S)$ where $V_1 \subseteq V$, $\Sigma_1 \subseteq \Sigma$ and $V - V_1$ is exactly the extra symbols of V. Clearly $G \triangleleft_{vg} G_1$, by the definition of interpretation.

(ii) Let $G = (V,\emptyset,P,S)$ where $V = \{[A,B]\text{: } A \text{ in } V_1,\ B \text{ in } V_2 \text{ and } A \text{ is in } \mu(B)\}$. Define μ_2 on V_2 by:

$$\mu_2(B) = \{[A,B]\text{: } A \text{ is in } \mu(B)\},$$

let $P = \mu_2(P_2)$ and $S = [S_1,S_2]$, then clearly $G \triangleleft_{vg} G_2(\mu_2)$.

Similarly define μ_1 on V_1 by:

$$\mu_1(A) = \{[A,B]\text{: } A \text{ is in } \mu(B)\}.$$

We need to show that $G \triangleleft_{vg} G_1(\mu_1)$.

It suffices to prove that $\mu_1(P_1) = \mu_2(P_2)$. Without loss of generality we may assume $P_1 \neq \emptyset$.

(a) $\mu_1(P_1) \subseteq \mu_2(P_2)$.

Assume $A_0 \to A_1 \ldots A_m$ is in P_1, then there exist $B_0, B_1, \ldots, B_m$ with A_i in $\mu(B_i)$, $0 \le i \le m$ such that

$$A_0 \to A_1 \ldots A_m \text{ is in } \mu(B_0 \to B_1 \ldots B_m)$$

hence $[A_0,B_0] \to [A_1,B_1] \ldots [A_m,B_m]$ is in $\mu_1(P_1)$ and also in $\mu_2(P_2)$.

(b) $\mu_2(P_2) \subseteq \mu_1(P_1)$ follows similarly. □

We now apply Theorem 3.9 to give:

Theorem 3.10

$$\acute{v}q\ \grave{v}g\ \acute{v}g = v\grave{g}\ \acute{v}q$$

Proof: Since we have already shown that $\acute{v}q\ \grave{v}g\ \acute{v}g \subseteq \grave{v}g\ \acute{v}q$ it remains to prove the reverse inclusion.

Claim 1: Let $G_i = (V_i, \Sigma_i, P_i, S_i)$ $i = 1, 2$ be such that $G_1 \underset{x}{\triangleleft} G_2(\mu)$, where $x = vg$ or vq, then there exists $G_i' = (V_i', \emptyset, P_i', S_i)$, $i = 1, 2$ with $G_i' \underset{vg}{\triangleleft} G_i$, $G_1' \underset{x}{\triangleleft} G_2'(\mu')$, and $V_i' = V_i - \Sigma_i$, $i = 1, 2$.

Simply define μ_i, $i = 1, 2$ by:

$\mu_i(a) = \lambda$, a in Σ_i and $\mu_i(A) = A$, A in $V_i - \Sigma_i$,

then $G_i' \underset{vg}{\triangleleft} G_i(\mu_i)$ and moreover restricting μ to the nonterminals of V_2 giving μ' implies

$G_1' \underset{x}{\triangleleft} G_2'(\mu')$,

in other words $\mu_1\mu(P_2) = \mu'\mu_2(P_2)$.

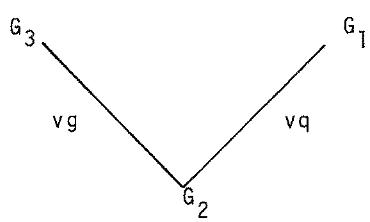
Claim 2: Let $G_i = (V_i, \emptyset, P_i, S_i)$, $i = 1, 2$ be such that $G_1 \underset{x}{\triangleleft} G_2(\mu)$ for $x = vg$ or vq, then there exists $G_i'' = (V_i'', \emptyset, P_i'', S_i)$, $i = 1, 2$ with $G_i \underset{vg}{\triangleleft} G_i''$, $i = 1, 2$, $G_1'' \underset{x}{\triangleleft} G_2''(\mu'')$, and $V_i'' = V_i$ - {the extra nonterminals in G_i}, $i = 1, 2$.

By Theorem 3.9 G_1'' and G_2'' exist, hence define μ'' by:

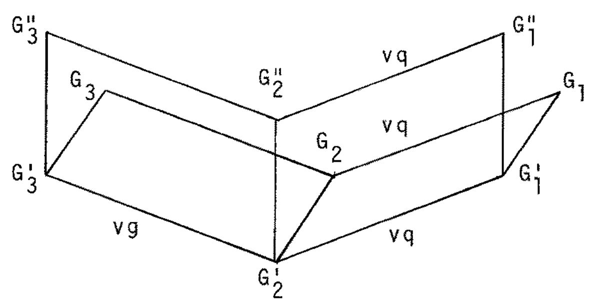
$\mu''(A) = \mu(A)$, for all A in V_2'',

and immediately $G_1'' \underset{x}{\triangleleft} G_2''(\mu'')$.

Now consider $G_i = (V_i, \Sigma_i, P_i, S_i)$, $i = 1, 2, 3$ such that $G_2 \underset{vg}{\triangleleft} G_3(\mu_3)$ and $G_2 \underset{vq}{\triangleleft} G_1(\mu_1)$, then diagramatically we have:

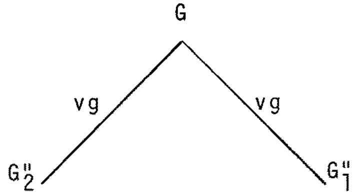


which by the above two claims we can transform into:

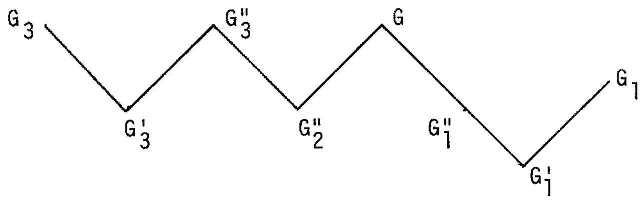


where the G_i'' have no terminals and no extra symbols, $G_2'' \underset{vg}{\vartriangle} G_3''(\mu_3'')$, $G_2'' \underset{vq}{\vartriangleleft} G_1''(\mu_1'')$ and all unmarked edges are vg-interpretations.

Since G_1'' and G_2'' fulfill the conditions of part (ii) of Theorem 3.9 then we can construct G such that



We can now relate G_3 and G_1 as follows:



Let this composition be denoted by ρ, that is $G_3 \rho G_1$. Since $\acute{v}g\grave{v}g \subseteq \grave{v}g\acute{v}g$ and $\acute{v}g\acute{v}g = \acute{v}g$ we obtain

$\rho \subseteq \grave{v}g\acute{v}g$.

By construction $\grave{v}g\acute{v}q \subseteq \rho$ and $\grave{v}g\acute{v}q \subseteq \acute{v}q\grave{v}g\acute{v}g$, hence the result. □

We have demonstrated as a result of Theorem 3.10 the following:

Theorem 3.11

Let $\mathcal{V} = \{ \mathcal{G}_x, \mathcal{G}_x^{-1} : x = g, q, vg, vq\}$. Then $\mathcal{V}^*/\equiv$ is finite.

To complete this investigation it is necessary to exhibit the equivalence classes of $\mathcal{V}$. This involves proving that the various "irreducible" compositions considered are indeed distinct. This straightforward but laborious task we leave to the reader.

To close this section we give an example of two pre-orders $\acute{x}$, $\grave{y}$ for which $\acute{x}\grave{y} \not\subseteq \grave{y}\acute{x}$ and $\grave{y}\acute{x} \not\subseteq \acute{x}\grave{y}$, thus demonstrating that $\mathcal{V}$ is not always as nicely behaved as those considered in this section. Let $\acute{f}g$ denote a full g-interpretation. Consider the pair $\grave{v}g$ and $\acute{f}g$. As in Theorem 3.6 we have $\grave{v}g\acute{f}g \not\subseteq \acute{f}g\grave{v}g$. Now consider the grammars:

G: $S \to SS$; $S \to Sa$

G_1: $S \to SS$;

G_2: $S \to SS$; $S \to S$

Then $G_1 \trianglelefteq_{\acute{f}g} G$ and $G_2 \trianglelefteq_{\grave{v}g} G$ but there is no F with $F \trianglelefteq_{\grave{v}g} G_1$ and $F \trianglelefteq_{\acute{f}g} G_2$, hence $\acute{f}g\grave{v}g \not\subseteq \grave{v}g\acute{f}g$.

II.3.3 Strong Form Equivalence and Lattices

Two grammar forms G_1 and G_2 are strong x-form equivalent if $\mathcal{G}_x(G_1) = \mathcal{G}_x(G_2)$. We write $G_1 \equiv_x G_2$. Immediately $G_1 \equiv_x G_2$ iff $G_1 \trianglelefteq_x G_2$ and $G_2 \trianglelefteq_x G_1$. It is easily verified that $\equiv_x$ is an equivalence relation over $\mathcal{G}$, hence for each grammar form G, let $[G]_x$ denote the equivalence class containing G modulo $\equiv_x$. For each grammar form G, let $\mathcal{E}_x(G) = \{[G']_x : G' \trianglelefteq_x G\}$ and $\mathcal{E}_x = \{[G]_x : G \text{ in } \mathcal{G}\}$. Define an "interpretation" relation $\le_x$, over $\mathcal{E}_x$ by: for all E_1, E_2 in $\mathcal{E}_x$, $E_1 \le_x E_2$ if there exist G_1 and G_2 with $E_i = [G_i]_x$, $i = 1, 2$ and $G_1 \trianglelefteq_x G_2$.

The relation $\le_x$ is again a pre-order. Further since $E_1 \le_x E_2$ and $E_2 \le_x E_1$ implies $E_1 = E_2$, $\le_x$ is anti-symmetric and therefore a partial order. We show in the following that $(\mathcal{E}_x, \le_x)$ and $(\mathcal{E}_x(G), \le_x)$ are distributive lattices.

Definition

Let M be a set, partially ordered under $\le$. An element z in M is said to be a greatest lower bound (glb) of the elements x, y in M, denoted by $z = x \wedge y$, if $z \le x$, $z \le y$ and $z' \le z$ for every element z' in M satisfying $z' \le x$ and $z' \le y$. An element z in M is said to be a least upper bound (lub) of the elements x, y in M denoted $z = z \vee y$ if $z \ge x$, $z \ge y$ and $z' \ge z$ for every element z' in M satisfying $z' \ge x$ and $z' \ge y$. A lattice is a pair $(M,\le)$, where M is a set partially ordered under $\le$ and each pair of elements of M have both a glb and an lub. A lattice $(M,\le)$ is distributive if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

holds for all elements x, y and z in M.

In the remainder of this section we assume x = s throughout, that is we only deal with strict interpretations. The interested reader can prove the analogous results for the other interpretations.

We begin by proving that the intersection of two grammar families is a grammar family.

Lemma 3.12

For all grammar forms G_1 and G_2 there exists a grammar form G satisfying $\mathcal{G}_s(G) = \mathcal{G}_s(G_1) \cap \mathcal{G}_s(G_2)$.

Proof: Let $G_i = (V_i, \Sigma_i, P_i, S_i)$, i = 1, 2. Construct a grammar form $G = (V, \Sigma, P, S)$ as follows:

$V = ((V_1 - \Sigma_1) \times (V_2 - \Sigma_2)) \cup (\Sigma_1 \times \Sigma_2)$

$\Sigma = \Sigma_1 \times \Sigma_2$,

$S = [S_1, S_2]$, and P consists of the following productions:

for each production $A \to X_1 \ldots X_n$ of P_1

and each production $B \to Y_1 \ldots Y_n$ of P_2

such that $[X_i, Y_i]$ is in V, $1 \le i \le n$ take the production:

$[A,B] \to [X_1, Y_1] \ldots [X_n, Y_n]$ into P.

The condition $[X_i, Y_i]$ in V ensures that nonterminals are paired with nonterminals and terminals with terminals. We now prove that $\mathcal{G}_s(G)$ is indeed equal to $\mathcal{G}_s(G_1) \cap \mathcal{G}_s(G_2)$. Consider $\mathcal{G}_s(G) \subseteq \mathcal{G}_s(G_1) \cap \mathcal{G}_s(G_2)$. We show that $G \triangleleft_s G_1$ which implies by transitivity that $\mathcal{G}_s(G) \subseteq \mathcal{G}_s(G_1)$. Consider a finite substitution μ defined on V_1^* by $\mu(A) = \{A\} \times (V_2 - \Sigma_2)$ and $\mu(a) = \{a\} \times \Sigma_2$, for each A in $V_1 - \Sigma_1$ and each a in Σ_1. Observe that $\mu(X) \cap \mu(Y) = \emptyset$ for $X \neq Y$, and $[S_1, S_2]$ is in $\mu(S_1)$. Let p: $[A,B] \to [X_1, Y_1] \ldots [X_n, Y_n]$ be an arbitrary production of P. Now $p_1 : A \to X_1 \ldots X_n$ is in P_1, by construction, hence p is in $\mu(p_1)$. Therefore $G \triangleleft_s G_1(\mu)$. We can prove similarly that $G \triangleleft_s G_2$ hence $\mathcal{G}_s(G)$ is in both $\mathcal{G}_s(G_1)$ and $\mathcal{G}_s(G_2)$.

We turn to the reverse inclusion. Suppose $G' = (V', \Sigma', P', S')$ is in $\mathcal{G}_s(G_1) \cap \mathcal{G}_s(G_2)$. There exist substitutions μ_1 and μ_2 such that $G' \triangleleft_s G_i(\mu_i)$, i = 1, 2. Let μ be the substitution on V^* defined by:

$\mu([X,Y]) = \mu_1(X) \cap \mu_2(Y)$ for all $[X,Y]$ in V.

Note that S' is in $\mu([S_1, S_2]) = \mu_1(S_1) \cap \mu_2(S_2)$. Moreover

$$\mu([X_1,Y_1]) \cap \mu([X_2,Y_2]) = (\mu_1(X_1) \cap \mu_2(Y_1)) \cap (\mu_1(X_2) \cap \mu_2(Y_2))$$
$$= (\mu_1(X_1) \cap \mu_1(X_2)) \cap (\mu_2(Y_1) \cap \mu_2(Y_2)) = \emptyset$$

for $[X_1, Y_1] \neq [X_2, Y_2]$.

Finally, let p': $C \to Z_1 \ldots Z_n$ be a production in P', then by assumption there are productions

p_1: $A \to X_1 \ldots X_n$ in P_1

and

p_2: $B \to Y_1 \ldots Y_n$ in P_2

such that p' is in $\mu_1(p_1) \cap \mu_2(p_2)$. Thus, C is in $\mu_1(A) \cap \mu_2(B)$ and Z_i is in $\mu_1(X_i) \cap \mu_2(Y_i)$, $1 \le i \le n$. Hence
p: $[A,B] \to [X_1,Y_1] \ldots [X_n,Y_n]$ is a production in P and p' is in $\mu(p)$ since $\mu([A,B]) = \mu_1(A) \cap \mu_2(B)$ and $\mu([X_i,Y_i]) = \mu_1(X_i) \cap \mu_2(Y_i)$, $1 \le i \le n$. Therefore $G' \trianglelefteq_s G(\mu)$. □

We are now in a position to prove that $\mathcal{E}_s$ and $\mathcal{E}_s(G)$ are distributive lattices.

Theorem 3.13

$\mathcal{E}_s$ is a distributive lattice under $\le_s$.

For all grammar forms G, $\mathcal{E}_s(G)$ is a distributive lattice under $\le_s$.

Proof: We will only prove the first statement, the second follows by similar arguments.

We must first show that $\mathcal{E}_s$ is a lattice. Since $\mathcal{E}_s$ is partially ordered under $\le_s$ it suffices to show the existence of a glb and lub for every pair of equivalence classes in $\mathcal{E}_s$. Let E_1 and E_2 be two equivalence classes. Then $E_i = [G_i]_s$, for some grammar form G_i, i = 1, 2. Now by Lemma 3.12 there exists a grammar form G such that $\mathcal{G}_s(G) = \mathcal{G}_s(G_1) \cap \mathcal{G}_s(G_2)$. Hence [G] = E, say, is a lower bound of E_1 and E_2. Consider G' such that [G'] = E' is also a lower bound of E_1 and E_2. We show that $E' \le_s E$.

Now since $E' \le_s E_i$, i = 1, 2 we have $G' \trianglelefteq_s G_i$, i = 1, 2. Immediately $\mathcal{G}_s(G') \subseteq \mathcal{G}_s(G_1) \cap \mathcal{G}_s(G_2)$, therefore $\mathcal{G}_s(G') \subseteq \mathcal{G}_s(G)$ and $[G']_s \le_s [G]_s$. Hence $[G]_s$ is the glb of $[G_1]$ and $[G_2]$.

Consider the least upper bound. Let $G_1 = (V_1,\Sigma_1,P_1,S_1)$ and $G_2 = (V_2,\Sigma_2,P_2,S_1)$ where $V_1 \cap V_2 = \{S_1\}$. We may assume G_1 and G_2 fulfill these conditions without any loss of generality. Define $G = (V,\Sigma,P,S)$ by:

$V = V_1 \cup V_2$, $\Sigma = \Sigma_1 \cup \Sigma_2$, $P = P_1 \cup P_2$ and $S = S_1$.

Clearly $G_i \trianglelefteq_s G$, i= 1, 2. Therefore $[G_i]_s \le_s [G]_s$, i = 1, 2, and $[G]_s$ is an upper bound. Consider a grammar form $G' = (V',\Sigma',P',S')$ such that $[G']_s$ is an upper bound of $[G_1]_s$ and $[G_2]_s$. We show $[G]_s \le_s [G']_s$. Now there exist substitutions μ_1 and μ_2 such that $G_i \trianglelefteq_s G'(\mu_i)$, i = 1, 2. Define a new substitution μ by: for all X' in V', $\mu(X') = \mu_1(X') \cup \mu_2(X')$. We have S_1 in $\mu(S')$ and for all X', Y' in V', $X' \ne Y'$ implies

$$\mu(X') \cap \mu(Y') = (\mu_1(X') \cup \mu_2(X')) \cap (\mu_1(Y') \cup \mu_2(Y'))$$
$$= (\mu_1(X') \cap \mu_1(Y')) \cup (\mu_1(X') \cap \mu_2(Y')) \cup (\mu_2(X') \cap \mu_1(Y')) \cup (\mu_2(X') \cap \mu_2(Y'))$$
$$= (\mu_1(X') \cap \mu_2(Y')) \cup (\mu_2(X') \cap \mu_1(Y')),$$
since $\mu_i(X') \cap \mu_i(Y') = \emptyset$ when $X' \neq Y'$
$$= \emptyset \text{ since } V_1 \cap V_2 = \{S_1\} \text{ and } X' \neq Y'.$$

Finally $P = P_1 \cup P_2 \subseteq \mu_1(P') \cup \mu_2(P') \subseteq \mu(P')$, thus $G \trianglelefteq_s G'(\mu)$ and therefore $[G]_s \leq_s [G']_s$ and $[G]_s$ is the lub of $[G_1]_s$ and $[G_2]_s$. Consequently, $\mathcal{E}_s$ is lattice under $\leq_s$. It remains to show that it is distributive.

Let $G_i = (V_i,\Sigma_i,P_i,S_1)$, $i = 1, 2, 3$ be three grammar forms with a common sentence symbol and apart from S_1, pairwise disjoint alphabets. There is again no loss of generality in this assumption since we are interested in $[G_i]_s$. Define $G = (V,\Sigma,P,S)$ by:

$V = (V_1-\Sigma_1) \times ((V_2-\Sigma_2) \cup (V_3-\Sigma_3)) \cup (\Sigma_1 \times (\Sigma_2 \cup \Sigma_3))$

$\Sigma = \Sigma_1 \times (\Sigma_2 \cup \Sigma_3)$,

$S = [S_1,S_1]$, and P is defined by:

for each production $A \to X_1 \ldots X_n$ of P_1

and each production $B \to Y_1 \ldots Y_n$ of $P_2 \cup P_3$ with $[X_i,Y_i]$ in V

take the production $[A,B] \to [X_1,Y_1] \ldots [X_n,Y_n]$ into P.

It is readily proved that $[G]_s = [G_1]_s \wedge ([G_2]_s \vee [G_3]_s)$ and $[G]_s = ([G_1]_s \wedge [G_2]_s) \vee ([G_1]_s \wedge [G_3]_s)$. Therefore $\mathcal{E}_s$ is distributive. □

Corollary 3.14

For arbitrary grammar forms G_1 and G_2: $[G_1]_s \wedge [G_2]_s = [G]_s$, where G is such that $\mathcal{G}_s(G) = \mathcal{G}_s(G_1) \cap \mathcal{G}_s(G_2)$.

However the corresponding implication for $\vee$ is not true in general. More precisely if G_1, G_2 and G are such that $[G_1]_s \vee [G_2]_s = [G]_s$ then the strongest implication is that $\mathcal{G}_s(G_1) \cup \mathcal{G}_s(G_2) \subseteq \mathcal{G}_s(G)$. In fact equality is only obtained when either $G_1 \trianglelefteq_s G_2$ or $G_2 \trianglelefteq_s G_1$. Consider the simple example G_1: $S \to a$, G_2: $S \to aa$ and G: $S \to a$; $S \to aa$, then $[G]_s = [G_1]_s \vee [G_2]_s$ but G is not an interpretation of either G_1 or G_2, hence $\mathcal{G}_s(G) \neq \mathcal{G}_s(G_1) \cup \mathcal{G}_s(G_2)$. Observe that when $G_1 \trianglelefteq_s G_2$ we obtain $[G_1]_s \vee [G_2]_s = [G_2]_s$, as expected.

We are now in a position to study minimality, which we do in the following section.

II.3.4 Minimal Grammar Forms

With the notion of strong form equivalence in mind we can ask for a given grammar form G whether G is the "smallest" grammar form which gives rise to $\mathcal{G}_x(G)$, for some x-interpretation. In this section we demonstrate that there is an essentially unique smallest grammar form for each grammar family, modulo strong x-form equivalence. This question may, of course be asked for form equivalence as well. Is there a "smallest" grammar form generating each grammatical family? For example, $S \to a$ generating the finite sets and $S \to \lambda|aS$ generating the regular sets, are surely minimal. However apart from these simple cases little seems to be known.

Minimality modulo strong form equivalence is therefore of interest and, more so, since the construction of the minimal form of a given form is effective.

Definition

A grammar form G is (production) minimal if there is no grammar form F such that $F \equiv_s G$ and F has fewer productions than G. Similarly G is symbol minimal if there is no grammar form F such that $F \equiv_s G$ and F has fewer symbols than G (by fewer we mean $\#V_F < \#V_G$). We say G is symbol tight if each symbol apart from the sentence symbol appears in a production of G.

We first have:

Theorem 3.15

For every grammar form $G = (V,\Sigma,P,S)$ there exists a production minimal grammar form $G' = (V,\Sigma,P',S)$ strongly s-form equivalent to G (strongly g-form equivalent to G) such that $P' \subseteq P$.

Proof: Clearly there exists at least one minimal grammar form $G'' = (V'',\Sigma'',P'',S'')$ with $G'' \equiv_s G$. Now G" is an s-interpretation of G modulo some μ, therefore for each production p" in P" let p be a particular production in P such that p" is in $\mu(p)$ and let P' be the set of all such productions p. Consider the grammar form G'. G' is a subgrammar of G, hence $G' \trianglelefteq_s G$. However by construction $G'' \trianglelefteq_s G'$ and by assumption $G \trianglelefteq_s G''$, thus $G' \equiv_s G''$, and because G" is minimal and G' contains no more productions then G", G' is minimal. The case of strong g-form equivalence is proved similarly. □

We now obtain the main theorem for s-interpretations.

Theorem 3.16

Let $G = (V,\Sigma,P,S)$ be a grammar form. For every two grammar forms F_1 and F_2 which are s-interpretations of G and are production minimal, symbol tight and strongly s-form equivalent to G, F_1 and F_2 are isomorphic.

Proof: Let $F_i = (V_i,\Sigma_i,P_i,S_i)$, i = 1, 2 be two arbitrary production minimal, symbol tight, strongly s-form equivalent grammar forms of G. Clearly $F_1 \equiv_s F_2$. Therefore there exists μ_2 such that $F_1 \triangleleft_s F_2(\mu_2)$. Clearly $P_1 \subseteq \mu_2(P_2)$. Since F_1 is symbol tight every symbol X_1 in V_1 apart from S_1 appears in some production in P_1, that is, X_1 is the image of some X_2 in V_2 under μ_2, for all X_1 in V_1. Now observe that $\mu_2(X) \neq \emptyset$, for all X in V_2. Otherwise all productions involving X could be removed from P_2 yielding a grammar form strongly s-form equivalent to F_1 but having fewer productions, a contradiction.

Since $\mu_2(X) \cap \mu_2(Y) = \emptyset$ for all $X \neq Y$ in V_2 we have $\#V_2 \leq \#V_1$. Similarly there exists μ_1, such that $F_2 \triangleleft_s F_1(\mu_2)$. By similar arguments $\#V_1 \leq \#V_2$. Hence $\#V_1 = \#V_2$. Therefore μ_2 is an isomorphism and $\mu_1 = \mu_2^{-1}$. □

We now extend this result to g-interpretations. In this case, essentially unique is pseudo-isomorphism, rather than isomorphism.

Definition

Let $G_i = (V_i,\Sigma_i,P_i,S_i)$ i = 1, 2 be two grammar forms. Then G_1 and G_2 are <u>pseudo-isomorphic with respect to μ_1 and μ_2</u>, where $G_1 \triangleleft_g G_2(\mu_2)$ and $G_2 \triangleleft_g G_1(\mu_1)$, if:

(i) $\mu_1: V_1 - \Sigma_1 \to V_2 - \Sigma_2$, $\mu_2: V_2 - \Sigma_2 \to V_1 - \Sigma_1$ are surjections and $\mu_2 = \mu_1^{-1}$ on this restricted domain, and

(ii) $\bar{\mu}_1: P_1 \to P_2$ defined by $\bar{\mu}_1(p) = \mu_1(p) \cap P_2$ for all p in P_1, and $\bar{\mu}_2$ defined analogously are surjections. Further $\bar{\mu}_2 = \bar{\mu}_1^{-1}$.

We say G_1 and G_2 are <u>pseudo-isomorphic</u> if there are substitutions μ_1 and μ_2 such that G_1 and G_2 are pseudo-isomorphic with respect to μ_1 and μ_2.

Let $G_i = (V_i,\Sigma_i,P_i,S_i)$, i = 1, 2 be two grammar forms with $G_2 \triangleleft_g G_1(\mu_1)$ and $\bar{\mu}_1$ defined as above. We say $\bar{\mu}_1$ is <u>terminal word preserving</u> if:

for each production p: $A \to x_0A_1 \ldots A_nx_n$ in P_1 (n=0 is interpreted as $A \to x_0$) $\bar{\mu}_1(p)$ has the form $B \to y_0B_1 \ldots B_ny_n$ where $x_i = \lambda$ iff $y_i = \lambda$, $0 \le i \le n$.

Condition (i) in the definition means that the nonterminals of G_1 and G_2 are isomprphic. Condition (ii) on the other hand implies that P_1 and P_2 are almost isomorphic. When condition (i) holds for both terminals and nonterminals of G_1 and G_2, condition (ii) is clearly satisfied as in Theorem 3.16 for s-interpretations. Let us consider two examples of pseudo-isomorphism.

Example 3.1

Let G_1 and G_2 be defined by the productions:

G_1: $A \to abAaa$; $A \to a$

G_2: $S \to aSb$; $S \to aba$

Clearly $G_1 \triangleleft_g G_2(\mu_2)$ and $G_2 \triangleleft_g G_1(\mu_1)$ where:

$\mu_1(A) = S$; $\mu_1(a) = \{\lambda, a, b, aba\}$ and $\mu_1(b) = \{\lambda\}$;

$\mu_2(S) = A$; $\mu_2(a) = \{\lambda, ab\}$ and $\mu_2'(b) = \{a, aa\}$.

Hence condition (i) is satisfied. Further

$\bar{\mu}_1(A \to abAaa) = S \to aSb$; $\bar{\mu}_1(A \to a) = S \to aba$; and

$\bar{\mu}_2(S \to aSb) = A \to abAaa$; $\bar{\mu}_2(S \to aba) = A \to a$

therefore $\bar{\mu}_1$ and $\bar{\mu}_2$ fulfill condition (ii) and G_1 and G_2 are pseudo-isomorphic. Note that they are not isomorphic.

Example 3.2

Let G_1 and G_2 be defined by the productions:

G_1: $S \to aS$; $S \to bS$; and

G_2: $S \to abS$; $S \to baS$

Then letting $\mu_1(a) = \{ab\}$, $\mu_1(b) = \{ba\}$ and $\mu_1(S) = S$, $G_2 \triangleleft_g G_1(\mu_1)$ and letting $\mu_2(a) = \{a,b\}$, $\mu_2(b) = \{\lambda\}$ and $\mu_2(S) = S$ we have $G_1 \triangleleft_g G_2(\mu_2)$. However $\bar{\mu}_1(S \to aS) = S \to abS$; $\bar{\mu}_1(S \to bS) = S \to baS$; but $\bar{\mu}_2(S \to abS) = \{S \to aS, S \to bS\}$ and $\bar{\mu}_2(S \to baS) = \{S \to aS; S \to bS\}$ hence $\bar{\mu}_2$ is not 1:1 onto and condition (ii) does not hold. Moreover there is no μ_2 such that $\bar{\mu}_2$ fulfills condition (ii) since $\mu_2(ab) = \mu_2(a)\mu_2(b) = \mu_2(b)\mu_2(a) = \mu_2(ba)$ in this situation.

Before turning to the main theorem of this section, we need the following straightforward result.

Lemma 3.17

Let $G_i = (V_i, \Sigma_i, P_i, S_i)$, 1 = 1, 2 be two pseudo-isomorphic grammar forms with respect to μ_1 and μ_2, then $\bar{\mu}_1$ and $\bar{\mu}_2$ are terminal word preserving.

Proof: Since G_1 and G_2 are pseudo-isomorphic with respect to μ_1 and μ_2, then $\bar{\mu}_1^{-1} = \bar{\mu}_2$. If $\bar{\mu}_1$ is not terminal word preserving, then there is a production:

$$p: A_0 \rightarrow x_0 A_1 \ldots A_n x_n \text{ in } P_1$$

and its corresponding production

$$q: B_0 \rightarrow y_0 B_1 \ldots B_n y_n \text{ in } P_2$$

such that $\bar{\mu}_1(p) = q$ and there is an $x_i \neq \lambda$, $y_i = \lambda$ for some i, $0 \leq i \leq n$. In which case $\bar{\mu}_1^{-1}(q) = \bar{\mu}_2(q) \neq p$, since for any substitution μ, $\mu(\lambda) = \{\lambda\}$. This is a contradiction, therefore $\bar{\mu}_1$ is terminal word preserving. Similarly $\bar{\mu}_2$ is terminal word preserving. □

We now have:

Theorem 3.18

Let G_1 and G_2 be two symbol tight, minimal, strong g-form equivalent grammar forms with $G_2 \triangleleft_g G_1(\mu_1)$ and $G_1 \triangleleft_g G_2(\mu_2)$. Then G_1 and G_2 are pseudo-isomorphic with respect to μ_1 and μ_2.

Proof: That condition (i) of pseudo-isomorphism holds follows in a similar manner to the proof of Theorem 3.16, except only nonterminals are considered. It remains to show condition (ii) holds. Let $G_i = (V_i, \Sigma_i, P_i, S_i)$, i = 1, 2. Define $\bar{\mu}_1: P_1 \rightarrow 2^{P_2}$ and $\bar{\mu}_2: P_2 \rightarrow 2^{P_1}$ by: $\bar{\mu}_1(p) = \mu_1(p) \cap P_2$ for all p in P_1, and $\bar{\mu}_2(p) = \mu_2(p) \cap P_1$ for all p in P_2. Now $\bigcup_{p \text{ in } P} \bar{\mu}_1(p) = P_2$ and $\bigcup_{p \text{ in } P_2} \bar{\mu}_2(p) = P_1$, since $G_2 \triangleleft_g G_1(\mu_1)$ and $G_1 \triangleleft_g G_2(\mu_2)$. Suppose $\bar{\mu}_1(p) = \emptyset$ for some p in P_1. Consider $G_p = (V_1, \Sigma_1, P_1 - \{p\}, S_1)$. Now $G_p \triangleleft_g G_1$, $G_2 \triangleleft_g G_p(\mu_1)$ and $G_1 \triangleleft_g G_2(\mu_2)$ hence G_p is strongly g-form equivalent to G_1 and has fewer productions than G_1, a contradiction. Hence $\bar{\mu}_1(p) \neq \emptyset$ for each p in P_1. It now suffices to show that $\#\bar{\mu}_1(p) = 1$ for each p in P_1.

Number the productions of P_1 as $p = p_1, \ldots, p_n$. Define subsets $M_1, M_2, \ldots$ of P_2 recursively by:

$$M_1 = \bar{\mu}(p_1) \text{ and}$$

$$M_{i+1} = M_i \cup \bar{\mu}_1(p_{i+1}), \text{ for } i \geq 1.$$

Since $\#P_1 = \#P_2$, $P_2 = \bigcup_{j=1}^{n} \bar{\mu}_1(p_j)$, $M_j \subseteq M_{j+1}$ for each j < n, and $\#M_1 \geq 2$ it follows that $M_{i+1} = M_i$ for some i < n. Hence $G_{p_{i+1}} = (V_1, \Sigma_1, P_1 - \{p_{i+1}\}, S_1)$ is also production minimal; a contradiction.

Therefore $\bar{\mu}_1$ is surjective. Analogously $\bar{\mu}_2$ is surjective. It remains to show that $\bar{\mu}_2 = \bar{\mu}_1^{-1}$.

Consider $\bar{\mu}_1\bar{\mu}_2$ defined by $\bar{\mu}_1\bar{\mu}_2(p) = \bar{\mu}_1(\bar{\mu}_2(p))$ for all p in P_2. It suffices to prove $\bar{\mu}_1\bar{\mu}_2$ is the identity map on P_2. Assume otherwise, that is there exists p in P_2 such that $\bar{\mu}_1\bar{\mu}_2(p) = q$ and $p \neq q$. Now for nonterminals μ_1 and μ_2 are surjective and $\mu_2 = \mu_1^{-1}$, therefore letting p be $A_0 \to x_1A_1 \ldots A_nx_n$, $n \geq 0$ then q is $A_0 \to y_0A_1 \ldots A_ny_n$ for some y_i, $0 \leq i \leq n$. Moreover q is in $\mu_1(\mu_2(p))$ therefore $y_i \neq \lambda$ implies $x_i \neq \lambda$, $0 \leq i \leq n$. Consider $G_q = (V_2, \Sigma_2, P_2 - \{q\}, S_2)$. As above G_q can be shown to be production minimal and strongly g-form equivalent to G_2, giving a contradiction. □

II.4 Normal Form, Closure and Characterization Results

Normal forms have always been a central topic in the study of grammars and their languages. Grammar forms not only enable the notion of normal form to be made rigorous but also allow its generalization in a natural way. Let G be a grammar form for which $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(CF)$, then any grammar F is said to be in G-normal form if $F \trianglelefteq_s G$. We consider a restricted version of this notion in which G is a two-symbol form. We are able to prove a super-normal form theorem for context-free grammars and grammar forms, which includes Chomsky Normal Form and Greibach two-standard form as special cases. This we do in Section 4.1, where we also derive other reduction results.

In Section 4.2 we demonstrate that $\mathcal{L}_g(G,\Rightarrow)$ is a full semi-AFL for every infinite G and demonstrate that $\mathcal{L}_s(G,\Rightarrow)$ is in the worst case only closed under intersection with regular sets and dfl-substitutions. Some further specialized closure properties are to be found in Section 6.1.

Finally in Section 4.3 three results are proved: for every infinite G, $\mathcal{L}_g(G,\Rightarrow)$ is a full principal semi-AFL, every g-grammatical family apart from the finite sets is s-grammatical and we characterize those grammar forms G for which $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(REG)$ or $\mathcal{L}(LIN)$.

II.4.1 Reduction Results

In order to prove that each g-grammatical family is a full principal semi-AFL, it is convenient to first derive some specific normal form theorems for grammar forms under both g- and s-interpretations. Most of these, which are of interest in their own right, are stated only for s-interpretations since if $\mathcal{L}_s(G_1,\Rightarrow) = \mathcal{L}_s(G_2,\Rightarrow)$ then by Corollary 1.2, $\mathcal{L}_g(G_1,\Rightarrow) = \mathcal{L}_g(G_2,\Rightarrow)$ that is s-form equivalence implies g-form equivalence. Our first result is stated without proof.

Proposition 4.1

For every grammar form G there exists an s-form equivalent grammar form G_1 which is reduced.

Simply discard from G the useless terminals, nonterminals and productions to give the reduced grammar form G_1.

Recall that a single production is a production of type $A \to B$. We say a grammar is single-free if it has no single productions.

Theorem 4.2

For every grammar form $G = (V,\Sigma,P,S)$ there exists a single-free s-form equivalent grammar form $G_1 = (V,\Sigma,P_1,S)$.

Proof: We carry out the standard construction to remove single productions. Without loss of generality assume G is reduced. For each A in $V-\Sigma$, let $s(A) = \{B: A \Rightarrow^+ B$ by way of single productions$\}$. Clearly G is single-free if for all A in $V-\Sigma$, $s(A) = \emptyset$. In this case G_1 is identified with G. Otherwise construct $G_2 = (V,\Sigma,P_2,S)$ where $P_2 = P \cup \{A \to \beta: B$ is in $s(A)$ and $B \to \beta$ is in $P\}$. By Corollary 2.10, $\mathcal{L}_s(G_2,\Rightarrow) = \mathcal{L}_s(G,\Rightarrow)$. Now letting $P_1 = P_2 - (V \times V)$, $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G_2,\Rightarrow)$. It remains to show that $\mathcal{L}_s(G_2,\Rightarrow) \subseteq \mathcal{L}_s(G_1,\Rightarrow)$.

Consider $G_2' = (V',\Sigma',P_2',S') \triangleleft_s G_2(\mu)$. It suffices to exhibit an s-interpretation G_1' of G_1 with $L(G_1',\Rightarrow) = L(G_2',\Rightarrow)$. We can assume that for all A', B' in $V'-\Sigma'$ with B' in s(A') that $A' \to \beta'$ is in P_2' for all β' such that $B' \to \beta'$ is in P_2'. This follows by the definition of G_2 and by Corollary 2.10 for grammars rather than grammar forms. Construct $G_1' = (V',\Sigma',P_1',S')$ by taking $P_1' = P_2' - (V' \times V')$. That $L(G_1',\Rightarrow) = L(G_2',\Rightarrow)$ is a standard result. □

Recall that a grammar (form) is <u>λ-free</u> if no production in the grammar has the empty word on its right hand side.

Theorem 4.3

Let $G = (V,\Sigma,P,S)$ be a nontrivial grammar form. Then there exists a grammar form H such that H is λ-free and H is s-form equivalent to G.

Proof: Since G is nontrivial $L(G,\Rightarrow)$ contains at least one non-empty word. This enables the standard construction to be carried out.

First define a substitution τ on V^* by:

$\tau(a) = a$, for all a in Σ,

$\tau(A) =$ either $\{A,\lambda\}$ if $A \Rightarrow^+ \lambda$ in G
or $\{A\}$ otherwise.

Consider $F = (V,\Sigma,P_F,S)$ defined by:

$P_F = \{A \to \beta: A \to \alpha$ is in P and β is in $\tau(\alpha)\}$.

By Corollary 2.10 $\mathcal{L}_s(F,\Rightarrow) = \mathcal{L}_s(G,\Rightarrow)$.

Second, consider $H = (V,\Sigma,P_H,S)$ defined by:

$P_H = P_F - (V \times \{\lambda\})$.

Since H is a subgrammar of F it is clear that $\mathcal{L}_s(H,\Rightarrow) \subseteq \mathcal{L}_s(F,\Rightarrow)$.

We now show that $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(H,\Rightarrow)$ completing the proof. Consider an arbitrary interpretation $G' \lhd_s G(\mu)$, where $G' = (V',\Sigma',P',S')$. It is sufficient to construct an interpretation $H' \lhd_s H(\mu)$ with $L(H',\Rightarrow) = L(G',\Rightarrow)$.

However the construction detailed above to obtain H from G is the standard λ-removal construction. Since, under s-interpretation, $A' \Rightarrow^+ \lambda$ in G' only if $\mu^{-1}(A') \Rightarrow^+ \lambda$ in G, then we can define τ', giving $F' \lhd_s F$ with $L(F',\Rightarrow) = L(G',\Rightarrow)$. Now $H' \lhd_s H$, since $P_{H'} = P_{F'} - (V' \times \{\lambda\})$ and $L(H',\Rightarrow) = L(G',\Rightarrow)$ by standard techniques. □

We say that $G = (V,\Sigma,P,S)$ is <u>c-reduced</u> if it is:

(i) reduced,

(ii) λ-free,

(iii) single-free, and

(iv) for all A in $V-\Sigma$, $A \neq S$, there exists a production $A \to xAy$ in P such that $xy \neq \lambda$.

<u>Lemma 4.4</u>

Each nontrivial grammar form $G = (V,\Sigma,P,S)$ has an s-form equivalent c-reduced grammar form H.

<u>Proof</u>: We may assume G is reduced, λ-free and single-free. A in $V-\Sigma$ is said to be <u>partially self-embedding</u> in G, if either $A = S$ or there exists x and y, $xy \neq \lambda$ such that $A \Rightarrow^+ xAy$.

Assume each nonterminal $A \neq S$ is partially self-embedding, then there is a derivation $A \Rightarrow^+ xAy$ with $xy \neq \lambda$ and by Corollary 2.10 we can assume $A \to xAy$ is in P without any loss of generality. In this case take H to be equal to G.

Otherwise suppose that there are $k > 0$ nonterminals which are non-partially self-embedding. Carry out an iterative construction based on the number of non-partially self-embedding nonterminals.

Consider $G_1 = (V_1,\Sigma,P_1,S)$ defined as follows:
$V_1 = V - \{A\}$, where A is a non-partially self-embedding nonterminal, and

$P_1 = \{B \to \bar{\beta}$: $B \to \beta$ is in P, $B \neq A$, and $\bar{\beta}$ is obtained from β by replacing each occurrence of A in β by some word in $r(A)\}$,

where $r(A) = \{\alpha: A \to \alpha$ is in P$\}$.
Clearly G simulates G_1, therefore $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$. That $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(G_1,\Rightarrow)$ follows from the back-substitution lemma, Lemma 2.12. Note that G_1 is reduced if G is reduced, G_1 is λ-free and G_1 is single-free.

Further, G_1 has k-1 non-partially self-embedding nonterminals. Therefore, iterate the construction k times to obtain G_k all of whose nonterminals are partially self-embedding. Let H be G_k to complete the lemma. □

A grammar $G = (V,\Sigma,P,S)$ is sequential if its nonterminals can be numbered $S = A_1,\ldots,A_n$, so that $A_i \to \alpha A_j \beta$ belongs to P implies $i \leq j$.

We say G is s-reduced if it both c-reduced and sequential.

It is well known that not all context-free languages are sequential (that is, can be generated by a sequential grammar). The language generated by the grammar G, defined by the productions:

$S \to aAa$; $S \to \lambda$

$A \to aAa$; $A \to bBb$

$B \to aBa$; $B \to bSb$

is an example of such a non-sequential language.

Under the two interpretation mechanisms we are discussing we consider whether each g- or s-grammatical family is sequential, that is, generated by a sequential grammar form. Our first result, for s-interpretations is negative, while our second, for g-interpretations is positive.

Theorem 4.5

Let G be defined by the productions:

$S \to baAab$; $S \to bacadedacab$

$A \to aAa$; $A \to caBac$

$A \to aAa$; $A \to caBac$

$B \to aBa$; $B \to dSd$

Then $\mathcal{L}_s(G,\Rightarrow)$ is not a sequential s-grammatical family.

Proof: First note that $L(G,\Rightarrow) = \{x e\,mi(x)$: x is in $(ba^+ca^+d)^*bacad\}$. It is straightforward using a result due to Shamir to demonstrate that $L(G,\Rightarrow)$ is not sequential. Second, observe that every word in every language in $\mathcal{L}_s(G,\Rightarrow)$ has at least 5 distinct symbols. This follows from the fact that the only terminating production $S \to bacadedacab$, contains each symbol of $L(G,\Rightarrow)$ at least once. Third, observe that if L is in $\mathcal{L}_s(G,\Rightarrow)$ and $L \subseteq \{a,b,c,d,e\}^*$ then there exists a permutation π of the symbols a, b, c, d, e such that

$L \subseteq \pi(L(G,\Rightarrow))$

where $\pi(L(G,\Rightarrow))$ denotes the language obtained from $L(G,\Rightarrow)$ by applying the permutation π to all its words. The validity of this observation follows by examining the productions of G. Since $L = L(G',\Rightarrow)$ for some $G' \underset{s}{\lhd} G$, each of the symbols a, b, c, d, e in L must be an interpretation of exactly one of the symbols a, b, c, d, e in the alphabet of G. The inclusion, $L \subseteq \pi(L(G,\Rightarrow))$ now follows because derivations in G' must follow the same pattern as those according to G, that is an interpretation of S yields that of A, which, in turn, yields that of B, after which we return to an interpretation of S.

We now establish the Theorem by contradiction. Assume there is a grammar form F, which is sequential and s-form equivalent to G. There is an $F' \underset{s}{\lhd} F(\mu)$ such that F' is reduced and $L(F',\Rightarrow) = L(G,\Rightarrow)$. Let F_1 be the smallest subgrammar of F such that $F' \underset{s}{\lhd} F_1(\mu)$, that is every production of F_1 is used in defining some production in F'. Immediately it follows that the terminal alphabet of F_1 consists of at most 5 symbols. Since each L in $\mathcal{L}_s(G,\Rightarrow)$ has at least 5 symbols, F_1 has exactly 5 terminal symbols. Rename the terminal alphabet of F_1 without changing its generative capacity, in such a way that μ becomes the identity on $\{a,b,c,d,e\}$ and is everywhere else unchanged. Hence $L(G,\Rightarrow) = L(F',\Rightarrow) \subseteq L(F_1,\Rightarrow)$. Since $L \subseteq \pi(L(G,\Rightarrow))$ for all $L \subseteq \{a,b,c,d,e\}^*$ and L in $\mathcal{L}_s(G,\Rightarrow)$, we have $L(F_1,\Rightarrow) = L(F',\Rightarrow)$. Finally, since F is sequential so is F_1, thus $L(G,\Rightarrow)$ is sequential, a contradiction. □

However under g-interpretation we can obtain a sequential g-form equivalent grammar form when given an arbitrary grammar form.

<u>*Theorem 4.6*</u>

Let $G = (V,\Sigma,P,S)$ be a grammar form. There exists a g-form equivalent grammar form H which is s-reduced.

<u>*Proof:*</u> By previous results we can assume G is c-reduced. Define a relation $\sim$ on $V-\Sigma$ by:

$A \sim B$ if $A \Rightarrow^+ \alpha_1 B \beta_1$ and $B \Rightarrow^+ \alpha_2 A \beta_2$,
for some α_i, β_i, $i = 1, 2$.

Clearly $\sim$ is an equivalence relation of finite index. Let [A] denote the equivalence class of A and $X_{[A]}$ be a distinguished element of [A] with $X_{[S]} = S$. Let $\bar{V} = \{X_{[A]}: A \text{ is in } V-\Sigma\}$. Define a grammar form $H = (V_H,\Sigma,P_H,S)$ as follows: $V_H = \bar{V} \cup \Sigma$ and P_H is obtained by replacing every nonterminal in every production of P by its corresponding distinguished element.

Immediately $G \underset{g}{\lhd} H$, and in fact $G \underset{s}{\lhd} H$. Hence $\mathcal{L}_g(G,\Rightarrow) \subseteq \mathcal{L}_g(H,\Rightarrow)$ and $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(H,\Rightarrow)$.

Consider the converse inclusion for g-interpretations. Define a new grammar form $F = (V,\Sigma,P \cup \{A \to B\colon A \neq B \text{ and } A \sim B\},S)$. F simulates H, therefore $\mathcal{L}_g(H,\Rightarrow) \subseteq \mathcal{L}_g(F,\Rightarrow)$. We need to show that $\mathcal{L}_g(F,\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$ to complete the proof.

Since for each A, B in $V-\Sigma$ with $A \neq B$ and $A \sim B$ there are derivations $A \Rightarrow^+ \alpha_1 B \beta_1$ and $B \Rightarrow^+ \alpha_2 A \beta_2$ in G, for some α_i, β_i, i = 1, 2, there are derivations $A \Rightarrow^+ u_1 B v_1$ and $B \Rightarrow^+ u_2 A v_2$ in G, for some u_i, v_i in Σ^*, i = 1, 2. Hence we can assume $A \to u_1 B v_1$ and $B \to u_2 A v_2$ are in P by Corollary 2.10. Hence $F \triangleleft_g G$ and $\mathcal{L}_g(F,\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$. □

We now prepare the way for demonstrating that left recursion may be removed from a grammar form without disturbing its grammatical family. This result will then be used later to derive a super-normal form result.

Lemma 4.7

Let $G = (V,\Sigma,P,S)$ be a grammar form and $A \to \alpha_1 B \alpha_2$ a production in P, where B is a nonterminal in $V-\Sigma$. Then $G_1 = (V,\Sigma,P_1,S)$, where $P_1 = (P - \{A \to \alpha_1 B \alpha_2\}) \cup \{A \to \alpha_1 \beta \alpha_2\colon B \to \beta \text{ is in } P\}$, is s-form equivalent to G.

Proof: Observe that G simulates G_1, hence we have immediately that $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$. Conversely, given an arbitrary interpretation $G' \triangleleft_s G$ it is straightforward to construct $G_1' \triangleleft_s G_1$ such that $L(G',\Rightarrow) = L(G_1',\Rightarrow)$ by the corresponding standard result for grammars. Hence the lemma follows. □

We also need:

Lemma 4.8

Let $G = (V,\Sigma,P,S)$ be a c-reduced grammar form and A a nonterminal in V. Let the A-productions in P be denoted by:

$$A \to A\alpha_1 | \dots | A\alpha_r | \beta_1 | \dots | \beta_s,$$

where the β_i do not begin with an A. Construct $G_1 = (V_1,\Sigma,P_1,S)$ where $V_1 = V \cup \{X\}$, X a new nonterminal and P_1 is the same as P except that the A-productions are replaced by the productions:

$$A \to \beta_1 | \dots | \beta_s | \beta_1 X | \dots | \beta_s X$$
$$X \to \alpha_1 | \dots | \alpha_r | \alpha_1 X | \dots | \alpha_r X.$$

Then $\mathcal{L}_s(G_1,\Rightarrow) = \mathcal{L}_s(G,\Rightarrow)$.

Proof: First observe that we cannot use simulation techniques to prove this lemma since we have interchanged left linear productions with right linear ones.

Claim 1: $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(G_1,\Rightarrow)$.

Consider an arbitrary interpretation $G' \trianglelefteq_s G(\mu)$, where $G' = (V',\Sigma',P',S')$ and $\mu(A) = \{A_1,\ldots,A_t\}$ say. We may assume that G' is reduced since G is reduced. Construct a grammar $G_1' = (V_1',\Sigma',P',S')$ where $V_1' = V' \cup \{A_i^{(j)}: 1 \le i, j \le t\}$ and the $A_i^{(j)}$ are new nonterminals. Define P_1' to be the same as P' except for the images of A-productions, which are replaced by the following productions:

(i) if $A_i \to \beta'$ is in P' then take $A_i \to \beta'$ into P_1',

(ii) if $A_i \to A_j\alpha'$ is in P' then take $A_j^{(k)} \to \alpha' A_i^{(k)}$ into P_1', for all k, $1 \le k \le t$

(iii) if $A_i \to A_j\alpha'$ is in P' then take $A_j^{(i)} \to \alpha'$ into P_1',

(iv) if $A_j \to \beta'$ is in P' and $A_i \overset{L}{\Rightarrow}{}^+ A_j$, then take $A_i \to \beta' A_j^{(i)}$ into P_1'.

This is essentially the standard left linear to right linear grammar construction, modified to take into account the fact that any one of the A_i may be the starting symbol. Hence we keep track of which A_i began the derivation in G by use of the superscript i. We claim that $L(G_1',\Rightarrow) = L(G',\Rightarrow)$.

Consider an arbitrary G'-derivation for x in Σ'^*,

$$S \overset{L}{\Rightarrow}{}^* x.$$

Either no A_i, $1 \le i \le t$ occurs in this derivation, in which case

$$S \overset{L}{\Rightarrow}{}^* x \text{ in } G_1'$$

since all productions in P' which are not images of A-productions are taken unchanged into P_1' or some A_i occurs in the derivation. In this latter case consider the first appearance of an A_i, $1 \le i \le t$,

$$S \overset{L}{\Rightarrow}{}^* uA_{i_0}\alpha_{i_0} \overset{L}{\Rightarrow}{}^* x \text{ in } G'.$$

Now this A_{i_0} yields, via images of A-productions of the type $A \to A\alpha_j$, a sequence of A_k symbols, $1 \le k \le t$ terminated by the application of the image of an A-production of the type $A \to \beta_\ell$. We write this as:

$$S \overset{L}{\Rightarrow}{}^* uA_{i_0}\alpha_{i_0} \overset{L}{\Rightarrow} uA_{i_1}\alpha_{i_1}\alpha_{i_0} \overset{L}{\Rightarrow} \cdots \overset{L}{\Rightarrow} uA_{i_{m-1}}\alpha_{i_{m-1}} \cdots \alpha_{i_1}\alpha_{i_0}$$

$$\overset{L}{\Rightarrow} u\beta_{i_m}\alpha_{i_{m-1}} \cdots \alpha_{i_1}\alpha_{i_0} \overset{L}{\Rightarrow}{}^* x$$

in G'. Now by definition of G_1' the derivation

(*) $A_{i_0} \overset{L}{\Rightarrow} A_{i_1}\alpha_{i_1} \overset{L}{\Rightarrow} \cdots \overset{L}{\Rightarrow} A_{i_{m-1}}\alpha_{i_{m-1}} \cdots \alpha_{i_1} \overset{L}{\Rightarrow} \beta_{i_m}\alpha_{i_{m-1}}\cdots\alpha_{i_1}$

in G' can be "simulated" by the derivation

(**) $A_{i_0} \overset{R}{\Rightarrow} \beta_{i_m} A_{i_{m-1}}^{(i_0)} \overset{R}{\Rightarrow} \beta_{i_m}\alpha_{i_{m-1}} A_{i_{m-2}}^{(i_0)} \overset{R}{\Rightarrow} \cdots \overset{R}{\Rightarrow} \beta_{i_m}\alpha_{i_{m-1}}\cdots\alpha_{i_2} A_{i_0}^{(i_0)}$

$\overset{R}{\Rightarrow} \beta_{i_m}\alpha_{i_{m-1}} \cdots \alpha_{i_1}$.

Clearly we can prove by induction on the number of such A_k sequences in a G'-derivation $S \overset{L*}{\Rightarrow} x$ in Σ'^* that $S \Rightarrow^* x$ in G_1'. Hence $L(G',\Rightarrow) \subseteq L(G_1',\Rightarrow)$.

Moreover, for each rightmost derivation $S \overset{R*}{\Rightarrow} x$ in G_1', x in Σ'^*, it can be observed that if any A_k sequence occurs then it must, by definition of G_1', be of type (**). Immediately such a (**)-derivation can be "simulated" in G' by the corresponding (*)-derivation. Crucial to this simulation, in both directions, is the information carried in the superscript position of the $A_i^{(j)}$ symbols in (**).

Hence $L(G_1',\Rightarrow) = L(G',\Rightarrow)$ and Claim 1 is proved.

Claim 2: $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$.

This proof is similar to that of Claim 1, therefore we merely sketch the technique leaving the details to the reader. Consider an arbitrary interpretation $G_1' \triangleleft_s G_1(\mu)$, where $G_1' = (V_1',\Sigma',P_1',S')$, $\mu(A) = \{A_1,\ldots,A_p\}$ and $\mu(X) = \{X_1,\ldots,X_q\}$, say, for some p, q > 0. As in Claim 1 we construct an interpretation $G' \triangleleft_s G(\mu')$ such that $L(G_1',\Rightarrow) = L(G',\Rightarrow)$. This is accomplished by defining μ' to be identical to μ except that for X, $\mu'(X) = \{X_i^{(j)}: 1 \le i \le q,\ 1 \le j \le p\}$ and introducing appropriate productions in P' such that

(+) $A_i \overset{R}{\Rightarrow} \beta_{i_0} X_{i_0} \overset{R}{\Rightarrow} \beta_{i_0}\alpha_{i_0} X_{i_1} \overset{R}{\Rightarrow} \cdots \overset{R}{\Rightarrow} \beta_{i_0} \cdots \alpha_{i_m} X_{i_{m-1}} \overset{R}{\Rightarrow} \beta_{i_o} \cdots \alpha_{i_m}$

if G_1', iff

(++) $A_i \overset{L}{\Rightarrow} X_{i_{m-1}}^{(i)}\alpha_{i_m} \overset{L}{\Rightarrow} \cdots \overset{L}{\Rightarrow} X_{i_0}^{(i)}\alpha_{i_1}\cdots\alpha_{i_m} \overset{L}{\Rightarrow} \beta_{i_0}\alpha_{i_1} \cdots\alpha_{i_m}$

in G'. □

We can now state and prove the left recursion removal theorem, namely:

Theorem 4.9

Let $G = (V,\Sigma,P,S)$ be a c-reduced grammar form. Then there exists a c-reduced grammar form G_1 such that G_1 is non-left recursive and $\mathcal{L}_s(G_1,\Rightarrow) = \mathcal{L}_s(G,\Rightarrow)$.

Proof: Based on Lemmas 4.7 and 4.8 the standard technique for removing left recursion from G can be used. Because of Lemmas 4.7 and 4.8 the resulting grammar form H is s-form equivalent to G and moreover H is non-left recursive, again by standard techniques. Clearly an s-form equivalent grammar form G_1 which is c-reduced and non-left recursive can be obtained from H, completing the proof. □

We now prove a powerful normal form result for grammar forms which is also a normal form result for context-free grammars encompassing Chomsky and Greibach two-standard normal forms and generalizing them both considerably.

A grammar form $G = (V,\Sigma,P,S)$ is said to be a unary form if $\Sigma = \{a\}$ for some terminal a. Note that every grammar form G has a g-form equivalent unary form, simply by identifying all the terminals in G. Similarly a grammar form $G = (V,\Sigma,P,S)$ is said to be a two-symbol form if $\Sigma = \{a\}$ for some terminal a and $V = \{S,a\}$.

Theorem 4.10 -- The Super-Normal Form Theorem

Let $G = (\{S,a\},\{a\},P,S)$ be a two-symbol form such that

(i) $L(G,\Rightarrow) = a^*$, and

(ii) there exists $S \to \alpha$ in P with $|\alpha|_S \geq 2$.

Then $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(CF)$.

Proof: We establish the theorem in three steps, making use of the following result from Salomaa and Soittala (1978).

Proposition: Let $E = (\{S,a\},\{a\},P_E,S)$ be a two-symbol form for which P_E includes the productions $S \to aSa$ and $S \to aSaSa$ and $L(E,\Rightarrow) = a^*$. Then $\mathcal{L}_s(E,\Rightarrow) = \mathcal{L}(CF)$.

Claim 1: Let $i \geq 0$ be an integer, $F = (\{S,a\},\{a\},P_F,S)$ be a two-symbol form, $L(F,\Rightarrow) = a^*$ and P_F contain the productions $S \to a^iSa^i$ and $S \to a^iSa^iSa^i$. Then $\mathcal{L}_s(F,\Rightarrow) = \mathcal{L}(CF)$.

Proof of Claim 1: Without loss of generality we may assume $S \to a^i$ is in P_F, $1 \leq i \leq 2i$. We proceed in three steps.

Step 1: If $L \subseteq \Sigma^*$ is an arbitrary context-free language with the property that x is in L implies $|x| \equiv 0 \pmod{i}$ then we show that L is in $\mathcal{L}_s(F,\Rightarrow)$.

Define a new alphabet $\Delta = \{[a_1,a_2,\ldots,a_i]: a_j \text{ is in } \Sigma, 1 \leq j \leq i\}$ and a homomorphism h from Δ to Σ^* by:

$$h([a_1,a_2,\ldots,a_i]) = a_1 \ldots a_i.$$

Let $L' = h^{-1}(L)$. Now L' is in $\mathcal{L}(CF)$ since $\mathcal{L}(CF)$ is closed under inverse homomorphism. Therefore by the Proposition there is an $E' = (V',\Delta,P_E',S') \trianglelefteq_s E$ such that $L(E',\Rightarrow) = L'$.

Now extend h to all symbols of V' by defining $h(A) = A$, for all A in $V'-\Delta$ and define $F' = (V'',\Sigma,P_F',S')$ by: $V'' = (V'-\Delta) \cup \Sigma$ and $P'' = \{A \to h(\alpha): A \to \alpha \text{ in } P_E'\}$. Since $E' \trianglelefteq_s E$ it should be clear that $F' \trianglelefteq_s F$. Moreover $L(F',\Rightarrow) = h(L(E',\Rightarrow)) = h(L') = L$. Hence L is in $\mathcal{L}_s(F,\Rightarrow)$ as desired.

Step 2: If L is an arbitrary language in $\mathcal{L}_s(F,\Rightarrow)$ and x, y, z, w are arbitrary words over some alphabet $\bar{\Sigma}$ with $|x| = |z| = |w| = i$ and $|y| = j$, for some j, $0 \leq j \leq i - 1$, then we show that $L' = xyzLw$ is also in $\mathcal{L}_s(F,\Rightarrow)$.

By assumption there is an $F' = (V',\Sigma',P_F',S') \trianglelefteq_s F$ with $L(F',\Rightarrow) = L$. Let A and $\bar{S}$ be two new nonterminals and consider $F'' = (V' \cup \{\bar{S},A\} \cup \bar{\Sigma}, \Sigma' \cup \bar{\Sigma}, P_{F''},\bar{S})$, where $P_F'' = P_F' \cup \{\bar{S} \to xAzS'w; A \to y\}$. Clearly $F'' \trianglelefteq_s F$ and $L(F'',\Rightarrow) = xyzLw = L'$.

Step 3: We now show that if L is an arbitrary context-free language then L is in $\mathcal{L}_s(F,\Rightarrow)$.

We can write $L = \bar{L} \cup \bigcup_{j=0}^{i-1} L_j$, where each L_j is the finite union of languages $L' = xyzLw$, where $|x| = |z| = |w| = i$, $|y| = j$ and u is in L implies $|u| \equiv 0 \pmod{i}$. Since all such languages L' are in $\mathcal{L}_s(F,\Rightarrow)$ by Step 2, $\mathcal{L}_s(F,\Rightarrow)$ contains all finite languages and $\mathcal{L}_s(F,\Rightarrow)$ is closed under union the result follows.

This completes the proof of Claim 1. □

Claim 2: Let i, j ≥ 0 be integers, $H = (\{S,a\},\{a\},P_H,S)$, $L(H,\Rightarrow) = a^*$ and P_H contain $S \to a^iSa^jSa^k$. Then $\mathcal{L}_s(H,\Rightarrow) = \mathcal{L}(CF)$.

Proof of Claim 2: We reduce this claim to the previous one. H contains the productions $S \to a$ and $S \to a^iSa^jSa^k$, for some i, j, k ≥ 0. Hence we have:

(*) $S \Rightarrow^* a^{i_1}Sa^{i_2}Sa^{i_3}Sa^{i_4}Sa^{i_5}Sa^{i_6}$ in H, for some $i_1,\ldots,i_6$.

Let $t = \max(\{i_1,\ldots,i_6\})$ and $m = 4t + 3$. Since $S \Rightarrow^* a^k$, for all $k \geq 1$ there exist derivations

$a^{i_1}Sa^{i_2} \Rightarrow^* a^m$, and

$a^{i_3}Sa^{i_4}Sa^{i_5}Sa^{i_6} \Rightarrow^* a^m$ in H.

This follows because $i_i + i_2 \leq 4t$ and $i_3 + i_4 + i_5 + i_6 \leq 4t$.
Hence $S \Rightarrow^* a^mSa^m$ in H is obtained from (*).
Second, there exist derivations

$a^{i_1}Sa^{i_2} \Rightarrow^* a^m$,

$a^{i_3}Sa^{i_4} \Rightarrow^* a^m$ and

$a^{i_5}Sa^{i_6} \Rightarrow^* a^m$ in H.

Hence, from (*) we obtain:

$S \Rightarrow^* a^mSa^mSa^m$ in H.

We have shown that H simulates E. Thus $\mathcal{L}_s(E,\Rightarrow) \subseteq \mathcal{L}_s(H,\Rightarrow)$ and therefore $\mathcal{L}_s(H,\Rightarrow) = \mathcal{L}(CF)$. Claim 2 is thereby established. □

Returning to the proof of the theorem, we consider the grammar form G.

Now $S \rightarrow \alpha$ is in P with $|\alpha|_S \geq 2$. Hence $\alpha = a^{i_1}Sa^{i_2}\ldots a^{i_t}Sa^{i_{t+1}}$, for some $t \geq 2$ and $i_p \geq 0$, $i \leq p \leq t + 1$. Immediately, we obtain the derivation:

$S \Rightarrow^* a^iSa^jSa^k$ in G

for some i, j, k ≥ 0. Therefore G simulates the grammar form H of Claim 2 giving the desired result. □

We now extend Theorem 4.10 to g-grammatical families.

We say $F = (\{S,a\},\{a\},P,S)$, a two-symbol form, is a <u>normal form grammar</u> if $L(F,\Rightarrow) = a^*$ and there is a production $S \rightarrow \alpha$ in P with $\alpha = a^{i_1}S\ldots a^{i_m}S$, with $m \geq 2$.

Our major theorem for g-interpretations now follows.

Theorem 4.11

Let F be a normal form grammar and G a grammar form. Then there exists H, g-form equivalent to G such that $H \trianglelefteq_s F$.

Proof: Without loss of generality, we may assume $G = (V_G, \{a\}, P_G, S_G)$. There are two cases to consider.

Case 1: G is finite. Then $\mathcal{L}_g(G,\Rightarrow) = \{\emptyset\}$ or $\mathcal{L}(FIN)$. In both cases there exists $H \trianglelefteq_s F$ g-form equivalent to G since F can be assumed to contain the production $S \to a$.

Case 2: G is infinite. There are two subcases.

Case 2.1: G is a two-symbol form. Then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(REG)$, $\mathcal{L}(LIN)$ or $\mathcal{L}(CF)$, by Theorem 2.18. If $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(CF)$ take H to be equal to F. If $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(REG)$ let the productions $S \to a$; $A \to a$; $S \to a^{i_1}Aa^{i_2}\ldots a^{i_{m-1}}Aa^{i_m}S$ define H. Clearly $H \trianglelefteq_s F$ and $\mathcal{L}_g(H,\Rightarrow) = \mathcal{L}(REG)$ since H is non-self-embedding. In the case $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(LIN)$ take the following productions in H: $S \to a$; $A \to a$; $S \to a^{i_1}Ba^{i_2}Aa^{i_3}A\ldots a^{i_m}A$; and $B \to a^{i_1}A^{i_2}\ldots a^{i_{m-1}}Aa^{i_m}S$. The two step delay in reproducing S ensures S is self-embedding, even when $i_1 = \ldots = i_m = 0$.

Case 2.2: G is not a two-symbol form. Since G is unary this implies G has at least two nonterminals. By Theorems 4.6 and 4.9 we can assume G is both s-reduced and non-left recursive. Observe that if G is expansive then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(CF)$ and we can take H equal to F. Hence we only need deal with the case of G being non-expansive. Letting $\#(V_G - \{a\}) = n$ we proceed by induction on n. The basis n = 1 has been dealt with under Case 2.1. Assume the theorem is true for all $\bar{G}$ with $1 \leq \#(V_{\bar{G}} - \{a\}) \leq \ell$, for some $\ell \geq 1$. Consider the case of G with $\#(V_G - \{a\}) = n = \ell + 1$.

By the discussion above each reduced subgrammar G_A for $A \neq S_G$ and A appearing on the right side of an S_G-production has fewer than n nonterminals. Therefore by the inductive assumption there is an $F_A \trianglelefteq_s F$ with $\mathcal{L}_g(F_A,\Rightarrow) = \mathcal{L}_g(G_A,\Rightarrow)$ and by Lemma 2.19 we can assume G_A and F_A are identical.

We construct H by first taking all productions of P_G into P_H with the exception of the S_G-productions. Second for each S_G-production $S_G \to \alpha$ we add appropriate productions to P_H as follows:

(i) If $\alpha = a^j$ for some $j \geq 1$ take $S_G \to a$ into P_H.

(ii) If $\alpha = a^{j_1}A_1a^{j_2}\ldots a^{j_k}A_ka^{j_{k+1}}$ for some $k \geq 1$ add sufficient productions to P_H to "simulate" this production. Letting p be the least integer such that $p(m-1) \geq 2k$ consider the p-step derivation

$$S \overset{R}{\Rightarrow} a^{i_1}Sa^{i_2}\ldots a^{i_m}S \overset{R}{\Rightarrow} a^{i_1}S\ldots a^{i_m+i_1}S\ldots a^{i_m}S \overset{R}{\Rightarrow} \ldots$$

$$\ldots \overset{R}{\Rightarrow} a^{i_1}S\ldots\ldots a^{i_m}S$$

in F using the production $S \Rightarrow a^{i_1}S\ldots a^{i_m}S$. An isolated version of this derivation is used to "simulate" $S_G \rightarrow \alpha$. First add the following productions to P_H:

$$S_G \rightarrow a^{i_1}D_1a^{i_2}\ldots a^{i_{m-1}}D_{m-1}a^{i_m}B_1,$$

$$B_1 \rightarrow a^{i_1}D_ma^{i_2}\ldots a^{i_{m-1}}D_{2m-2}a^{i_m}B_2,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$B_{p-2} \rightarrow a^{i_1}D_{(p-2)(m-1)+1}a^{i_2}\ldots a^{i_{m-1}}D_{(p-1)(m-1)}a^{i_m}B_{p-1}.$$

Second, the production

$$B_{p-1} \rightarrow a^{i_1}D_{(p-1)(m-1)+1}a^{i_2}\ldots D_{2k-1}a^{i_{2k-1-(p-1)(m-1)}}C_1\ldots$$

$$\ldots \rightarrow a^{i_{m-1}}C_{p(m-1)-2k+1}a^{i_m}D_{2k},$$

if $j_{k+1} = 0$ and the production

$$B_{p-1} \rightarrow a^{i_1}D_{(p-1)(m-1)+1}a^{i_2}D_{(p-1)(m-1)+2}\ldots$$

$$\ldots \quad D_{2k+1}a^{i_{2k+1-(p-1)(m-1)}}C_1\ldots$$

$$\ldots \quad a^{i_{m-1}}C_{p(m-1)-2k-1}a^{i_m}C_{p(m-1)-2k},$$

$$\ldots \quad a^{i_{m-1}}C_{p(m-1)-2k-1}a^{i_m}C_{p(m-1)-2k},$$

otherwise.

Third, add the productions:

$C_q \rightarrow a$, for all new nonterminals C_q

and

$D_{2q-1} \rightarrow a$, for all new nonterminals D_{2q-1}, $q \geq 1$.

The B_q are also new nonterminals, however the even subscripted D_{2q} are not new, in fact $D_{2q} = A_q$, $1 \leq q \leq k$.

The basic idea behind this construction is that the D_{2q-1} represent the a^{j_q}, the D_{2q} are the A_q and if $j_{k+1} = 0$ then we must ensure that $D_{2k} = A_k$ also appears rightmost in the simulation, in case $A_k = S_G$. On the other hand if $j_{k+1} \neq 0$ then A_k must not appear rightmost in the simulation, hence in this case we also add D_{2k+1} to represent $a^{j_{k+1}}$. We can do this since there are at least $p+1 \leq 2k+1$ nonterminals in the underlying p-step derivation in F.

Clearly $H \triangleleft_s F$ by construction. Therefore it only remains to prove that H and G are g-form equivalent.

First construct $H' \triangleleft_g H$ in which all the productions in P_H which occur in P_G are taken unchanged into $P_{H'}$. This leaves the productions in P_H which are used to "simulate" the S_G-productions of G. For each production $S_G \to \alpha$ in G:

(i) If $\alpha = a^j$, for some $j \geq 1$, then take $S_G \to \alpha$ into $P_{H'}$. This is possible since $S_G \to a$ is in P_H.

(ii) If $\alpha = a^{j_1}A_1 \ldots A_k a^{j_{k+1}}$ for some $k \geq 1$ take into $P_{H'}$ the simulating productions introduced above except that a is replaced by λ, $C_q \to \lambda$ is taken for all q and $D_{2q-1} \to a^{j_q}$ is taken for all q.

It should be clear that H' simulates G, hence $\mathcal{L}_g(G,\Rightarrow) \subseteq \mathcal{L}_g(H',\Rightarrow) \subseteq \mathcal{L}_g(H,\Rightarrow)$. We now prove that $\mathcal{L}_g(H,\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$. We do this in two steps.

Step 1: Construct $\bar{H}$ from H be removing all the nonterminals not in V_G-{a} using the back-substitution lemma. Hence $\bar{H} = (V_G,\{a\},\bar{P},S_G)$ and by Lemma 2.12 is g-form equivalent to H. Moreover all the non-S_G-productions in $\bar{P}$ are identical to those in P_G and the S_G-productions in $\bar{P}$ are "similar" to those in P_G.

Step 2: Consider the "similarity" of the S_G-productions in $\bar{P}$ and P_G. Let $S_G \to \alpha$ be a production in $\bar{P}$. Then either

(i) $\alpha = a$ and there is some production $S_G \to a^i$, $i \geq 1$ in P_G,

or

(ii) $\alpha = a^{\ell_1}A_1 \ldots A_k a^{\ell_{k+1}}$ for some $k \geq 1$ and by the construction each of the ℓ_q, $1 \leq q \leq k+1$ are non-zero. By the construction there is some production $S_G \to \beta$ in P_G with $\beta = a^{j_1}A_1 \ldots A_k a^{j_{k+1}}$ and $j_q \geq 0$, $1 \leq q \leq k+1$.
If $j_q > 0$, $1 \leq q \leq k+1$, then we have found the corresponding production in P_G. Otherwise we modify $S_G \to \beta$ in $\bar{P}$ so that there is a corresponding production $S_G \to \alpha$ in P. We need the following:

Claim: Let $S_G \to \beta = a^{\ell_1}A_1 \ldots A_k a^{\ell_{k+1}}$, $k \geq 1$, $1 \leq q \leq k + 1$, be a production in $\bar{H}$, and r an integer $1 \leq r \leq k + 1$. Replace $S_G \to \beta$ by $S_G \to a^{\ell_1}A_1 \ldots A_{r-1}A_r a^{\ell_{k+1}}$ where a^{ℓ_r} has been erased in β. Let this be $\bar{H}'$. Then $\bar{H}$ and $\bar{H}'$ are g-form equivalent if either A_{r-1} or A_r is not equal to S_G.

Proof of Claim: Assume A_r is not equal to S_G. Then, by the assumptions of the theorem the subgrammar form $\bar{H}_{A_r}$ defines a g-grammatical family, which is a full semi-AFL (see Section 4.2) and such families are closed under pre- or post-product with a finite set. Hence taking a^{ℓ_r} into the subgrammar defined by A_r does not lose any generative capacity, hence $\bar{H}$ and $\bar{H}'$ are g-form equivalent. The full details of this proof are left to the reader. □

Returning to step 2 consider a production $S_G \to \alpha$ which contains $A_1, \ldots, A_k$ in this order, that is $\alpha = a^{j_1}A_1 \ldots A_k a^{i_{k+1}}$. Since $S_G \to \alpha$ does not correspond to $S_G \to \beta$ this immediately implies there are some $j_r = 0$, $1 \leq r \leq k + 1$. Replace $S_G \to \beta_{\ell_r}$ in $\bar{P}$ by the production $S_G \to \bar{\beta}$ where $\bar{\beta}$ is the same as β except that a^{ℓ_r} is replaced by λ if $a^{j_r} = \lambda$. Now since β can contain at most one appearance of S_G then such an a^{ℓ_r} will be adjacent to a non-S_G nonterminal with the exception of the following two cases: (a) $r = 1$ and $A_1 = S_G$ and (b) $r = 2$, $A_1 = S_G$ and $k = 1$. However case (a) would imply $i_1 = 0$ and $A_1 = S_G$ in $S_G \to \alpha$ in G. But by assumption S_G is non-left recursive hence this case will not occur. Similarly in case (b) since $a^{\ell_2} \neq \lambda$ the construction must have given either $a^{j_2} \neq \lambda$ or there exists another S_G-production in G of the form $S_G \to a^{q_1}S_G a^{q_2}$ with $q_1, q_2 > 0$.

Hence in all cases we can modify the S_G-productions in $\bar{H}$ without losing any generative capacity to obtain a production which has a corresponding production in G.

It should be clear that $G \underset{g}{\lhd} \bar{H}$ and we have also shown that $\bar{H} \underset{g}{\lhd} G$, hence G and $\bar{H}$ are strong g-form equivalent. Therefore H and G are g-form equivalent, completing the proof. □

Immediate consequences of Theorem 4.11 are the following normal form results.

Let F_1 be defined by: $S \to a$; $S \to aS$; $S \to aSS$,
F_2 be defined by: $S \to a$; $S \to SS$
and F_3 be defined by: $S \to a$; $S \to aS$; $S \to aSaSaS$
then each F_i is a normal form grammar, $1 \le i \le 3$. F_1 is the Greibach two-standard form grammar, and F_2 the Chomsky normal form grammar. Hence

Theorem 4.12

Let G be a non-empty grammar form, then there exist grammar forms H_1 and H_2 g-form equivalent to G such that
(i) H_1 is in Greibach two-standard normal form.
(ii) H_2 is in Chomsky normal form.

However consider F_4 defined by $S \to a^i$, $1 \le i \le 4$; $S \to aSaSa$, the Griebach-Nivat "normal form" grammar. Note that F_4 does not fulfi the conditions of Theorem 4.11. Although $\mathcal{L}_g(F_4,\Rightarrow) = \mathcal{L}(CF)$ not all g-grammatical families can be characterized by s-interpretations of F_4. Since every infinite interpretation $F_4' \triangleleft_s F_4$ is self-embedding, then $\mathcal{L}_g(F_4',\Rightarrow) \neq \mathcal{L}(REG)$. Hence $\mathcal{L}(REG)$ cannot be obtained as the g-grammatical family of a grammar form H with H in F_4-normal form. Whether this is the only family that cannot be obtained from F_4 is an open question. Moreover whether there exist normal form grammars F which miss an arbitrary g-grammatical family also remains an open question.

II.4.2 Closure Properties

In this section we demonstrate that $\mathcal{L}_s(G,\Rightarrow)$ is closed under intersection with regular sets and, in general, is not closed under any of the other AFL operations. In contrast we prove that $\mathcal{L}_g(G,\Rightarrow)$ is a full semi-AFL for all infinite grammar forms G.

Consider the grammar form G defined by the production $S \to ab$, then $\mathcal{L}_s(G,\Rightarrow) \neq \mathcal{L}(FIN)$ by Theorem 2.5 and further for each L in $\mathcal{L}_s(G,\Rightarrow)$, L contains only words of length two in each word of which the first symbol is not equal to the second symbol. Moreover for each a_1a_2, b_1b_2 in L, $a_1 \neq b_2$ and $a_2 \neq b_1$ because s-interpretations are given by dfl-substitutions. Hence $\mathcal{L}_s(G,\Rightarrow)$ is not closed under homomorphism, since $\{aa\}$ is not in $\mathcal{L}_s(G,\Rightarrow)$, and is not closed under union since $\{ab,ba\}$ is not in $\mathcal{L}_s(G,\Rightarrow)$. Since G only gives rise to finite sets $\mathcal{L}_s(G,\Rightarrow)$ is not closed under inverse homomorphism.

Finally, since all words in each L in $\mathcal{L}_s(G,\Rightarrow)$ are of length two, $\mathcal{L}_s(G,\Rightarrow)$ is not closed under catenation. Clearly $\mathcal{L}_s(G,\Rightarrow)$ is closed under intersection with regular sets.

These preliminary remarks give rise to the following:

Theorem 4.13

Let $G = (V,\Sigma,P,S)$ be a grammar form. Then $\mathcal{L}_s(G,\Rightarrow)$ is closed under intersection with regular sets and under dfl-substitution, but in general, under no other AFL operations.

Proof: Since the standard construction assumes closure under union it has to be modified slightly. Assume G is in Chomsky normal form without any loss of generality (Theorem 4.12). Let $G' = (V',\Sigma',P',S') \triangleleft_s G$ and $M = (Q,\Sigma',\delta,q_0,F)$ be an arbitrary finite state acceptor. We construct an interpretation $G'' \triangleleft_s G(\mu)$ such that $L(G'',\Rightarrow) = L(G',\Rightarrow) \cap L(M)$.

Let $V'' = \{S'\} \cup \Sigma' \cup \{[p,A,q]: A \text{ in } V' - \Sigma',\ p,\ q \text{ in } Q\}$ and $G'' = (V'',\Sigma',P'',S')$. Define P" as those productions obtained by taking:

(a) for each production $S' \to AB$ in P', all productions $S' \to [q_0,A,p][p,B,q]$ with p in Q and q in F,

(b) for each production $S' \to a$ in P', the production $S' \to a$ only if a is in L(M),

(c) for each production $A \to BC$ in P', the productions $[p,A,q] \to [p,B,r][r,C,q]$ for all p, q, r in Q, and

(d) for each production $A \to a$ in P', the productions $[p,A,q] \to a$ for all p, q in Q with $\delta(p,a) = q$.

Clearly $L(G'',\Rightarrow) = L(G',\Rightarrow) \cap L(M)$ by standard methods, and moreover letting $\mu(a) = \{a\}$, for all a in Σ', $\mu(A) = \{A\} \cup \{[p,A,q]: p,\ q \text{ in } Q\}$ then $G'' \triangleleft_s G'(\mu)$ and therefore $L(G'',\Rightarrow)$ is in $\mathcal{L}_s(G,\Rightarrow)$ as required.

Closure under dfl-substitution follows directly from the definition of s-interpretation. The third part of the Theorem follows from the preliminary remarks, completing the proof. □

In contrast we now prove that every g-grammatical family is not only closed under intersection with regular sets but also under union and homomorphism.

Theorem 4.14

Let $G = (V,\Sigma,P,S)$ be a grammar form. Then $\mathcal{L}_g(G,\Rightarrow)$ is closed under intersection with regular sets, union and homomorphism and, in general, under no other AFL operations.

Proof: By Corollary 1.2 $\mathcal{L}_g(G,\Rightarrow) = \mathcal{H}(\mathcal{L}_s(G,\Rightarrow))$, therefore $\mathcal{H}(\mathcal{L}_g(G,\Rightarrow)) = \mathcal{H}(\mathcal{H}(\mathcal{L}_s(G,\Rightarrow))) = \mathcal{L}_g(G,\Rightarrow)$. Let $G_i = (V_i,\Sigma_i,P_i,S_i)$, $i = 1, 2$ be two g-interpretations of G, $G_i \triangleleft_g G(\mu_i)$, $i = 1, 2$. Without loss of generality we may assume $S_1 = S_2$ and $(V_1-\Sigma_1) \cap (V_2-\Sigma_2) = \{S_1\}$, and further assume that S_1 does not appear on the right hand side of any production in $P_1 \cup P_2$ (this transformation can easily be accomplished within $\mathcal{G}_g(G)$). Now letting $G' = (V_1 \cup V_2, \Sigma_1 \cup \Sigma_2, P_1 \cup P_2, S_1)$ we have $G' \triangleleft_g G$ and $L(G',\Rightarrow) = L(G_1,\Rightarrow) \cup L(G_2,\Rightarrow)$. We can prove closure under intersection with regular sets using a slightly modified version of the construction given in Theorem 4.13.

Consider G defined by $S \rightarrow a$, then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\mathrm{FIN})$ and is therefore not closed under catenation closure nor under inverse homomorphism. Finally, G defined by $S \rightarrow a$ and $S \rightarrow aSa$ has $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\mathrm{LIN})$ by Theorem 2.18 and $\mathcal{L}(\mathrm{LIN})$ is not closed under catenation. □

However if we now restrict our attention to infinite grammar forms, we obtain full semi-AFLs under g-interpretation, although under s-interpretation we cannot strengthen Theorem 4.13 in this case. For example, let G be defined by the productions

$S \rightarrow ab$; $S \rightarrow abSab$

then by similar arguments to those used previously $\mathcal{L}_s(G,\Rightarrow)$ is not closed under union nor under homomorphism, and since $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}(\mathrm{LIN})$ and contains non-regular languages, $\mathcal{L}_s(G,\Rightarrow)$ is not closed under either catenation or catenation closure. Let h be defined by $h(a) = ab$ and $h(b) = \lambda$ then $h^{-1}(ab) = b^*ab^*$ is not in $\mathcal{L}_s(G,\Rightarrow)$, hence $\mathcal{L}_s(G,\Rightarrow)$ is not closed under inverse homomorphism.

Theorem 4.15

Let $G = (V,\Sigma,P,S)$ be an infinite grammar form, then $\mathcal{L}_g(G,\Rightarrow)$ is a full semi-AFL.

Proof: Because of Theorem 4.14 it suffices to show that $\mathcal{L}_g(G,\Rightarrow)$ is closed under regular substitution. By Theorem 4.6 we may assume that G is s-reduced and that G is a unary form without loss of generality. We proceed by induction on the number of nonterminals in $V - \Sigma$. Suppose G has k nonterminals, $k \geq 1$. If $k = 1$ then G is a two-symbol form, hence $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\mathrm{REG})$, $\mathcal{L}(\mathrm{LIN})$ or $\mathcal{L}(\mathrm{CF})$ by Theorem 2.18. In each case $\mathcal{L}_g(G,\Rightarrow)$ is closed under regular substitution. Assume the result holds for all $k \leq t$, $t \geq 1$, we now prove that it holds for the case $k = t + 1$.

Let $V-\Sigma = \{A_1,\ldots,A_k\}$ where $A_1 = S$ and $A_i \Rightarrow^+ \alpha A_j \beta$ in G implies $i \leq j$. Let $G' = (V',\Sigma',P',S') \underset{g}{\lhd} G(\mu)$ and assume G' is reduced. Let τ be a regular substitution on Σ'^* and Σ_τ be the image alphabet. Our aim is to construct a $G_\tau = (V_\tau,\Sigma_\tau,P_\tau,S')$ such that $L(G_\tau,\Rightarrow) = \tau(L(G',\Rightarrow))$ and $G_\tau \underset{g}{\lhd} G$. Observe that for all i, $2 \leq i \leq k$, $G_i = (V,\Sigma,P,A_i)$ has at most $k - 1$ nonterminals when reduced and hence $\mathcal{L}_g(G_i,\Rightarrow)$ is closed under regular substitution, by the inductive assumption. Also note that for all $S \to \alpha$ in P we can assume α contains at most one S, otherwise $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(\mathrm{CF})$ by Theorem 2.15 and the result follows trivially.

Since for each i, $2 \leq i \leq k$, $\mathcal{L}_g(G_i,\Rightarrow)$ is closed under regular substitution, it is a full semi-AFL and therefore it is closed under pre- and post-product with a regular set, that is for all L in $\mathcal{L}_g(G_i,\Rightarrow)$ and R a regular set, LR and RL are in $\mathcal{L}_g(G_i,\Rightarrow)$.

If there is a derivation $S \Rightarrow^+ x\,Sy$, $xy \neq \lambda$ in G then we may assume there are productions $S \to xSy$ and $S \to \lambda$ in G, and therefore letting $G_S = (\{S,a\},\{a\},\{S\to\lambda,S\to xSy\},S)$, $\mathcal{L}_g(G_S,\Rightarrow) \supseteq \mathcal{L}(\mathrm{REG})$ by Lemma 2.13 and if $x \neq \lambda$ and $y \neq \lambda$, $\mathcal{L}_g(G_S,\Rightarrow) \supseteq \mathcal{L}(\mathrm{LIN})$ by Theorem 2.16.

We use each of these remarks in the construction of G_τ. Given G' and τ, consider the effect of replacing P' by $\tau(P')$, where $\tau(A) = \{A\}$, for all A in $V'-\Sigma'$, and Σ' by Σ_τ, giving $\tau(G')$ say. Clearly $L(\tau(G',\Rightarrow)) = \tau(L(G',\Rightarrow)$. However G' has an infinite set of productions. We now modify $\tau(G')$ to give a G_τ with a finite set of productions.

Let $A \to \alpha$ belong to P', where A is in $\mu(S)$. Now $A \to \alpha$ may be written as:

either (i) $A \to x$, x is in Σ'^*

or (ii) $A \to \alpha_1\ldots\alpha_m$, $m \geq 1$ and $\alpha_i = x_i B_i y_i$.

Construct μ_τ and P_τ such that:

Case (i) gives rise to derivations $A \overset{+}{\Rightarrow} y$ in G_τ, y in $\tau(x)$. This can be done since G is infinite.

Case (ii) gives rise to derivations
$A \Rightarrow^{+} z_1 \ldots z_m$ in G_τ, where z_i is in $\tau(x_i)\tau(L(B_i,\Rightarrow))\tau(y_i)$, $1 \le i \le m$. If B_i is not in $\mu(S)$ then by the inductive assumption and the remarks above, replace α_i by α_i' the sentence symbol of a sub-grammar generating $\tau(x_i)\tau(L(B_i,\Rightarrow))\tau(y_i)$. This sub-grammar can be obtained as a g-interpretation of G_j, where B_i is in $\mu(A_j)$, $j \ge 2$, since $\mathcal{L}_g(G_i,\Rightarrow)$ is a full semi-AFL by the inductive assumption.

If B_i is in $\mu(S)$, then four cases occur:

(a) $x_i = y_i = \lambda$, then take $\alpha_i' = B_i$,

(b) $x_i \neq \lambda$, $y_i = \lambda$, then we may assume there is a production $S \to xS$ in P, $x \neq \lambda$. Take $\alpha_i' = \bar{B}_i$, where $\bar{B}_i \Rightarrow^{+} yB_i$, y in $\tau(x_i)$, via interpretations of $S \to xS$,

(c) $x_i = \lambda$, $y_i \neq \lambda$, similar to (b),

(d) $x_i \neq \lambda \neq y_i$, then we may assume there is a production $S \to xSy$ in P, with $x \neq \lambda$ and $y \neq \lambda$. Take $\alpha_i' = \bar{B}_i$, where $\bar{B}_i \Rightarrow^{+} x'B_iy'$, x' in $\tau(x_i)$, y' in $\tau(y_i)$, via interpretations of $S \to xSy$. In all cases we have replaced $A \to \alpha_1 \ldots \alpha_m$ by $A \to \alpha_1' \ldots \alpha_m'$, where the α_i' are new nonterminals equal to either B_i, $\bar{B}_i$ or the sentence symbol of the sub-grammar generating $\tau(x_i)\tau(L(B_i,\Rightarrow))\tau(y_i)$.

Thus the accumulated productions form P_τ, and μ_τ is defined by the construction, hence $G_\tau \triangleleft_g G(\mu_\tau)$ and $L(G_\tau,\Rightarrow) = \tau(L(G',\Rightarrow))$ as desired completing the theorem. □

II.4.3 Characterization Theorems

It is the aim of this section to demonstrate three results. First we establish that every infinite grammar form generates a full principal semi-AFL under g-interpretation. This enables us to exhibit many sub-context-free families which are not g-grammatical. Second we show that the g-grammatical family of every infinite grammar form is also an s-grammatical family. Hence the only family not generated under s-interpretations is $\mathcal{L}$(FIN). Since there are s-grammatical families which are not g-grammatical this reinforces our earlier remark that s-interpretation is more general than g-interpretation. Third, we consider when $\mathcal{L}_s(G,\Rightarrow)$ equals $\mathcal{L}$(REG) or $\mathcal{L}$(LIN).

We now turn to our first result. Letting L be in $\mathcal{L}_g(G,\Rightarrow)$ for some infinite grammar form G, it follows that $\mathcal{F}(L) \subseteq \mathcal{L}_g(G,\Rightarrow)$ where $\mathcal{F}(L)$ is the smallest full semi-AFL containing L. When equality occurs we say L is a full generator for $\mathcal{L}_g(G,\Rightarrow)$.

It is easy to see that under g-interpretations changing an appearance of a terminal symbol into an appearance of a new terminal symbol does not affect the generative capacity of a grammar form. Likewise changing a nonempty terminal word into a single terminal symbol on the right hand side of some production does not affect the generative capacity. Similarly in an s-reduced non-expansive grammar form $G = (V,\Sigma,P,S)$ given a production $S \rightarrow \alpha$ we can assume that only the following possibilities occur for S-productions:

(i) α is in $\Sigma \cup \{\lambda\}$

(ii) α is in $(\Sigma \cup \{\lambda\})\{S\}(\Sigma \cup \{\lambda\})$

(iii) $\alpha = A_1 \ldots A_m Sa$, $m > 0$ where all the A_i are nonterminals different from S and a is terminal,

(iv) $\alpha = aSB_1 \ldots B_n$, $n > 0$, where all the B_j are nonterminals different from S and a is terminal,

(v) $\alpha = A_1 \ldots A_m SB_1 \ldots B_n$, $m + n > 0$ where all A_i and B_j are nonterminals different from S, and

(vi) $\alpha = A_1 \ldots A_m$, $m > 0$ where the A_i are nonterminals different from S.

Now each nonterminal different from S defines a sub-grammar form which generates a full semi-AFL(compare Theorem 4.15). Hence in an S-production containing such a nonterminal, X say, X can be replaced by aX, Xb or aXb where a and b are terminals without any increase in the generative capacity of G. This gives the following proposition.

Proposition 4.16

Let $G = (V,\Sigma,P,S)$ be an infinite s-reduced non-expansive grammar form with at least two nonterminals. Then there exists a g-form equivalent grammar form $H = (V_H,\Sigma_H,P_H,S)$ which is also infinite, s-reduced, non-expansive and satisfies the following conditions:

(a) There exists disjoint terminal alphabets Σ_a, Σ_b and Σ_c such that $\Sigma_H = \Sigma \cup \Sigma_a \cup \Sigma_b \cup \Sigma_c$.

(b) $V_H = \Sigma_H \cup (V - \Sigma)$.

(c) Each production $S \rightarrow x$ in P_H with x in Σ_H^* has x in $\Sigma_c \cup \{\lambda\}$.

(d) Each symbol in $\Sigma_a \cup \Sigma_b \cup \Sigma_c$ occurs in one and only one production in P_H and there only once.

(e) Each production $S \to \alpha S\beta$ in P_H satisfies α is in $\Sigma_a \cup (\Sigma_a(V-\Sigma))^*$ and β is in $\Sigma_b \cup (\Sigma_b(V-\Sigma))^*$.

(f) Each production $S \to \alpha$ in P_H, where S is not in α satisfies α is in $(\Sigma_c(V-\Sigma))^+$.

(g) All productions for all nonterminals different from S are taken unchanged into P_H.

Essentially the terminals from Σ_c are used to mark the terminating (with respect to S) S-productions and each of the terminals in $\Sigma_a \cup \Sigma_b \cup \Sigma_c$ is used to either uniquely mark the appearance of a nonterminal different from S or the absence of such nonterminals.

We now state and prove our main theorem.

Theorem 4.17

Let $G = (V,\Sigma,P,S)$ be an infinite grammar form. Then $\mathcal{L}_g(G,\Rightarrow)$ is a full principal semi-AFL and furthermore a full generator for $\mathcal{L}_g(G,\Rightarrow)$ can be effectively constructed from G.

Proof: We may assume that G is an s-reduced grammar form without any loss of generality. Now if G is expansive $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(CF)$ and hence $\mathcal{L}_g(G,\Rightarrow)$ is a full principal semi-AFL. Full generators for $\mathcal{L}(CF)$ are well known and an interpretation G' of G can easily be constructed, which defines such a generator.

Having solved the case of G being expansive assume from now on that G is non-expansive.

Case 1: S is the only nonterminal. Then $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(REG)$ or $\mathcal{L}(LIN)$. Both $\mathcal{L}(REG)$ and $\mathcal{L}(LIN)$ are full principal semi-AFLs and full generators for them are well known. Hence we are left with:

Case 2: $V-\Sigma$ contains at least two nonterminals. We proceed by induction on the number of nonterminals. The basis $\#(V-\Sigma) = 1$ is subsumed under Case 1. Hence assume the theorem is true for all G with $\#(V-\Sigma) \le k - 1$, for some $k > 1$. Consider the case that G has k nonterminals.

Recalling that $\mathcal{F}(L)$ is the smallest full semi-AFL generated by L, it suffices to show that:

(*) there exists a language L in $\mathcal{L}_g(G,\Rightarrow)$ such that $\mathcal{L}_g(G,\Rightarrow) \subseteq \mathcal{F}(L)$.

Moreover this can be further reduced to showing that

(**) there exists a language L in $\mathcal{L}_s(G,\Rightarrow)$ such that

$\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{F}(L)$, since this implies that
$\mathcal{H}(\mathcal{L}_s(G,\Rightarrow)) = \mathcal{L}_g(G,\Rightarrow) \subseteq \mathcal{H}(\mathcal{F}(L)) = \mathcal{F}(L)$.
Let us assume that G satisfies the conditions for H in Proposition 4.16, not only for S but also for each nonterminal. In other words each terminal symbol in Σ appears once and only once in some production in P and moreover we can split Σ into three disjoint subsets, namely those terminals used in "terminating" productions, those deposited to the left by each recursive nonterminal and those deposited to the right. We say a production $A \to \alpha$ for a nonterminal is terminating, for the purpose of this proof, if α does not contain A. That we can assume G to be in this "terminal-distinct" form follows from the observation that each $A \neq S$ defines a subgrammar form which satisfies the conditions of Proposition 4.17.

Without loss of generality assume $A \to \lambda$ is in P for each A in $V-\Sigma$.

Letting the left, middle and right terminals be denoted by Σ_ℓ, Σ_m and Σ_r where $\Sigma = \Sigma_\ell \cup \Sigma_m \cup \Sigma_r$ we add to G:

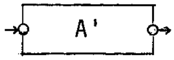
(i) the terminals $\bar{\Sigma}_\ell = \{\bar{a}\text{: a in } \Sigma_\ell\}$ and $\bar{\Sigma}_r = \{\bar{a}\text{: a in } \Sigma_r\}$ giving $\bar{\Sigma} = \Sigma \cup \bar{\Sigma}_\ell \cup \bar{\Sigma}_r$, and

(ii) the productions $A \to \bar{\alpha}A\bar{\beta}$ for all $A \to \alpha A\beta$ in P for all A in $V-\Sigma$, where $\bar{\alpha}(\bar{\beta})$ is $\alpha(\beta)$ with each left and right terminal replaced by its "barred" version.

Let the resulting grammar form be $\bar{G} = (\bar{V},\bar{\Sigma},\bar{P},\bar{S})$, where $\bar{V} = (V-\Sigma) \cup \bar{\Sigma}$. Clearly $\bar{G} \trianglelefteq_s G$ and also $G \trianglelefteq_s \bar{G}$, hence G and $\bar{G}$ are s-form equivalent (in fact strong s-form equivalent).

We claim that $L(\bar{G},\Rightarrow)$ is a full generator for $\mathcal{L}_g(G,\Rightarrow)$.

To show this consider an arbitrary interpretation $G' = (V',\Sigma',P',S') \trianglelefteq_s G(\mu)$. We need to prove that $L(G',\Rightarrow)$ is in $\mathcal{F}(L(\bar{G},\Rightarrow))$. We do this by sketching the construction of an a-transducer $M_{G'}$ which satisfies $L(G',\Rightarrow) = M_{G'}(L(\bar{G},\Rightarrow))$.

Let 

represent an a-transducer with a single input state $\sigma(A')$ and a single accepting state $\sigma_a(A')$ (this is no loss of generality).

First observe that for each A in $V - (\Sigma \cup \{S\})$ by the inductive assumption $L(\bar{G}_A,\Rightarrow)$ is a full generator for $\mathcal{L}_g(G_A,\Rightarrow)$. Although this has not been proved directly under Case 1, the proof sketch we now outline can be adapted for this purpose. Hence for each A' in $V' - \Sigma'$ with A' in $\mu(A)$, $A \neq S$ there is an a-transducer $M_{G'_{A'}}$

such that $L(G'_{A'},\Rightarrow) = M_{G'_{A'}}(L(\bar{G}_A,\Rightarrow))$. The construction we now give for $M_{G'}$ is inductive and whenever we refer to a copy of the a-transducer for an A', A' not in $\mu(S)$, we assume that its state set is unique. To construct $M_{G'}$ we only consider the S"-productions of G' for all S" in $\mu(S)$. Assume initially the existence of an initial state σ and an accepting state σ_a in M_G'. Let the left, middle and right terminal symbols for the S-productions be $\Sigma_a \cup \bar{\Sigma}_a$, Σ_c and $\Sigma_b \cup \bar{\Sigma}_b$ respectively. Enumerate the images of the S-productions from $1,2,\ldots,n_S$ say.

For each production j: $S'' \to \alpha$ in P', where S" is in $\mu(S)$ and $1 \le j \le n_S$, add the following transitions to $M_{G'}$:

(i) if $\alpha = c'$, then (σ,c,c',σ_a), where $S \to c$ is in $\bar{P}$.

(ii) if $\alpha = a_1'A_1'\ldots a_m'A_m'S''b_1'B_1'\ldots b_n'B_n'$ where m, n ≥ 0, a_p, b_q are in Σ' and the A_p' and B_q' are nonterminals then there exists a production $S \to a_1A_1\ldots a_mA_mSb_1\ldots b_nB_n$ in $\bar{P}$ and also the production $S \to \bar{a}_1\ldots\bar{a}_mA_mS\bar{b}_1\ldots\bar{b}_nB_n$. Since $A \to \lambda$ is in $\bar{P}$ for each A in $\bar{V} - \bar{\Sigma}$, we can encode the production j: $S'' \to \alpha$ with the following transitions:

$(\sigma,\bar{a}_1\ldots\bar{a}_m(a_1\ldots a_m)^j\bar{a}_1,\ a_1',\sigma(A_1'),$

$(\sigma_a(A_1'),\bar{a}_2,a_2',\sigma(A_2')),$

.

.

.

$(\sigma_a(A_{m-1}'),\bar{a}_m,a_m',\sigma(A_m'))$

$(\sigma_a(A_m'),\lambda,\lambda,\sigma)$

and

$(\sigma,\bar{b}_1\ldots\bar{b}_n(b_1\ldots b_n)^j\bar{b}_1\ ,b_1',\sigma(B_1')),$

.

.

.

$(\sigma_a(B_{n-1}'),\bar{b}_n,b_n',\sigma(B_n'))$

$(\sigma_a(B_n'),\lambda,\lambda,\sigma),$

$(\sigma_a(B_n'),\lambda,\lambda,\sigma_a).$

Because of the structure of $\bar{G}$ a word u derived from $a_1'A_1'\ldots a_m'A_m'$ will be output by $M_{G'}$ iff a word v derived from $b_1'B_1'\ldots b_n'B_n'$ is output by $M_{G'}$. Moreover this will only occur when there is a derivation

$$S' \Rightarrow^+ xS''y \Rightarrow x\alpha y \Rightarrow^+ xuS''vy \text{ in } G'.$$

The encoding technique is crucial to the construction, since otherwise $M_{G'}$ will create erroneous output.

(iii) $\alpha = a'S''$, $a'S''b'$, $S''b'$, $a'S''b_1'B_1'\ldots b_n'B_n'$,
$a_1'A_1'\ldots a_m'A_m'S''b'$ or $\alpha = c_1'C_1'\ldots c_r'C_r'$.
Each of these is dealt with in a similar way to case (ii).

Finally add the transitions $(\sigma_a,\lambda,\lambda,\sigma)$ and $(\sigma,\lambda,\lambda,\sigma_a)$ to $M_{G'}$. Note that not all words in $L(G',\Rightarrow)$ are accepted by $M_{G'}$ and secondly note that there is a λ-transition (outputting λ) not only from σ_a to σ but also from σ to σ_a, and this is true for each a-transducer corresponding to each subgrammar of G'.

Consider each word w in $L(G',\Rightarrow)$. There is a derivation

$$S' \Rightarrow^{+} w' \Rightarrow^{*} w \text{ in } G'$$

such that S' derives w' using only images of S-productions and further w' does not contain any image of S. There is a corresponding derivation $S \Rightarrow^{+} x$ in $\bar{G}$ of which $S' \Rightarrow^{+} w'$ is the image. Since $S' \Rightarrow^{+} w'$ can only contain the image of exactly one terminating S-production, the other productions used are images of recursive S-productions. Now the encoding suggested above results in using some multiple of each recursive S-production, rather than the original single applications. This clearly can always be carried out and each word of $L(\bar{G},\Rightarrow)$ either gives rise to a unique decoding into a word of $L(G',\Rightarrow)$ or it cannot be decoded. In the former case $M_{G'}$ accepts the word and gives the correct output and in the latter case $M_{G'}$ rejects the word.

This completes the proof sketch that for G' an arbitrary s-interpretation of G, $L(G',\Rightarrow)$ is in $\mathcal{S}(L(G',\Rightarrow))$. □

As immediate applications of this Theorem we have:

Corollary 4.18

The families of metalinear, nonterminal bounded and derivation bounded languages are not g-grammatical families.

We now consider our second result namely the comparison of the collections of g-grammatical and s-grammatical families. By Lemma 2.8 and Theorem 2.5 we know that $\mathcal{L}$(FIN) is both a g-grammatical family and not an s-grammatical one. Since the only other g-grammatical family "below" $\mathcal{L}$(FIN) is {∅}, which is also s-grammatical, we turn our attention to g-grammatical families generated by infinite grammar forms.

Theorem 4.19

Let $G = (V,\Sigma,P,S)$ be an infinite grammar form. Then $\mathcal{L}_g(G,\Rightarrow)$ is s-grammatical.

Proof: The basic idea of the proof is to construct a grammar from H from G such that H and G are g-form equivalent and $\mathcal{L}_s(H,\Rightarrow) = \mathcal{L}_g(H,\Rightarrow)$. Without loss of generality we can assume $\Sigma = \{a\}$ and hence $\mathcal{L}_s(G,\Rightarrow)$ is closed under union and intersection with regular sets. Let $H = (V_H,\{a\},P_H,S)$ where $V_H = V \cup \{A\}$, where A is some new nonterminal and P_H contains the productions of P modified by replacing each appearance of a with an A. Finally the productions $A \to \lambda$ and $A \to aA$ are added to P_H. By simulation $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(H,\Rightarrow)$ and $\mathcal{L}_g(G,\Rightarrow) \subseteq \mathcal{L}_g(H,\Rightarrow)$. We now show that for each $H' = (V_H',\Sigma',P_H',S') \triangleleft_s H(\mu)$ there is a $G' = (V',\Sigma'',P',S') \triangleleft_s G(\mu')$ and a regular substitution τ such that $L(H',\Rightarrow) = \tau(L(G',\Rightarrow))$. In other words $\mathcal{L}_s(H,\Rightarrow) \subseteq \mathcal{L}_g(G,\Rightarrow)$ which shows that $\mathcal{L}_g(H,\Rightarrow) = \mathcal{L}_g(G,\Rightarrow)$.

Consider such an H'. Essentially we let G' consist of the G-portion of H' together with a different terminal symbol for each image of A. Let $\Sigma'' = \{[A']: A' \text{ is in } \mu(A)\}$ and $V' = V_H' - (\mu(A) \cup \Sigma') \cup \Sigma''$. For each production $B \to \alpha$ in P_H' where B is not in $\mu(A)$ take the production $B \to [\alpha]$ into P' where $[\alpha]$ denotes α with each image A' of A replaced by [A'].

For each A' in $\mu(A)$ the language $\{x: A' \Rightarrow^* x \text{ in } H'\}$ is clearly regular, since it is obtained from interpretations of $A \to aA$ and $A \to \lambda$. Let this be $\tau([A'])$ for each A' in $\mu(A)$. By construction it should be clear that $L(H',\Rightarrow) = \tau(L(G',\Rightarrow))$, hence $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}_g(H,\Rightarrow)$.

It remains to prove that $\mathcal{L}_s(H,\Rightarrow) = \mathcal{L}_g(H,\Rightarrow)$. It is sufficient to show that $\mathcal{H}(\mathcal{L}_s(H,\Rightarrow)) \subseteq \mathcal{L}_s(H,\Rightarrow)$. Let $H' = (V_H',\Sigma',P_H',S') \triangleleft_s H(\mu)$ and $h: \Sigma'^* \to \Sigma''^*$ be a homomorphism. We construct an $H'' \triangleleft_s H(\mu')$ such that $L(H'',\Rightarrow) = h(L(H',\Rightarrow))$.

Let $\mu(A) = \{A_1,\ldots,A_m\}$, $m \geq 1$ and $\Sigma' = \{a_1,\ldots,a_n\}$, $n \geq 1$. Let $h(a_n) = b_1 \ldots b_{n_1}$, $n_1 \geq 0$, b_i in Σ''. The terminal a_n appears in at most two kinds of productions, either $A_i \to a_n$ or $A_i \to a_n A_j$, for some i and j.

(i) Replace each production of the form $A_i \to a_n$ by either $A_i \to \lambda$ if $n_1 = 0$ or $A_i \to b_1 B_{i,1}$, $B_{i,1} \to b_2 B_{i,2}, \ldots, B_{i,n_1-1} \to b_{n_1}$, where the $B_{i,k}$ are new nonterminals, otherwise.

(ii) For each production of the form $A_i \to a_n A_j$ either remove it and replace A_i everywhere by A_j if $n_1 = 0$ or replace it by $A_i \to b_1 B_{i,j,1}, B_{i,j,1} \to b_2 B_{i,j,2}, \ldots, B_{i,j,n_1-1} \to b_{n_1} A_j$, where the $B_{i,j,k}$ are new nonterminals, otherwise.

Let the resulting grammar form be H_1, then we claim that $L(H_1,\Rightarrow) = h_1(L(H',\Rightarrow))$ where $h_1(a_i) = a_i$, $1 \le i < n$ and $h_1(a_n) = h(a_n)$. If $h(a_n) = \lambda$ then we have indeed erased a_n in $L(H',\Rightarrow)$ to give $L(H_1,\Rightarrow)$, otherwise we have included unique productions to generate $h(a_n)$ rather than a_n at each position that a_n would have been generated in H'. Hence $L(H_1,\Rightarrow) = h_1(L(H',\Rightarrow))$. Assuming without loss of generality that $\Sigma' \cap \Sigma'' = \emptyset$ we can iterate the construction to obtain H" such that $L(H'',\Rightarrow) = h(L(H',\Rightarrow))$ obtaining the desired result. □

We have therefore demonstrated that each g-grammatical family apart from $\mathcal{L}(FIN)$ is also s-grammatical.

To close this section we consider our final area for discussion, that is when does $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(REG)$ or $\mathcal{L}(LIN)$? We first have:

Theorem 4.20

Let $G = (V,\Sigma,P,S)$ be a grammar form. Then $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(REG)$ iff there is an a in Σ such that $a^* \subseteq L(G,\Rightarrow)$.

Proof: if: Let $R \subseteq \Delta^*$ be an arbitrary regular language. We show that R is in $\mathcal{L}_s(G,\Rightarrow)$. Since $a^* \subseteq L(G,\Rightarrow)$ for some a in Σ, there exists $G_1 \triangleleft_s G$ with $L(G_1,\Rightarrow) = a^*$, since $\mathcal{L}_s(G,\Rightarrow)$ is closed under intersection with regular sets by Theorem 4.13. Since a^* is in $\mathcal{L}_s(G,\Rightarrow)$, Δ^* is in $\mathcal{L}_s(G,\Rightarrow)$ since s-grammatical families are closed under dfl-substitution (Theorem 4.13). Finally, since $R = \Delta^* \cap R$, then R is in $\mathcal{L}_s(G,\Rightarrow)$ by Theorem 4.13.
only if: Since $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(REG)$, b^* must be in $\mathcal{L}_s(G,\Rightarrow)$ for some b, hence by Lemma 2.1 $a^* \subseteq L(G,\Rightarrow)$ for some a in Σ. □

This also yields

Corollary 4.21

Let G be a grammar form.
Then $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(REG)$ iff $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(FIN)$.

We can now characterize those grammar forms which give exactly $\mathcal{L}(REG)$ under s-interpretations.

Theorem 4.22

Let $G = (V,\Sigma,P,S)$ be a reduced grammar form. Then $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(REG)$ iff (i) G is non-self-embedding and (ii) $a^* \subseteq L(G,\Rightarrow)$ for some a in Σ.

Proof: if: Condition (ii) implies $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(REG)$ and condition (i) implies $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}(REG)$.
only if: Since $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(REG)$ condition (ii) must hold and since $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}(REG)$ condition (i) must hold. □

We now turn to the question of when $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(LIN)$. For this case we will not give any proofs, but only the main theorems and appropriate definitions.

We will first consider unary reduced grammar forms.

A nonterminal A in a unary reduced grammar form $G = (V,\{a\},P,S)$ is left pumping (respectively, right pumping) if for some fixed $m, n \geq 0$, there are infinitely many values i such that:

$A \Rightarrow^* a^{i+m}Aa^n$ (respectively, $A \Rightarrow^* a^mAa^{n+i}$).

A nonterminal is pumping if it is both left and right pumping.

Let $A_1,\ldots,A_m$ be all the pumping nonterminals in G. For each i, $1 \leq i \leq m$, the lengths j of the terminal words a^j generated by A_i constitute an ultimately periodic set. Denote its period by $p(A_i)$ and let p be the least common multiple of the periods $p(A_1),\ldots,p(A_m)$. Denote the residue classes modulo p by $R_0, R_1,\ldots,R_{p-1}$.

We say the residue class R_j is A_i reachable if there are integers r, s and t such that:

$S \Rightarrow^* a^rA_ia^s$, $A_i \Rightarrow^* a^{t+np}$, for all $n \geq 0$,

where $r + s + t \equiv j$ modulo p.

The pumping spectrum of G consists of all integers in all A_i-reachable residue classes, where $1 \leq i \leq m$.

For linear grammar forms we now obtain:

Theorem 4.23

Let $G = (V,\{a\},P,S)$ be a reduced unary linear grammar form. Then $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(LIN)$ iff (i) $L(G,\Rightarrow) = a^*$ and (ii) the pumping spectrum of G consists of all non-negative integers.

This can now be extended to arbitrary grammars by way of the following definition.

Let $G = (V,\Sigma,P,S)$ be a grammar form and a be in Σ, then G_a, the <u>a-restriction of G</u>, is the subgrammar form $G_a = (V_a,\{a\},P_a,S)$ of G where $P_a \subseteq P$ is the set of all those productions in P which do not contain the letter a.

We now have:

Theorem 4.24

Let $G = (V,\Sigma,P,S)$ be a grammar form.
Then $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(LIN)$ iff (i) and (ii) hold.

(i) $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}(LIN)$.

(ii) There exists an a in Σ such that $a^* \subseteq L(G,\Rightarrow)$ and the pumping spectrum of the reduced a-restriction of G consists of all non-negative integers.

II.5 Syntax Analysis

The existence of "universal" parsing algorithms for a given grammar family is the focal point of this section. Given a grammar form G, can a parsing algorithm M for the grammar G be supplied such that whenever $G' \underset{x}{\lhd} G(\mu)$, for some grammar G', a parsing algorithm for G' is immediate. In Section 5.1 we examine a technique for s-interpretations, while in Section 5.2 some results on conflict freeness are presented which are a necessary adjunct to a study of precedence parsing for grammar forms. Finally, in Section 5.3 we consider the pushdown acceptor approach, showing that every grammatical family is a pushdown acceptor family and vice versa.

II.5.1 Syntax Analysis of s-grammatical Families

We are concerned in this section with the following problem:

Given a parsing method M_G for a grammar G can M_G be used as a parsing method for G', where $G' \underset{x}{\lhd} G(\mu)$? If so, then how economic is this indirect parsing technique?

Under s-interpretation we demonstrate that this is indeed the case and, further, when only recognition is required for G', the time taken to recognize an arbitrary word x' in Σ'^* is of the same order as the time taken to parse words y, $|y| = |x'|$, y in Σ^*, by M_G.

Under g-interpretation on the other hand the relationship between words in Σ'^* and words in Σ^* is not as pleasant as for s-interpretations. The set $\{x: x' \text{ is in } \mu(x)\}$, for arbitrary x' in Σ'^* is a singleton set in the case of s-interpretations, whereas in the case of g-interpretations it may be infinite, since erasing and identification is allowed. Therefore in the following attention is restricted to s-interpretation families only.

The central idea is based upon Lemma 2.1, namely, if a word x' over Σ' is in $L(G',\Rightarrow)$ then there is a derivation $S \Rightarrow^+ \mu^{-1}(x')$ in G. Of course, the converse does not hold, therefore the major difficulty occurs when $\mu^{-1}(x')$ is found to be in $L(G,\Rightarrow)$, since it is still necessary to decide whether x' is in $L(G',\Rightarrow)$ or not. However in this case the extra information that $\mu^{-1}(x')$ is in $L(G,\Rightarrow)$ is sufficient to allow a fast recognition procedure to be carried out on a bracketed version of x', based upon the fast recognition of parenthesis languages.

Let $G = (V,\Sigma,P,S)$ be a grammar form and M_G a parsing method for G, which for each word x in Σ^* either yields all leftmost derivations of x, one after another, if x is in $L(G,\Rightarrow)$ or rejects x if it is not in $L(G,\Rightarrow)$. Let $G' = (V',\Sigma',P',S') \underset{s}{\lhd} G(\mu)$. Then a parsing method $M_{G'}$ for G' can be obtained as follows. For each word x' in Σ'^*, $M_{G'}$

consists of at most three major steps. These are:
(1) $M_{G'}$ determines the unique word $x = \mu^{-1}(x')$,
(2) $M_{G'}$ parses x using M_G, and
(3) $M_{G'}$ checks to see if there are G-derivations of x, which via interpretation yield G'-derivations of x'.

For the purposes of the algorithm we need to introduce bracketed versions of G and G'. Assume the elements of P are arbitrarily but uniquely numbered from 1 to #P. Let $\Delta = \{(_i,)_i : 1 \le i \le \#P\}$ and let the bracketed version of G, $\hat{G}$ be defined by:

$$\hat{G} = (V \cup \Delta, \Sigma \cup \Delta, \hat{P}, S)$$

where $\hat{P} = \{A \to (_i\alpha)_i : A \to \alpha$ is in P and $A \to \alpha$ is numbered i$\}$. The bracketed version of G', $\hat{G}'$ is defined as:

$$\hat{G}' = (V' \cup \Delta, \Sigma' \cup \Delta, \hat{P}', S')$$

where $\hat{P}' = \{A' \to (_i\alpha')_i : A' \to \alpha'$ is in $\mu(A \to \alpha)$ for some $A \to \alpha$ and $A \to \alpha$ is numbered i$\}$.

Finally, define $\hat{\mu}$ by $\hat{\mu}(X) = \mu(X)$, for all X in V, $\hat{\mu}((_i) = (_i$ and $\hat{\mu}()_i) =)_i$, $1 \le i \le \#P$. Clearly $\hat{G}' \trianglelefteq_s \hat{G}(\hat{\mu})$.

We now expand the three steps of $M_{G'}$ mentioned above.

Step 1:

$M_{G'}$ determines the unique word $x = \mu^{-1}(x')$. Note that this is a linear (in $|x'|$) time operation.

Step 2:

$M_{G'}$ parses x by M_G. If x is rejected by M_G then x' is rejected by $M_{G'}$, otherwise suppose M_G determines that $\delta_1, \ldots, \delta_m$, $m > 0$, are the leftmost derivations of x in G.

Step 3:

<u>For</u> $\delta := \delta_1, \ldots, \delta_m$ <u>do</u>
(a) $M_{G'}$ determines the derivation $\hat{\delta}$ obtained by replacing each occurrence of a production $A \to \alpha$ in δ by $A \to (_i\alpha)_i$ from $\hat{P}$. Let $\hat{x}$ denote the bracketed version of x generated by $\hat{\delta}$.
(b) $M_{G'}$ determines $\hat{x}'$ the bracketed version of x' from $\hat{x}$ by consulting x'. Observe that x' is in $L(G',\Rightarrow)$ iff $\hat{x}'$ is in $L(\hat{G}',\Rightarrow)$.
(c) $M_{G'}$ now parses $\hat{x}'$ in a bottom-up manner. Furthermore, it simultaneously performs all parses of $\hat{x}'$ which conform to the shape of the derivation tree determined by $\hat{\delta}$.

If during these bottom-up reductions the "sentential form"

$\alpha({}_i y_0 Y_1 y_1 \ldots Y_q y_q)_i \beta$
occurs, where $q \geq 0$, $Y_1, \ldots, Y_q$ are sets of nonterminals, $y_0, \ldots, y_q$ are terminal words and α, β are words over $V' \cup \Delta$, then we obtain
$\alpha X \beta$
if $X = \{A_0' : A_0' \to ({}_i y_0 A_1' y_1 \ldots A_q' y_q)_i$ is in $\hat{P}'$ and A_j' is in Y_j, $1 \leq j \leq q\}$.

Clearly x' is in $L(G',\Rightarrow)$ iff $\hat{x}'$ determines a set of nonterminals including S'.

(d) If S' is in the set of nonterminals determined by $\hat{x}'$, then $M_{G'}$ constructs all leftmost G'-derivations from x', which conform with $\hat{\delta}$ by repeatedly traversing in a top-down manner the output of (c). At each traversal single nonterminals are chosen from the sets of nonterminals, which conform with those already chosen. If there are at most k nonterminals in each set of nonterminals and the parse tree produced by (c) has r internal nodes, then at most k^r distinct derivation trees can be produced by (d).

Let us clarify this algorithm by working through an example.

Example 5: Let G be defined by the following numbered productions:
1: $S \to AB$: 2: $S \to CD$; 3: $A \to aAb$; 4: $A \to ab$; 5: $B \to cB$;
6: $B \to c$; 7: $C \to aC$; 8: $C \to a$; 9: $D \to bDc$; 10: $D \to bc$

Let $\Delta = \{({}_i,)_i : 1 \leq i \leq 10\}$ and $G' \mathrel{\triangleleft_s} G(\mu)$ be defined by:
$S' \to A'B'$; $S' \to A'B''$; $S' \to C'D'$; $A' \to dA''e$; $A' \to df$;
$A'' \to dA'e$; $A'' \to df$; $B' \to gB'$; $B' \to h$; $B'' \to hB''$;
$B'' \to gB'$; $C' \to dC'$; $C' \to d$; $D' \to fD'h$; $D' \to eg'$; $D' \to fg$;
where $\mu(S) = S'$, $\mu(A) = \{A',A''\}$, $\mu(B) = \{B',B''\}$, $\mu(C) = C'$, $\mu(D) = D'$, $\mu(a) = d$, $\mu(b) = \{e,f\}$ and $\mu(c) = \{g,h\}$. Now $L(G,\Rightarrow) = \{a^i b^i c^j : i,j \geq 1\} \cup \{a^i b^j c^j : i,j \geq 1\}$, a well known inherently ambiguous language. Note that for each word in $L(G,\Rightarrow)$, G gives at most two distinct derivation trees.
Consider $x' = ddfegh$.

Step 1: $x = \mu^{-1}(x') = aabbcc$.

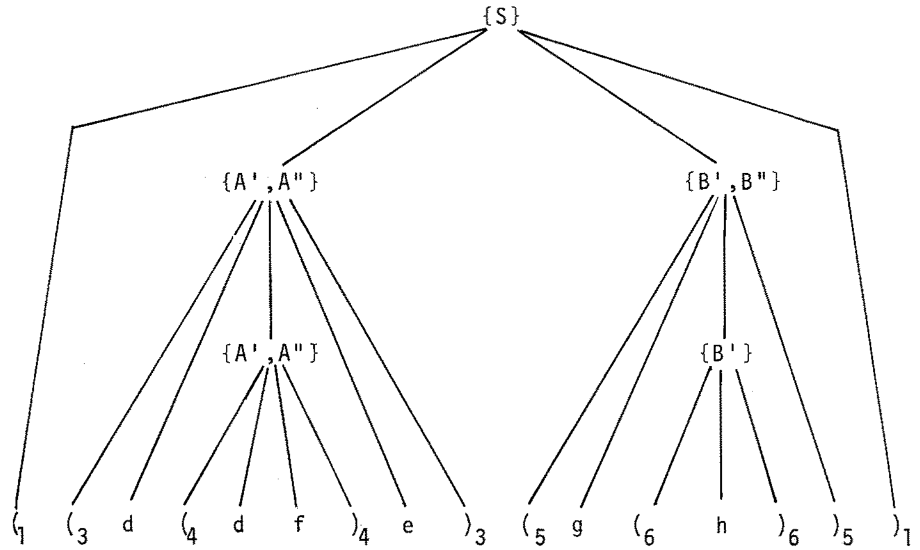
Step 2: M_G produces two leftmost derivations for x, namely
δ_1: $S \overset{L}{\Rightarrow} AB \overset{L}{\Rightarrow} aAbB \overset{L}{\Rightarrow} aabbB \overset{L}{\Rightarrow} aabbcB \overset{L}{\Rightarrow} aabbcc$
and
δ_2: $S \overset{L}{\Rightarrow} CD \overset{L}{\Rightarrow} aCD \overset{L}{\Rightarrow} aaD \overset{L}{\Rightarrow} aabDc \overset{L}{\Rightarrow} aabbcc$
which may be represented as 1, 3, 4, 5, 6 and 2, 7, 8, 9, 10 respectively.

Step 3: Consider $\delta = \delta_1$.

(a) Determine $\hat{\delta}$: $S \stackrel{L}{\Rightarrow} (_1AB)_1 \stackrel{L}{\Rightarrow} (_1(_3aAb)_3B)_1$
$\stackrel{L}{\Rightarrow} (_1(_3a(_4ab)_4b)_3B)_1 \stackrel{L}{\Rightarrow} (_1(_3a(_4ab)_4b)_3(_5cB)_3)_1$
$\stackrel{L}{\Rightarrow} (_1(_3a(_4ab)_4b)_3(_5c(_6c)_6)_5)_1 = \hat{x}$.

(b) Determine $\hat{x}'$ from $\hat{x}$.
$\hat{x}' = (_1(_3d(_4df)_4e)_3(_5g(_6h)_6)_5)_1$

(c) We leave it to the reader to construct $\hat{G}'$. Consider the "parse tree" obtained from $\hat{x}'$ using $\hat{G}'$.



Since S' is in the set of nonterminals determined by $\hat{x}'$, $\hat{x}'$ is in $L(G',\Rightarrow)$, that is, x' is in $L(G',\Rightarrow)$.

(d) Construct all possible leftmost derivations in G'.
Observe that although $S' \stackrel{L}{\Rightarrow} A'B'$ and $S' \stackrel{L}{\Rightarrow} A'B''$ are valid initially, the combinations $S' \stackrel{L}{\Rightarrow} A''B'$ or $S' \stackrel{L}{\Rightarrow} A''B''$ given by the "parse tree" are invalid, since they do not correspond to any productions in P'. Hence we obtain
$S' \stackrel{L}{\Rightarrow} A'B' \stackrel{L}{\Rightarrow} dA''eB' \stackrel{L}{\Rightarrow} ddfeB' \stackrel{L}{\Rightarrow} ddfegB' \stackrel{L}{\Rightarrow} ddfegh$
and
$S' \stackrel{L}{\Rightarrow} A'B'' \stackrel{L}{\Rightarrow} dA''eB'' \stackrel{L}{\Rightarrow} ddfeB'' \stackrel{L}{\Rightarrow} ddfegB' \stackrel{L}{\Rightarrow} ddfegh$
We leave to the reader the consideration of $\delta = \delta_2$.

Note that if we merely require the recognition of x' rather than the parsing of x', then Step 3(d) is unnecessary.

Let us now state and prove the theorems governing the time and space complexity of $M_{G'}$ relative to that of M_G.

Theorem 5.1

Let G be an arbitrary grammar form and suppose that there is a parsing method M_G for G and a function t(n), n ≥ 0, such that for each word x of length n M_G outputs all leftmost derivations of x in no more than t(n) steps. Let $G' \trianglelefteq_s G(\mu)$ and p(n), n ≥ 0, be a function such that for each word x' of length n in L(G',⇒) there are at most p(n) equally shaped derivations of that word. Then for some constant c and each word x' in Σ'^*, $|x'| = n$, $M_{G'}$, as defined above, yields either all leftmost derivations of x' if x' is in L(G',⇒) or rejects x' if x' is not in L(G',⇒) in at most c.t(n).p(n) steps.

Proof: The number of steps required by $M_{G'}$ for (1) and (2) is n and t(n) respectively. Assuming x is in L(G,⇒), let $\ell_1,\ldots,\ell_m$ be the lengths of $\delta_1,\ldots,\delta_m$ the leftmost derivations of x. Since $\delta_1,\ldots,\delta_m$ are produced in time at most t(n), we have

$\sum_{i=1}^{m} \ell_i \le t(n)$. Consider the number of steps required in (3). Now

(3a) involves a single scan of each δ_i, each replacement of a production involving c_1 steps, say. Hence (3a) takes at most

$\sum_{i=1}^{m} \ell_i c_1$ steps, that is at most $c_1.t(n)$ steps. Now (3b) takes $c_2.n$

steps, while in step (3c) each reduction can be carried out in constant time, c_3 say, since the number of possible "righthand sides" is bounded. Recall that the reduction process introduces sets of nonterminals in general. Clearly there are at most $(\#V')^q$ such reductions at each step (3c) where q is the maximum number of nonterminals in the right side of a production. Thus for each δ_i, $\hat{x}'$ can be parsed using a number of steps proportional to ℓ_i. Hence the number of steps in (3c) is at most

$\sum_{i=1}^{m} c_3.\ell_i \le c_3 t(n)$ steps. Finally in (3d) $M_{G'}$ need only traverse the

output of (3c) at most p(n) times and the number of steps required for each traversal is proportional to the length of δ_i. Step (3d) requires therefore at most

$\sum_{i=1}^{m} c_4.\ell_i.p(n) \le c_4.p(n).t(n)$ steps.

Thus by adding the bounds for steps (1), (2) and (3) we obtain the result. □

Eliminating step (3d) from M_G, we obtain a recognizer for G'. In this case we have:

Corollary 5.2

Let G be an arbitrary grammar form and suppose that there is a parsing method M_G for G and a function t(n), n ≥ 0, such that for each word x of length n M_G outputs all leftmost derivations of x in no more than t(n) steps and let $G' \trianglelefteq_s G(\mu)$. Then for some constant c and each word x' in Σ' |x'| = n, $M_{G'}$, as defined above, either accepts x' if x' is in L(G',⇒) or rejects x' if x' is not in L(G',⇒) in at most c.t(n) steps.

In particular if G is unambiguous this Corollary indicates that the time taken to recognize whether a word x' is in L(G',⇒) or not, is of the same order as the time taken to recognize that $\mu^{-1}(x')$ is or is not in L(G,⇒).

We now consider the space requirements of $M_{G'}$.

Theorem 5.3

Let G = (V,Σ,P,S) be an arbitrary grammar form. Suppose there exist functions s(n) and ℓ(n), n ≥ 0 and a parsing procedure M_G for G with the following properties:

(1) For each word x in Σ* and using at most s(|x|) space, M_G consecutively determines each leftmost G-derivation of x if x is in L(G,⇒) and rejects x if x is not in L(G,⇒).

(2) The length of each G-derivation of each word x in L(G,⇒) is at most ℓ(|x|).

Then for each $G' \trianglelefteq_s G(\mu)$ there exists a constant c such that for each word x' in Σ'*, $M_{G'}$ using at most s(|x'|) + c.ℓ(|x'|) space, yields either a leftmost derivation of x' if x' is in L(G',⇒) or rejects x' if x' is not in L(G',⇒).

Proof: Without loss of generality assume that step (3) is executed for each derivation δ_i of x as soon as δ_i is determined in Step (2).

Steps (1) and (2) of $M_{G'}$ require |x'| and s(|x'|) space respectively. Step (3a) requires at most $c_1\ell(|x'|)$ space since each derivation step adds exactly one pair of parentheses. Step (3b) can also be carried out in $c_1.\ell(|x'|)$ space, while Step (3c) requires at most $c_2.\ell(|x'|)$ space since at each derivation step $M_{G'}$ has only to record a finite set of nonterminals. Now Step (3d) can be carried out in at most $c_3.\ell(|x'|)$ space, since to generate all possible G'-derivations of x', $M_{G'}$ adds at each derivation step only a finite set of pointers. Finally by adding the above bounds and noting that $|x'| \leq c_4.\ell(|x'|)$ for some c_4, the upper bound

$$s(|x'|) + c.\ell(|x'|)$$

on the space requirement of $M_{G'}$ is obtained. □

II.5.2 Precedence Relations

It has already been mentioned in Section 5.1 that under g-interpretations the relationship of a parsing method M_G for a master grammar G to an "indirect" parsing method $M_{G'}$ for a g-interpretation grammar $G' \triangleleft_g G$ is unclear. In this section the property of conflict freeness is considered and it is demonstrated that at least for precedence parsing, only s-interpretations make sense in the parsing context. We consider non-decreasing and length preserving interpretations as well as g- and s-interpretations.

We first need to define some restricted kinds of g-interpretation.

Definition

Let $G = (V,\Sigma,P,S)$, $G' = (V',\Sigma',P',S')$ and $G' \triangleleft_g G(\mu)$. Then

(i) G' is a <u>nondecreasing</u> interpretation, $G' \triangleleft_{nd} G(\mu)$, if for all X in V and for all α in $\mu(X)$, $|\alpha| \geq 1$.

(ii) G' is a <u>length preserving</u> interpretation, $G' \triangleleft_{\ell p} G(\mu)$, if $\mu(X) \subseteq V'$ for all X in V.

(iii) G' is a <u>unitary</u> interpretation, $G' \triangleleft_{ut} G(\mu)$ if $\#\mu(X) = 1$ for all X in V, $\mu(A) = A$, for all A in $V - \Sigma$ and $P' = \mu(P)$.

(iv) G' is a <u>unary</u> interpretation, $G' \triangleleft_{un} G(\mu)$ if $G' \triangleleft_{ut} G(\mu)$ and $\#\Sigma' = 1$, that is G' is a unary grammar.

(v) G' is a unary length preserving interpretation, $G' \triangleleft_{u\ell p} G(\mu)$ if it is both a unary and a length preserving interpretation.

If G is a reduced grammar form, then for each a there is exactly one $u\ell p$-interpretation with $\Sigma' = \{a\}$. Further when $G' \triangleleft_{u\ell p} G$, G' is strong form equivalent to G.

The notion of unitary interpretation is commonly encountered in the definition of programming languages. Usually a programming language is defined in terms of reference symbols, which in a particular implementation become implementation symbols. Hence the implementation grammar is a unitary interpretation of the reference grammar. This can be seen in ALGOL 60 or PASCAL, where begin, end, if and so on occur in the reference language, and BEGIN, END, IF for example, appear in some implementation languages.

We now recall the notions of (Wirth-Weber) precedence relations on context free grammars.

Definition

Let $G = (V,\Sigma,P,S)$ be a grammar (form). For all X, Y in V, write

(i) $X \doteq Y$ if there is a production $A \to \alpha XY\beta$ in P, for some A in $V - \Sigma$ and α, β in V^*.

(ii) $X \lessdot Y$ if there is a production $A \to \alpha XC\beta$ in P and a derivation $C \Rightarrow^+ Y\gamma$, for some A, C in $V - \Sigma$ and α, β, γ in V^*.

(iii) $X \gtrdot Y$ if Y is in Σ, there is a production $A \to \alpha BU\beta$ in P, and derivations $B \Rightarrow^+ \gamma X$ and $U \Rightarrow^* Y\delta$, for some A, B in $V - \Sigma$, U in V, α, β, γ, δ in V^*.

The relations $\doteq$, $\lessdot$, $\gtrdot$ are called (Wirth-Weber) precedence relations.

A grammar G is said to be conflict free if at most one precedence relation holds between every ordered pair (X,Y) of symbols in V. A collection $\mathcal{G}$ of grammars is said to be conflict free if each grammar in $\mathcal{G}$ is conflict free.

The importance of this notion of conflict freeness stems from the following definition of a simple precedence grammar. A grammar G is said to be a simple precedence grammar if it is conflict free and uniquely invertible. A grammar G is uniquely invertible if each right hand side of a production in G does not occur as the right hand side of more than one production in G.

However in the remainder of this section we only study the relation of conflict free grammar forms to their x-interpretation grammars.

We say a grammar is separated if the right hand side of each production is either a terminal word or a nonterminal word, but not a mixture. Formally, letting $G = (V,\Sigma,P,S)$, G is separated if for all $A \to \alpha$ in P, α is in $\Sigma^* \cup (V-\Sigma)^*$.

Theorem 5.4

Let $G = (V,\Sigma,P,S)$ be a grammar form with the property that for each nonterminal A in $V - \Sigma$ there exists a word x in Σ^+ such that $A \Rightarrow^+ x$. Then the following three conditions are equivalent.

(i) $\mathcal{G}_g(G)$ is conflict free.

(ii) $\mathcal{G}_{nd}(G)$ is conflict free.

(iii) G is separated and whenever a production $A \to \alpha$ is in P with α in $(V - \Sigma)^+$, then α is in $V - \Sigma$.

Proof: Clearly (i) implies (ii).

(ii) implies (iii). Let x be a word in Σ^+ such that $S \Rightarrow^+ x$ in G, and let a be some symbol in x. Consider $G' \underset{nd}{\triangleleft} G(\mu)$ where $G' = (V',\{a\},P',S)$, $\mu(A) = A$ for all A in $V - \Sigma$, $\mu(b) = \{a,aa\}$ for all b in Σ, $P' = \mu(P)$ and $V' = (V - \Sigma) \cup \{a\}$. Then there is a production $A \to \alpha aa\beta$ in P' for some A in $V - \Sigma$ and α, β in V'^*. Therefore $a \doteq a$.

Define a homomorphism h on V^* by $h(A) = A$ for all A in $V - \Sigma$ and $h(b) = a$ for all b in Σ. We first demonstrate that G is separated. Assume otherwise, in which case there is a production $B \to \alpha Cb\beta$ or $B \to \alpha bC\beta$ in P for some B, C in $V - \Sigma$, b in Σ, and α, β in V^*. From the hypothesis there is a derivation $C \Rightarrow^+ y$ for some nonempty terminal word y. Then either

$$B \Rightarrow h(\alpha)Cah(\beta) \Rightarrow^+ h(\alpha)a^{|y|}ah(\beta) \text{ in } G'$$

or

$$B \Rightarrow h(\alpha)aCh(\beta) \Rightarrow^+ h(\alpha)a\,a^{|y|}h(\beta) \text{ in } G'.$$

Therefore either $a \gtrdot a$ or $a \lessdot a$ also holds in G'. Hence G' has conflicts, a contradiction, therefore G is separated. Now assume there is a production $B \to \alpha CD\beta$ in G, for some B, C, D in $V - \Sigma$ and α, β in V^*. By the hypothesis, there are derivations $C \Rightarrow^+ x_1$ and $D \Rightarrow^+ x_2$ in G, where both x_1 and x_2 are nonempty. Then

$$B \Rightarrow h(\alpha)CDh(\beta) \Rightarrow^+ h(\alpha)h(x_1)Dh(\beta) \Rightarrow^+ h(\alpha)h(x_1)h(x_2)h(\beta)$$

in G'. Since $h(x_i) = a^{|x_i|}$, $i = 1, 2$ we have $a \gtrdot a$. Thus G' has conflicts, a contradiction, therefore whenever there is a production $X \to \gamma$ in P with γ in $(V - \Sigma)^+$, then γ is in $V - \Sigma$.

(iii) implies (i)

Assume (iii) holds. Let $G' = (V',\Sigma',P',S') \underset{g}{\triangleleft} G(\mu)$. By the hypothesis, there are no A, B in $V' - \Sigma'$ and a in Σ' such that $A \doteq B$, $a \doteq A$ or $A \doteq a$. However it is possible that $a \doteq b$, for some terminals a and b. Now since there are no productions of type $A \to \alpha$ in P', α in $(V - \Sigma)^+$, $|\alpha| > 1$, there are no relations $X \lessdot Y$ or $X \gtrdot Y$ for any symbols X, Y in V'. Therefore G' is conflict-free. □

Consider the grammar G defined by the productions:

$S \to AB$; $A \to a$; $B \to \lambda$.

Then $\mathcal{G}_g(G)$ is conflict free, but G has a production $C \to \gamma$ with γ in $(V - \Sigma)^2$. This demonstrates that the condition: for all A in $V - \Sigma$, there is a nonempty terminal word x such that $A \Rightarrow^+ x$; cannot be removed and still have Theorem 5.4 valid.

Condition (iii) in Theorem 5.4 is very strong and implies G has no self-embedding variable. Therefore $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(FIN)$. In fact is can be proved that $\mathcal{L}_g(G,\Rightarrow) = \mathcal{L}(FIN)$ if either $\mathcal{G}_g(G)$ or $\mathcal{G}_{nd}(G)$ is conflict free and $L(G,\Rightarrow)$ contains at least one non-empty word.

Turning to s-interpretations we strengthen our claim that this is the appropriate interpretation mechanism in the parsing context, since the characterization result is particularly simple.

Theorem 5.5

For each grammar form G, $\mathcal{G}_s(G)$ is conflict free iff G is conflict free.

Proof: Clearly if $\mathcal{G}_s(G)$ is conflict free then G is conflict free, therefore consider the converse.

Let $G' \triangleleft_s G(\mu)$, where $G = (V,\Sigma,P,S)$ and $G' = (V',\Sigma',P',S')$. Now for each word x' in $\mu(V^*)$, the set $\mu^{-1}(x') = \{x\colon x \text{ in } V^* \text{ and } x' \text{ is in } \mu(x)\}$ contains exactly one element. Then for each derivation in G',

$A' \Rightarrow^+ \alpha'$

there is a derivation in G

$\mu^{-1}(A') \Rightarrow^+ \mu^{-1}(\alpha')$.

It immediately follows that if $X' \doteq, \lessdot, \gtrdot Y'$ in G', then $\mu^{-1}(X') \doteq, \lessdot, \gtrdot \mu^{-1}(Y')$ in G, respectively. Therefore if G' has conflicts so does G, a contradiction. Hence $\mathcal{G}_s(G)$ is conflict free. □

Having demonstrated that s-interpretations preserve conflict freeness we now turn to ℓp-interpretations.

Theorem 5.6

For each grammar form G, $\mathcal{G}_{\ell p}(G)$ is conflict free iff there is some conflict free G', where $G \triangleleft_{u\ell p} G$.

Proof: Clearly if $\mathcal{G}_{\ell p}(G)$ is conflict free then for all $G' \underset{u\ell p}{\lhd} G$, $G' \underset{\ell p}{\lhd} G$ hence G' is conflict free. Consider the converse implication. First note that when $G' \underset{u\ell p}{\lhd} G$ then $G \underset{\ell p}{\lhd} G'$, hence G' and G are strongly ℓp-form equivalent. Since G' has only one terminal symbol, $\mathcal{G}_{\ell p}(G') = \mathcal{G}_s(G')$, therefore by Theorem 5.5 $\mathcal{G}_{\ell p}(G')$ is conflict free. □

It is well known that for each context-free language L there exists a conflict free grammar G with $L(G,\Rightarrow) = L$. We now strengthen this result.

Theorem 5.7

For each context-free language L, there exists a grammar form G such that $L = L(G,\Rightarrow)$ and $\mathcal{G}_s(G)$ (and $\mathcal{G}_{\ell p}(G)$) is conflict free.

Proof: Without loss of generality assume L is λ-free. Now $L = L(F,\Rightarrow)$ for some λ-free grammar F and there exists a grammar G which fulfills the following conditions:

(i) $L(F,\Rightarrow) = L(G,\Rightarrow)$ and G is obtained effectively from F,

(ii) G is conflict free, and

(iii) G only has productions of types $A \to a$ and $A \to BC$, and further for each production of type $A \to BC$, the nonterminal B does not appear in the second position of the right hand side of any production in G.

Such a G is constructed in Theorem 5.9. Consider a unary ℓp-interpretation G' of G. Since neither $a \doteq b$ nor $a \lessdot b$ are possible in G, then G' is conflict free. The result follows by Theorems 5.5 and 5.6. □

In the above proof we have used the fact that for each grammar there is an equivalent conflict free grammar. We turn to the consideration of when this result can be generalized for strong x-form equivalence and x-form equivalence, x = g, nd, ℓp and s. For strong x-form equivalence we give necessary and sufficient conditions, while for x-form equivalence we show that every grammar has a g-form equivalent conflict free grammar.

Theorem 5.8

For each grammar form G, the following hold:

(1) Let F be the grammar obtained by replacing each occurrence of

a terminal by a new terminal in some minimal grammar form G' strongly g-form equivalent to G. Then G has a strongly g-form equivalent conflict free grammar form iff F is conflict free.

(2) For x = nd or ℓp, G has a strongly x-form equivalent conflict free grammar form iff the grammar form obtained from G by replacing each occurrence of a terminal by a new terminal is conflict free.

(3) G has a strongly s-form equivalent conflict free grammar form iff G is conflict free.

Proof: In all cases the "if" case is obvious, therefore we only consider the "only if" cases in the following.

(1) Suppose $G_1 = (V_1,\Sigma_1,P_1,S_1)$ is strongly g-form equivalent to G and is conflict free. Then by Theorem 3.15 there exists a minimal grammar form $G_2 = (V_1,\Sigma_1,P_2,S_1)$ strongly g-form equivalent to G_1 with $P_2 \subseteq P_1$. Since G_1 is conflict free so is G_2. Let G_3 be the grammar form obtained by replacing each occurrence of a terminal by a new terminal in the productions of G_2. Clearly G_3 is conflict free, minimal and strongly g-form equivalent to G. Let G_4 be the form obtained from G_3 by deleting all terminals immediately to the right of a terminal in P_3. Clearly G_4 is conflict free. Since both F and G_4 are minimal, symbol tight and strongly g-equivalent they are also pseudo-isomorphic by Theorem 3.18. Consider the following transformation:

(*) Let H_1 be a grammar with each occurrence of a terminal a different symbol. Let H_2 be obtained from H_1 by either replacing one terminal a by either ab or b, b a new terminal or conversely replacing an appearance of ab by a.
Clearly H_2 is conflict free iff H_1 is conflict free. Consider F and G_4. Since they are pseudo-isomorphic there exists a sequence of (*) transformations which applied to G_4 yields F. Hence F is conflict free.

(2) Let $G_1 = (V_1,\Sigma_1,P_1,S_1)$ be the form obtained from G by replacing each occurrence of a terminal by a new terminal. First consider the length preserving situation. Assume G has some strongly ℓp-form equivalent conflict free grammar form $G_2 = (V_2,\Sigma_2,P_2,S_2)$. Because of the transitivity of strong form equivalence we have $G_1 \underset{\ell p}{\lhd} G_2(\mu)$. Let V_3 be the symbols occurring in the productions of P_1. Since μ is length preserving, for each production

$$p: X_0 \rightarrow X_1 \ldots X_m,\ X_i \text{ in } V_1,$$

there is a production, say

$h(p)\colon Y_0 \to Y_1 \ldots Y_m$, Y_i in V_2,

such that X_i is in $\mu(Y_i)$ for all i. Let $h(X_i) = Y_i$, $0 \le i \le m$. The function h is well defined on V_3 since (i) for each nonterminal A in V_3 there is exactly one nonterminal B in V_2 such that A is in $\mu(B)$, and (ii) each terminal in V_3 occurs in exactly one production in P_1 and only once in that production. Hence h is a homomorphism on V_3^*. It is straightforward to verify that if $X \doteq, \lessdot, \gtrdot Y$ then $h(X) \doteq, \lessdot, \gtrdot h(Y)$, for all X, Y in V_3. Therefore G_2 has conflicts if G_1 has conflicts and since G_2 is conflict free G_1 must also be conflict free. Second consider the nondecreasing situation. Assume G has a strongly nd-form equivalent conflict free grammar form G_2. Let G_3 be the grammar form obtained from G_2 by replacing each terminal occurrence by a new terminal. As in the length preserving case G_3 is conflict free and also strongly nd-form equivalent to G_1. Now in both G_3 and G_1 replace terminal subwords in their productions by the leftmost symbol in the terminal subword giving G_4 and G_5 respectively. Since this may be accomplished by a sequence of (*)-transformations G_4 is conflict free and G_5 is conflict free iff G_1 is conflict free. Clearly G_4 and G_5 are strongly nd-form equivalent, moreover they are strongly ℓp-form equivalent since no production in either G_4 or G_5 contains terminal subwords of length greater than one. Hence by the ℓp case above, G_5 is conflict free and therefore G_1 is conflict free.

(3) Suppose G has a strongly s-form equivalent conflict free grammar form G'. By Theorem 5.5 $\mathcal{G}_s(G')$ is conflict free and moreover, since G is in $\mathcal{G}_s(G')$, G is conflict free. □

For the final result of this section we show that each grammar form has a g-form equivalent conflict free grammar form, thereby strengthening a well known result for context free grammars.

Theorem 5.9

Every grammar form $G = (V,\Sigma,P,S)$ has an x-form equivalent conflict free grammar form F, where x = g, nd, ℓp or s.

Proof: If $L(G,\Rightarrow) = \{\lambda\}$, then let $F = (\{S\},\emptyset,\{S\to\lambda\},S)$. Clearly F is conflict free and x-form equivalent to G. Consider the case that G is nontrivial. Then by Theorem 4.12 we may assume G is in Chomsky Normal Form. Construct $F = (V_F,\Sigma,P_F,S)$ as follows:

Let $V_F = V \cup \{\hat{A}\colon A \text{ is in } V - \Sigma\}$, and

$$P_F = \{A \to a,\ \hat{A} \to a\colon A \to a \text{ is in } P,\ a \text{ in } \Sigma\} \cup \{A \to \hat{B}C,\ \hat{A} \to \hat{B}C\colon A \to BC \text{ is in } P\}.$$

It is well known that F is conflict free. Since $F \underset{x}{\lhd} G$, immediately $\mathcal{L}_x(F,\Rightarrow) \subseteq \mathcal{L}_x(G,\Rightarrow)$. Conversely let $G' = (V',\Sigma',P',S') \underset{x}{\lhd} G(\mu)$. For each nonterminal A in V', let $\hat{A}$ be a new nonterminal. Define a substitution μ' on V_F by $\mu'(X) = \mu(X)$, for all X in V and $\mu'(\hat{A}) = \{\hat{B}\colon B \text{ is in } \mu(A)\}$ for all A in $V - \Sigma$. Let

$$P'_F = \{A \to w,\ \hat{A} \to w\colon A \to w \text{ is in } P',\ w \text{ in } \Sigma^*\} \cup \{A \to \hat{B}C,\ \hat{A} \to \hat{B}C\colon A \to BC \text{ is in } P',\ B, C \text{ in } V' - \Sigma'\}$$

and $F' = (V'_F,\Sigma',P'_F,S')$ where $V'_F = V' \cup \{\hat{A}\colon A \text{ is in } V' - \Sigma'\}$. Then $F' \underset{x}{\lhd} F(\mu')$ and $L(F',\Rightarrow) = L(G',\Rightarrow)$. Hence $\mathcal{L}_x(G,\Rightarrow) \subseteq \mathcal{L}_x(F,\Rightarrow)$ completing the proof. □

II.5.3 Pushdown Acceptor Forms

We develop the notion of a pda form under both g- and s-interpretations.

Recall that a pushdown acceptor (pda) is a sextuple $M = (Q,\Sigma,\Gamma,H,Z_0,q_0)$ where Q is a finite set of states, Σ is the input alphabet, Γ is the pushdown alphabet, H is a finite subset of $Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \times \Gamma^* \times Q$ of moves, Z_0 is the initial pushdown symbol and q_0 is the initial state. Often a set of final states is also given, but for our purposes this is an unncessary addition as will be seen. Also H the move set is usually given as a transition function $\delta\colon Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \to 2^{\Gamma^* \times Q}$, however this is clearly an equivalent formulation.

Given a pda M define the relation $\vdash$ on $Q \times \Sigma^* \times \Gamma^*$ by:

$(p,x,\gamma) \vdash (q,x',\gamma')$ if $x = zx'$, $\gamma = Z\delta$ and

$\gamma' = \delta'\delta$ where (p,z,Z,δ',q) is in H.

Observe that the stack γ is being read from left to right. Let $\overset{*}{\vdash}$ be the reflexive transitive closure of $\vdash$ and let $\text{Null}(M) = \{x\colon x \text{ in } \Sigma^*,\ (q_0,x,Z_0) \overset{*}{\vdash} (q,\lambda,\lambda) \text{ for some } q \text{ in } Q\}$. It is well known that the family of all Null(M) is identical to $\mathcal{L}(\text{CF})$.

A pda form is a pda, in the same way that a grammar form is a grammar.

Definition

Let M_1 and M_2 be pda forms, where $M_i = (Q_i,\Sigma_i,\Gamma_i,H_i,Z_{i,0},q_{i,0})$ $i = 1, 2$. We say M_2 is an s-interpretation of M_1 modulo μ, where μ is a dfl-substitution from $Q_1 \cup \Sigma_1 \cup \Gamma_1$ into $Q_2 \cup \Sigma_2 \cup \Gamma_2$, if

(i) $\mu(Q_1) \subseteq Q_2$,

(ii) $\mu(\Sigma_1) \subseteq \Sigma_2$,

(iii) $\mu(\Gamma_1) \subseteq \Gamma_2$,

(iv) $H_2 \subseteq \mu(H_1)$, where $\mu(H_1) = \bigcup_{h \text{ in } H_1} \mu(h)$ and

$\mu((p,x,Z,\gamma,q)) = \mu(p) \times \mu(x) \times \mu(Z) \times \mu(\gamma) \times \mu(q)$,

(v) $Z_{2,0}$ is in $\mu(Z_{1,0})$, and

(vi) $q_{2,0}$ is in $\mu(q_{1,0})$.

We denote this by $M_2 \trianglelefteq_s M_1(\mu)$, or simply $M_2 \trianglelefteq_s M_1$ when μ is understood. The family of pda forms defined by a pda form M is denoted $\mathcal{M}_s(M)$ and defined by:

$$\mathcal{M}_s(M) = \{M' \trianglelefteq_s M(\mu)\text{: for some substitution } \mu \text{ and pda } M'\}.$$

Similary by $\mathcal{L}_s(M)$ we denote the family of languages given by M, defined by:

$$\mathcal{L}_s(M) = \{\text{Null}(M')\text{: } M' \text{ is in } \mathcal{M}_s(M)\}.$$

These are known as the s-pda family and the s-pda language family of M, respectively.

Before proving our major result, namely that for each grammar form G there is a pda form M such that $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}_s(M)$ and vice versa, we prove a normal form result for s-pda families. For each 1-state pda form M and each s-interpretation M' of M there is an s-interpretation M" of M such that M" is a 1-state pda and Null(M") = Null(M').

Notation

For each pda $M = (Q,\Sigma,\Gamma,H,Z_0,q_0)$ and for $Z_1,\ldots,Z_k$ in Γ, $k \geq 1$, and q, q' in Q let

$$\langle q,Z_1 \ldots Z_k,q'\rangle = \{(q,Z_1,q_1)(q_1,Z_2,q_2) \ldots (q_{k-1},Z_k,q')\text{: } q_1,\ldots,q_{k-1} \text{ in } Q\}.$$

Let $\bar{M} = (Q,\Sigma,(Q\times\Gamma\times Q) \cup \{Z_0\}, \bar{H},Z_0,q_0)$ be the pda in which $\bar{H}$ is defined by:

(i) if $h = (q_0,x,Z_0,\lambda,q)$ is in H then h is in $\bar{H}$.

(ii) if (q,x,Z,λ,q') is in H then $(q,x,(q,Z,q'),\lambda,q')$ is in $\bar{H}$.

(iii) if (q_0,x,Z_0,γ,q) is in H, $\gamma \neq \lambda$, then (q_0,x,Z_0,γ',q) is in $\bar{H}$ for all γ' in $\langle q,\gamma,q'\rangle$ and all q' in Q.

(iv) if (q,x,Z,γ,q') is in H, $\gamma \neq \lambda$, then $(q,x,(q,Z,q"),\gamma',q')$ is in $\bar{H}$ for all γ' in $\langle q',\gamma,q"\rangle$ and all q" in Q.

Let $\bar{\bar{M}} = (\{q_0\},\Sigma,(Q\times\Gamma\times Q) \cup \{Z_0\},\bar{\bar{H}},Z_0,q_0)$ be the pda in which (q_0,x,Z,γ,q_0) is in $\bar{\bar{H}}$ if (q,x,Z,γ,q') is in $\bar{H}$ for some q, q' in Q.

Observe that, by construction, in $\bar{M}$ a move of the type $(q,x,(p,Z,p')\gamma) \vdash (q',x',\gamma')$ may only take place when p = q and, secondly, $\gamma = \gamma'$, that is an erasing move takes place, only when $p' = q'$.

We relate M, $\bar{M}$ and $\bar{\bar{M}}$ in the following lemma.

Lemma 5.10

Let M, $\bar{M}$ and $\bar{\bar{M}}$ be defined as above. Then $\text{Null}(\bar{M}) = \text{Null}(\bar{\bar{M}}) = \text{Null}(M)$.

Proof: We first sketch a proof that $\text{Null}(\bar{M}) = \text{Null}(M)$. Consider an accepting move sequence in M:

$$(q_0,x_1\ldots x_m,Z_0) \vdash (q_1,x_2\ldots x_m,\gamma_1) \vdash \ldots \vdash (q_{m-1},x_m,\gamma_{m-1}) \vdash (q_m,\lambda,\lambda)$$

where x_i is in $\Sigma \cup \{\lambda\}$, $1 \le i \le m$.

Now γ_{m-1} is in Γ and since it is erased, then either $\gamma_{m-1} = Z_0$ in which case $(q_{m-1},x_m,\gamma_{m-1}) \vdash (q_m,\lambda,\lambda)$ is in $\bar{M}$ also or $\gamma_{m-1} \ne Z_0$ in which case $(q_{m-1},x_m,(q_{m-1},\gamma_{m-1},q_m)) \vdash (q_m,\lambda,\lambda)$ is in $\bar{M}$ by the previous observations. Moreover, in both cases, the corresponding move in $\bar{M}$ is uniquely determined. By induction on i there is a unique δ_i in $\langle q_i,\gamma_i,q_m\rangle$ for each i, $1 \le i \le m-1$. Hence in $\bar{M}$ there is a uniquely determined accepting move sequence:

$$(q_0,x_1\ldots x_m,Z_0) \vdash (q_1,x_2\ldots x_m,\delta_1) \vdash \ldots \vdash (q_{m-1},x_m,\delta_{m-1}) \vdash (q_m,\lambda,\lambda)$$

with δ_i in $\langle q_i,\gamma_i,q_m\rangle$. Thus $\text{Null}(M) \subseteq \text{Null}(\bar{M})$.

Conversely, given an accepting move sequence:

$$(q_0,x_1 \ldots x_m,Z_0) \vdash \ldots \vdash (q_{m-1},x_m,\delta_{m-1}) \vdash (q_m,\lambda,\lambda)$$

in $\bar{M}$, then δ_i is in $\langle q_i,\gamma_i,q_m\rangle$ for some unique γ_i and therefore:

$$(q_0,x_1 \ldots x_m,Z_0) \vdash \ldots \vdash (q_{m-1},x_m,\gamma_{m-1}) \vdash (q_m,\lambda,\lambda)$$

is an accepting move sequence in M. We have now shown that $\text{Null}(\bar{M}) \subseteq \text{Null}(M)$ and hence $\text{Null}(\bar{M}) = \text{Null}(M)$.

We close by demonstrating that $\text{Null}(\bar{\bar{M}}) = \text{Null}(\bar{M})$. By definition of $\bar{\bar{M}}$, for each accepting move sequence:

$$(q_0,x_1 \ldots x_m,Z_0) \vdash (q_1,x_2 \ldots x_m,\delta_1) \vdash \ldots \vdash (q_m,\lambda,\lambda)$$

in $\bar{M}$ there is an accepting move sequence:

$$(q_0,x_1 \ldots x_m,Z_0) \vdash (q_0,x_2 \ldots x_m,\delta_1) \vdash \ldots \vdash (q_0,\lambda,\lambda)$$

in $\bar{\bar{M}}$. Conversely, given an accepting move sequence in $\bar{\bar{M}}$ the δ_i, $1 \le i \le m-1$ uniquely determine the corresponding q_i, $1 \le i \le m$. Hence there is a uniquely determined accepting move sequence in $\bar{M}$. This gives the result. □

Lemma 5.11

Let M, $\bar{M}$ and $\bar{\bar{M}}$ be given as above. Then $\bar{M} \underset{s}{\triangleleft} M$, $\bar{M} \underset{s}{\triangleleft} \bar{\bar{M}}$ and moreover if M is a 1-state pda and $M' \underset{s}{\triangleleft} M(\mu)$ for some pda M', then $\bar{\bar{M}}' \underset{s}{\triangleleft} M$.

Proof: (i) $\bar{M} \underset{s}{\triangleleft} M$. Define a dfl-substitution μ by:

$\mu(q) = q$, for all q in Q,

$\mu(a) = a$, for all a in Σ,

$\mu(Z_0) = \{Z_0\} \cup (Q \times \{Z_0\} \times Q)$ and

$\mu(Z) = Q \times \{Z\} \times Q$, for all Z in $\Gamma - \{Z_0\}$.

Clearly $\bar{H} \subseteq \mu(H)$, hence $\bar{M} \underset{s}{\triangleleft} M(\mu)$.

(ii) $\bar{M} \underset{s}{\triangleleft} \bar{\bar{M}}$. Define a dfl-substitution μ by:

$\mu(q_0) = Q$,

$\mu(X) = X$ for all X in $\Sigma \cup \Gamma$.

Then $\bar{H} \subseteq \mu(\bar{\bar{H}})$ by construction, hence $\bar{M} \underset{s}{\triangleleft} \bar{\bar{M}}(\mu)$.

(iii) If #Q > 1 then $\bar{\bar{M}}$, and hence $\bar{\bar{M}}'$, cannot be an s-interpretation of M. When M is a 1-state pda define a dfl-substitution $\bar{\bar{\mu}}$ by:

$\bar{\bar{\mu}}(q_0) = q_0$,

$\bar{\bar{\mu}}(a) = \mu(a)$, for all a in Σ,

$\bar{\bar{\mu}}(Z_0) = \{Z_0'\} \times (Q' \times \mu(Z_0) \times Q')$ and

$\bar{\bar{\mu}}(Z) = Q' \times \mu(Z) \times Q'$, for all Z in $\Gamma - \{Z_0\}$.

Immediately $\bar{\bar{H}}' \subseteq \bar{\bar{\mu}}(H)$, hence $\bar{\bar{M}}' \underset{s}{\triangleleft} M(\bar{\bar{\mu}})$. □

Our promised "normal form" result now follows:

Theorem 5.12

Let M be a 1-state pda form. Then for all $M' \underset{s}{\triangleleft} M$ there exists an $\bar{\bar{M}}' \underset{s}{\triangleleft} M$ with $Null(\bar{\bar{M}}') = Null(M')$ and $\bar{\bar{M}}'$ a 1-state pda.

This result for pda families corresponds to the well known result that the family of 1-state pda's generates the family of all pda languages, that is all context-free languages. We now proceed to strengthen this correspondence to show that every pda language family is a grammatical family and vice versa.

Consider the following well known 1-state pda M_G corresponding to a given grammar G.

Definition

Let $G = (V, \Sigma, P, S)$ be a grammar. Let M_G, the corresponding 1-state pda of G, be the pda $(\{q_0\}, \Sigma, V, H, S, q_0)$ where

$H = \{(q_0,a,a,\lambda,q_0)\colon a \text{ in } \Sigma\} \cup \{(q_0,\lambda,A,\alpha,q_0)\colon A \to \alpha \text{ is in } P\}$.

It is well known that $\mathrm{Null}(M_G) = L(G,\Rightarrow)$, for all context-free grammars G.

Theorem 5.13

For every grammar G, $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}_s(M_G)$.

Proof: Letting $G' \trianglelefteq_s G(\mu)$, where $G' = (V',\Sigma',P',S')$ the pda corresponding to G' is given by:

$M' = (\{q_0\},\Sigma',V',H',S',q_0)$, where $H' = \{(q_0,a',a',\lambda,q_0)\colon a \text{ in } \Sigma'\}$
$\cup \{(q_0,\lambda,A',\alpha',q_0)\colon A' \to \alpha' \text{ is in } P\}$.

Let $\bar{\mu}$ be a dfl-substitution defined by:

$\bar{\mu}(q_0) = q_0$ and

$\bar{\mu}(X) = \mu(X)$, for all X in V.

Clearly $M' \trianglelefteq_s M_G(\bar{\mu})$ and $\mathrm{Null}(M') = L(G',\Rightarrow)$ as pointed out above. Therefore $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(M_G)$.

Consider the reverse inclusion. Since M_G is a 1-state pda we need only consider 1-state s-interpretations of M_G by Theorem 5.12. Let $M' = (\{q_0\},\Sigma',\Gamma',H',Z_0',q_0)$ and $M' \trianglelefteq_s M(\mu)$. Since $H' \subseteq \mu(H)$ each move in H' is either of the form:

(i) (q_0,a',a'',λ,q_0), where a' and a" are in $\mu(a)$, for some a in Σ, or of the form

(ii) $(q_0,\lambda,A',\alpha',q_0)$, where $A' \to \alpha'$ is in $\mu(A \to \alpha)$ for some $A \to \alpha$ in P.

Type (i) moves mean that a" on the pushdown can match a symbol a' in the input. Define $\tau(a'') = \{a'\colon (q_0,a',a'',\lambda,q_0) \text{ is in } H'\}$, for all a" in Σ' and $\tau(X) = X$, for all X in $\{q_0\} \cup \Gamma$. Note that τ is not necessarily a dfl-substitution but $\tau(\mu(a)) \subseteq \mu(a)$ for all a in Σ, hence $\tau\mu$ is a dfl-substitution. Let $G' = (\Gamma' \cup \Sigma',\Sigma',P',Z_0')$ where $P' = \{A' \to \alpha''\colon \alpha'' \text{ is in } \tau(\alpha'),\ (q_0,\lambda,A',\alpha',q_0) \text{ is in } H'\}$. Then $G' \trianglelefteq_s G(\tau\mu)$. The proof that $\mathrm{Null}(M') = L(G',\Rightarrow)$ is straightforward and is omitted. Thus $\mathcal{L}_s(M_G) \subseteq \mathcal{L}_s(G,\Rightarrow)$ completing the proof. □

We now turn to the converse result.

Definition

Let $M = (Q,\Sigma,\Gamma,H,Z_0,q_0)$ be a pda and S a new symbol. Define the corresponding grammar $G_M = (\{S\} \cup \Sigma \cup (Q\times\Gamma\times Q),\Sigma,P,S)$ by specifying P as follows:

(i) if (q_0,x,Z_0,λ,q') is in H, then $S \to x$ is in P.

(ii) if (q,x,Z,λ,q') is in H, then $(q,Z,q') \to x$ is in P.

(iii) if $(q_0,x,Z_0,Z_1\ldots Z_r,q)$ is in H, then
$S \to x(q,Z_1,q_1)(q_1,Z_2,q_2)\ \ldots\ (q_{r-1},Z_r,q_r)$ is in P, for all $q_1,\ldots,q_r$ in Q.

(iv) if $(q,x,Z,Z_1\ldots Z_r,q')$ is in H, then
$(q,Z,q'') \to x(q',Z_1,q_1)(q_1,Z_2,q_2)\ \ldots\ (q_{r-1},Z_r,q'')$
is in P for all $q'',q_1,\ldots,q_r$ in Q.

It is well known that $L(G_M,\Rightarrow) = \mathrm{Null}(M)$.

Theorem 5.14

For every pda M, $\mathcal{L}_s(G_M,\Rightarrow) = \mathcal{L}_s(M)$.

Proof: Let $M' = (Q',\Sigma',\Gamma',H',Z_0',q_0')$ and $M' \triangleleft_s M(\mu)$. Then $G' = (\{S\} \cup \Sigma' \cup (Q'\times\Gamma'\times Q'),\Sigma',P',S)$ can be constructed in the manner specified above and it is clear that $G' \triangleleft_s G_M(\bar{\mu})$, where $\bar{\mu}$ is defined by:

$\bar{\mu}(S) = S$, $\bar{\mu}(a) = \mu(a)$, for all a in Σ, and

$\bar{\mu}((p,Z,q)) = \{(p',Z',q')\colon p'$ in $\mu(p)$, Z' in $\mu(Z)$ and q' in $\mu(q)\}$, for all (p,Z,q) in $Q\times\Gamma\times Q$.

Hence $\mathcal{L}_s(M) \subseteq \mathcal{L}_s(G_M,\Rightarrow)$. Consider the converse result. Let $G' \triangleleft_s G_M(\mu)$ where $G' = (V',\Sigma',P',S')$. Also let $\bar{M} = (Q,\Sigma,(Q\times\Gamma\times Q) \cup \{Z_0\},\bar{H},Z_0,q_0)$ be the specific s-interpretation pda of M introduced earlier. Since $\bar{M} \triangleleft_s M$ it suffices to demonstrate a pda $\bar{M}' \triangleleft_s \bar{M}(\bar{\mu})$ with $\mathrm{Null}(\bar{M}') = L(G',\Rightarrow)$. Define $\bar{\mu}$ as follows:

$\bar{\mu}(X) = X$, for all X in $Q \cup \Sigma$,

$\bar{\mu}(Z_0) = \mu(S)$ and $\bar{\mu}((p,Z,q)) = \mu((p,Z,q))$ for all (p,Z,q) in $Q\times\Gamma\times Q$.

Clearly $\bar{\mu}$ is a dfl-substitution. Let $\bar{M}' = (Q,\Sigma',V',H',S',q_0)$, where H' consists of the following moves:

(i) if $S' \to x'$ is in P', $S' \to x'$ is in $\mu(S \to x)$ and $S \to x$ is type (i) (by which we mean $S \to x$ is obtained from the move (q_0,x,Z_0,λ,q') in H, fulfilling condition (i) in the definition of G_M), then let (q_0,x',S',λ,q') be in H'.

(ii) if $A \to x'$ is in P', $A \to x'$ is the image of a type (ii) production coming from (q,x,Z,λ,q') then let (q,x',A,λ,q') be in H'.

(iii) if $S' \to x'A_1\ \ldots\ A_r$ is in P', the image of a type (iii) production coming from $(q_0,x,Z_0,Z_1\ldots Z_r,q)$ then let $(q_0,x',S',A_1\ldots A_r,q)$ be in H'.

(iv) if $A \to x'A_1\ldots A_r$ is in P', the image of a type (iv) production coming from $(q,x,Z,Z_1\ldots Z_r,q')$ in H, let $(q,x',A,A_1\ldots A_r,q')$ be in H'.

Clearly $\bar{M}' \underset{s}{\lhd} \bar{M}(\bar{\mu})$. Finally, note that in G_M, S only appears on the lefthand side of productions, all other nonterminals are in $Q \times \Gamma \times Q$. It follows that in a leftmost derivation in G', $S' \overset{L}{\Rightarrow}^* \alpha$ in G' implies that $\alpha = u\beta$ for some u in Σ'^* and β in $\mu(Q \times \Gamma \times Q)^*$. By induction on the length of the derivation, $n \geq 1$, it can be shown that:

$S' \overset{L}{\Rightarrow}^n u\beta$ in G' iff there is an n-step move sequence from (q_0, u, S') to (q, λ, β) for some q in Q.

Thus $\text{Null}(\bar{M}') = L(G', \Rightarrow)$, completing the proof. □

We can summarize the results so far in the following theorem.

Theorem 5.15

For each family of languages $\mathcal{L}$, $\mathcal{L}$ is an s-pda language family iff $\mathcal{L}$ is an s-grammatical family.

We now turn to g-interpretations of pda forms.

First observe that the definition of g-interpretation for pda forms, which is analogous to that for grammar forms, will normally produce acceptors as interpretations which are not pdas. This is because the image of an input symbol may be a word. We therefore allow a more general definition of a pda in which (p, x, Z, γ, q) is an allowable move even when $|x| > 1$. We call this a generalized pda or simply gpda. The definition of a move sequence is appropriately modified, in which case it is clear that if M_1 is a gpda then there exists a pda M_2 such that $\text{Null}(M_1) = \text{Null}(M_2)$. In other words no generative power is added.

We say M_1 is a g-interpretation of M_2 modulo μ, denoted $M_2 \underset{g}{\lhd} M_1(\mu)$, if $M_i = (Q_i, \Sigma_i, \Gamma_i, H_i, Z_{i,0}, q_{i,0})$, $i = 1, 2$ and μ is a finite substitution from $Q_1 \cup \Sigma_1 \cup \Gamma_1$ to $Q_2 \cup \Sigma_2^* \cup \Gamma_2$, which fulfills conditions (i), (iii), (iv), (v) and (vi) in the definition of s-interpretations and (ii) is replaced by:

(ii') $\mu(\Sigma_1) \subseteq \Sigma_2^*$.

We obtain $\mathcal{M}_g(M)$ and $\mathcal{L}_g(M)$ analogously to $\mathcal{M}_s(M)$ and $\mathcal{L}_s(M)$.

The following results are straightforward and are left to the reader.

Proposition 5.16

For each gpda form M_1 there is a strong g-form equivalent pda form M_2.

Proposition 5.17

For each pda form M, and hence each gpda form, $\mathcal{L}_g(M) = \mathcal{H}(\mathcal{L}_s(M))$.

Our main result now follows straightforwardly.

Theorem 5.18

For each language family $\mathcal{L}$ the following are equivalent statements:

(i) $\mathcal{L}$ is a g-grammatical family.

(ii) $\mathcal{L}$ is a g-pda language family.

(iii) $\mathcal{L}$ is a g-gpda language family.

Proof: (ii) ≡ (iii) follows from Proposition 5.16. We show that (i) ≡ (ii). Now by Theorem 5.15 a language family $\mathcal{L}$ is s-grammatical iff it is an s-pda language family.

Now $\mathcal{L} = \mathcal{L}_g(G,\Rightarrow)$, for some grammar form G
iff $= \mathcal{H}(\mathcal{L}_s(G,\Rightarrow))$
iff $= \mathcal{H}(\mathcal{L}_s(M))$, for some pda form M by Theorem 5.15
iff $= \mathcal{L}_g(M)$. Hence the result. □

We have demonstrated in this section that the analogue for pda's of s- and g-interpretations for grammar forms yields exactly the same families of languages. This means that pda forms correspond to grammar forms in the same way that pdas corresponds to grammars. Thus for each s- or g-grammatical family we immediately have available a parsing algorithm for the whole family. In the same way that deterministic pda (dpda) have been studied as models of realistic parsing techniques (linear in the length of the input word) for grammars, so the study of dpda within an s- or g-pda family may now be investigated with the same end in view. This is an area of investigation for which little is known at the time of writing.

In closing we mention one further result which the interested reader may prove for himself.

Proposition 5.19

For every gpda M_1 there is a pda M_2 such that $\mathcal{L}_s(M_1) = \mathcal{L}_s(M_2)$.

Thus even under s-interpretations gpda's are no more powerful than pda's.

II.6 Dense Collections of Grammatical Families

The framework of grammar forms provides us not only with the twin notions of a family of grammars and a grammatical family with respect to a given grammar but also it provides us with collections of these in a natural way. In Section II.3 the collection of the families of grammars defined by context-free grammar forms was studied. In the present section we study the corresponding collection of context-free grammatical families under s-interpretations. We show, in Section 6.2, that the collection of s-grammatical families is dense in the sense that given two families $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$ and $\mathcal{L}_1$ contains all finite sets then there always exists a family $\mathcal{L}_3$ properly in between, that is

$$\mathcal{L}_1 \subsetneqq \mathcal{L}_3 \subsetneqq \mathcal{L}_2.$$

Second, in Section 6.3 we abstract the notions that enable such a density result to be proved using the techniques of the first section. This leads in a natural way to collections of language families fulfilling some basic properties, which we call MSW spaces. We demonstrate that such collections are not a rare occurrence, in that given an arbitrary collection $\mathcal{M}$ we can always obtain an MSW space by closing $\mathcal{M}$ under some operators in a fixed finite sequence, which is exactly the closure of $\mathcal{M}$ under these particular operators. Finally, in Section 6.4, we demonstrate a density result for two-symbol-s-grammatical families, which leads to the decidability of form equivalence for sub-linear two-symbol-s-grammatical families. These two density results are of interest since they are established in two very different ways and also because such results have not been forthcoming in the past when generative devices have been studied.

II.6.1 Preliminary Notions

In order to prove the results on density it is convenient to introduce some notation and also a number of language and language family operations.

We say that a collection $\mathcal{M}$ of language families is <u>dense</u> if for any two language families $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{M}$ with $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$ there exists a language family $\mathcal{L}$ in $\mathcal{M}$ strictly in between, that is $\mathcal{L}_1 \subsetneqq \mathcal{L} \subsetneqq \mathcal{L}_2$. Two language families $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{M}$ with $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$ are said to be a <u>dense pair</u> with respect to $\mathcal{M}$ if $\mathcal{M}(\mathcal{L}_1, \mathcal{L}_2) = \{\mathcal{L} \text{ in } \mathcal{M} : \mathcal{L}_1 \subseteq \mathcal{L} \subseteq \mathcal{L}_3\}$ is dense, we normally write $(\mathcal{L}_1, \mathcal{L}_2)$ is a dense pair. If $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$ and there is no $\mathcal{L}_3$ in $\mathcal{M}$ such that $\mathcal{L}_1 \subsetneqq \mathcal{L}_3 \subsetneqq \mathcal{L}_2$, then $\mathcal{L}_2$ is a <u>successor</u> of $\mathcal{L}_1$. We say in $\mathcal{L}_1$ in $\mathcal{M}$ is <u>density forcing</u> with respect to $\mathcal{M}$ if

$\mathcal{M}[\mathcal{L}_1] = \{\mathcal{L} \text{ in } \mathcal{M}: \mathcal{L}_1 \subseteq \mathcal{L}\}$ is dense.

In the next section we will prove that $\mathcal{L}$(REG) is density forcing with respect to the collection of all s-grammatical families and also that ($\mathcal{L}$(REG), $\mathcal{L}$(CF)) is a dense pair.

We now introduce some necessary language and language family operations.

Let L_1 and L_2 be two languages over disjoint alphabets. Then the superdisjoint union of L_1 and L_2, denoted by $L_1 \uplus L_2$, is simply the union of L_1 and L_2. Note that the superdisjoint union is only defined if the two languages are over disjoint alphabets. Similarly we define a kind of inverse of this operation. Let L be a language over some alphabet Σ. Then a language L_1 is obtained by breaking L with respect to an alphabet $\Sigma_1 \subseteq \Sigma$ if $L_1 = L \cap \Sigma_1^*$ and $L - L_1 \subseteq (\Sigma-\Sigma_1)^*$, that is $L - L_1$ does not contain any word containing a symbol of Σ_1. We say that L is coherent if it cannot be broken in a non-trivial manner. So if L_1 is obtained by breaking L then either $L_1 = L$ or $L_1 = \emptyset$.

The operation of superdisjoint union can be extended to language families as follows.

The superdisjoint wedge of two language families $\mathcal{L}_1$ and $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2$, is defined by: $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2 = \{L_1 \uplus L_2: L_1$ in $\mathcal{L}_1$ and L_2 in $\mathcal{L}_2\}$.

Another useful operation for both languages and language families is that of removing all words of a given length from a language or from all languages in a family. Let L be a language and $\mathcal{L}$ a language family. Then for $i \geq 1$ we denote by L(i) the language defined by $\{x$ is in L: $|x| \neq i\}$. Similarly by $\mathcal{L}(i)$, we denote the language family $\{L(i) = L$ is in $\mathcal{L}\}$. We call L(i) an extraction of L and $\mathcal{L}(i)$ an extraction of $\mathcal{L}$.

A language family $\mathcal{L}$ is closed under covering if for every infinite language L, the fact that L(i) is in $\mathcal{L}$ for infinitely many i implies that L is in $\mathcal{L}$.

We will provide a grammatical characterization of superdisjoint wedge by way of the following operation on grammars.

Let $G_i = (V_i, \Sigma_i, P_i, S_i)$ be two (context-free) grammars such that S_i does not occur on the right hand side of any production in P_i for i = 1, 2 and suppose that $V_1 \cap V_2 = \emptyset$. Define a new grammar $G_1 \oplus G_2 = ((V_1 \cup V_2)-\{S_2\}, \Sigma_1 \cup \Sigma_2, P_1 \cup P_2', S_1)$, where P_2' is P_2 with S_2 replaced by S_1. We say $G_1 \oplus G_2$ is the direct sum of G_1 and G_2.

Note that we can always rename the alphabet of a grammar form without changing its grammatical family and moreover without any loss of generality we may assume its sentence symbol does not appear on the right hand side of any production. Therefore for any two grammar forms we can always define their direct sum.

Let us now state some straightforward but important results about grammar forms with respect to the operations defined above.

Lemma 6.1

Let $G_i = (V_i, \Sigma_i, P_i, S_i)$ fulfill the above conditions for $i = 1, 2$. Then

(i) $\mathcal{L}_s(G_1 \oplus G_2, \Rightarrow) = \mathcal{L}_s(G_1, \Rightarrow) \dot{\vee} \mathcal{L}_s(G_2, \Rightarrow)$, $L(G_1 \oplus G_2, \Rightarrow) = L(G_1, \Rightarrow) \dot{\cup} L(G_2, \Rightarrow)$ and if L is a coherent language in $\mathcal{L}_s(G_1 \oplus G_2, \Rightarrow)$ then L is either in $\mathcal{L}_s(G_1, \Rightarrow)$ or in $\mathcal{L}_s(G_2, \Rightarrow)$.

(ii) If $G_i \triangleleft_s G$ for some G, $i = 1, 2$, then $G_1 \oplus G_2 \triangleleft_s G$.

(iii) $\mathcal{L}_s(G, \Rightarrow)$ is closed under superdisjoint union for each grammar form G.

(iv) $\mathcal{M} = \{\mathcal{L}_s(G, \Rightarrow)$: G is a context-free grammar form$\}$ is closed under superdisjoint wedge.

(v) $\mathcal{L}_s(G, \Rightarrow)$ is closed under both breaking and extraction for each grammar form G.

(vi) $\mathcal{M}$ is closed under extraction.

Proof: (i) Follows from the definitions, noting that if $G \triangleleft_s G_1 \oplus G_2$ then G can be decomposed into the direct sum of G_1' and G_2' such that $G_1' \triangleleft_s G_1$, $G_2' \triangleleft_s G_2$ and $G = G_1' \oplus G_2'$. The final statement follows directly from the notion of a coherent language.

(ii) Clear.

(iii) Let L_1, L_2 be arbitrary languages in $\mathcal{L}_s(G, \Rightarrow)$ over disjoint alphabets and $G_i \triangleleft_s G$ be two grammars such that $L(G_i, \Rightarrow) = L_i$, $i = 1, 2$ and G_1 and G_2 fulfill the direct sum conditions. Then $G_1 \oplus G_2$ is well-defined, $L(G_1 \oplus G_2, \Rightarrow) = L_1 \dot{\cup} L_2$ and $G_1 \oplus G_2 \triangleleft_s G$ by (ii).

(iv) Consider two arbitrary s-grammatical families $\mathcal{L}_s(G_1, \Rightarrow)$ and $\mathcal{L}_s(G_2, \Rightarrow)$. We may assume by the previous remarks that G_1 and G_2 fulfill the direct sum conditions hence $\mathcal{L}_s(G_1 \oplus G_2, \Rightarrow) = \mathcal{L}_s(G_1, \Rightarrow) \dot{\vee} \mathcal{L}_s(G_2, \Rightarrow)$ by (i).

(v) Let $L_1 \subseteq \Sigma_1^*$ be an arbitrary language in $\mathcal{L}_s(G, \Rightarrow)$ and $\Sigma_2 \subseteq \Sigma_1$. Since s-grammatical families are closed under intersection with regular sets, $L_2 = L_1 \cap \Sigma_2^*$ is in $\mathcal{L}_s(G, \Rightarrow)$. Hence s-grammatical families are closed under breaking. Letting $L \subseteq \Sigma^*$ be an arbitrary language in $\mathcal{L}_s(G, \Rightarrow)$ and $i \geq 1$ be an integer then $L(i) = L \cap (\Sigma^* - \Sigma^i)$. Hence s-grammatical families are closed under extraction.

(vi) Consider an arbitrary grammar form $G = (V,\Sigma,P,S)$ and an integer $i \geq 1$. Letting $\mathcal{L} = \mathcal{L}_s(G,\Rightarrow)$ then it should be clear that $\mathcal{L}(i) = \mathcal{L}_s(G_i,\Rightarrow)$, where G_i is obtained from G by the usual intersection with regular sets construction, that is $L(G_i,\Rightarrow) = L(G,\Rightarrow) \cap (\Sigma^*-\Sigma^i)$. □

II.6.2 Denseness and s-grammatical Families

We have now prepared the way for the main result of this section, namely $\mathcal{L}$(REG) is density forcing for s-grammatical families.

Theorem 6.2

Let $\mathcal{M}$ be the collection of s-grammatical families and $\mathcal{L}$ in $\mathcal{M}$ be a family containing all the finite sets. Then $\mathcal{L}$ is density forcing.

Proof: We have to show that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are arbitrary families in $\mathcal{M}$ with $\mathcal{L} \subseteq \mathcal{L}_1 \subsetneqq \mathcal{L}_2$, then there exists $\mathcal{L}_3$ in $\mathcal{M}$ such that $\mathcal{L}_1 \subsetneqq \mathcal{L}_3 \subsetneqq \mathcal{L}_2$. We proceed by a number of claims.

Claim 1: $\mathcal{L}_2 - \mathcal{L}_1$ contains an infinite coherent language L.

Proof of Claim: Clearly $\mathcal{L}_2 - \mathcal{L}_1$ contains only infinite languages, since both $\mathcal{L}_1$ and $\mathcal{L}_2$ contain all the finite languages. Moreover since $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$ there must be at least one such language, say L. Let $L \subseteq \Sigma^*$ and assume L is not coherent, since if it is coherent L is the required language. Since L is not coherent we can break it in a nontrivial way into $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_3^*$ such that $\Sigma_2 = \Sigma - \Sigma_1$, $L_1 \cup L_2 = L$ and hence $L_1 \uplus L_2 = L$. Moreover $\Sigma_1 \neq \Sigma \neq \Sigma_2$ and at least one of L_1 and L_2 must be infinite; let this be L_1. Now $\#\Sigma_1 < \#\Sigma$ by definition and letting L_1 be L we can repeat this process if L_1 is not coherent. Clearly this construction must terminate after a finite number of steps with an L which is coherent. Furthermore the resulting L must be in $\mathcal{L}_2 - \mathcal{L}_1$ since s-grammatical families are closed under breaking. □

Claim 2: Let L be a language in $\mathcal{L}_2 - \mathcal{L}_1$, then there exists an integer $p \geq 1$ such that $L(p) \neq L$ and $L(p)$ is not in $\mathcal{L}_1$.

Proof of Claim 2: Since L is infinite there exist infinitely many values of p such that $L(p) \neq L$. Now assume for each such p that $L(p)$ is in $\mathcal{L}_1$. Let $G_1 = (V_1,\Sigma_1,P_1,S_1)$ be a grammar form such that $\mathcal{L}_s(G_1,\Rightarrow) = \mathcal{L}_1$ and let Δ be the alphabet of L. Now there are only finitely many dfl-substitutions mapping Σ_1 into the set of subsets of Δ. Since every $L(p) \neq L$ is in $\mathcal{L}_1$ and there are infinitely many such $L(p)$, then there are two distinct integers m and n such that $L(m)$ and $L(n)$ are obtained by the same dfl-substitution as far as terminals are concerned. Let H_m and H_n be the interpretations of G_1 generating $L(m)$ and $L(n)$ respectively. We may assume that the nonterminal alphabets of H_m and H_n are disjoint without any loss of generality. Now construct the sum of H_m and H_n as for the direct sum except that the terminal alphabets are identical and let H be the resulting grammar. Clearly $L(H,\Rightarrow) = L_m \cup L_n = L$, $H \trianglelefteq_s G_1$ and therefore L is in $\mathcal{L}_1$. This is in contradiction to the assumption that L is not in $\mathcal{L}_1$. □

We are now able to establish the theorem by way of our final claim.

Claim 3: Let L be a coherent language in $\mathcal{L}_2 - \mathcal{L}_1$, $p \geq 1$ be an integer such that $L(p) \neq L$ and $L(p)$ is not in $\mathcal{L}_1$ and $H_p \trianglelefteq_s G_2$ has $L(H_p,\Rightarrow) = L(p)$. Then $\mathcal{L}_3 = \mathcal{L}_s(G_1 \oplus H_p,\Rightarrow)$ is strictly in between $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof of Claim 3: First observe that we can always assume $G_1 \oplus H_p$ is well defined, by suitably renaming the alphabet of G_1 if necessary. Second $\mathcal{L}_1 = \mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_3$ follows from the definition of the direct sum. Proper inclusion follows since $L(p)$ is in $\mathcal{L}_3 - \mathcal{L}_1$. Third, consider the relationship of $\mathcal{L}_2$ and $\mathcal{L}_3$. Consider an arbitrary language L' in $\mathcal{L}_3$. L' can be expressed as $L'' \uplus L'''$ where L" is in $\mathcal{L}_1$ and L''' is in $\mathcal{L}_s(H_p,\Rightarrow)$, since $\mathcal{L}_3 = \mathcal{L}_1 \,\forall\, \mathcal{L}_s(H_p,\Rightarrow)$. However L" and L''' are both in $\mathcal{L}_2$, since $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and $H_p \trianglelefteq_s G_2$ implies $\mathcal{L}_s(H_p,\Rightarrow) \subseteq \mathcal{L}_2$. Hence $L'' \uplus L''' = L'$ is in $\mathcal{L}_2$ because s-grammatical families are closed under superdisjoint union (Lemma 6.1). Thus we have shown that $\mathcal{L}_3 \subseteq \mathcal{L}_2$. Proper inclusion follows from Lemma 6.1(i) since L is coherent and is neither in $\mathcal{L}_1$ nor in $\mathcal{L}_s(H_p,\Rightarrow)$. □

This completes the proof of the theorem. □

While we conjecture that the condition that $\mathcal{L}$ contain all finite languages is a necessary and sufficient condition for $\mathcal{L}$ to be density forcing in $\mathcal{M}$ this remains an open question. Assuming this conjecture to be true implies that $\mathcal{M}(\mathcal{L}(\text{REG}), \mathcal{L}(\text{CF}))$ forms a <u>maximal</u> dense pair with respect to $\mathcal{M}$. In other words there is no $\mathcal{L} \subsetneq \mathcal{L}(\text{REG})$ such that $\mathcal{M}(\mathcal{L}, \mathcal{L}(\text{CF}))$ is a dense pair.

We now show that the problem of maximality for $\mathcal{M}$, the collection of s-grammatical families, is reducible to a particular question about finite forms.

Consider any grammar form F_1 such that $\mathcal{L}_s(F_1,\Rightarrow) \subsetneq \mathcal{L}(REG)$. Letting the terminal alphabet of F_1 be $\{a_1,\ldots,a_n\}$, $n > 0$, then by Lemma 4.22 we know that there are positive integers k_i, $i = 1,\ldots,n$ such that:

$a_i^{k_i}$ is not in $L(F_1,\Rightarrow)$, for $i = 1,\ldots,n$.

Let $k = \max(\{k_i: 1 \le i \le n\}) + 1$ and add to F_1 new nonterminals and right linear productions generating the language

$\{a_1,\ldots,a_n\}^* - \{a_1^{k_1},\ldots,a_n^{k_n}\}$,

this can always be done since we may assume the sentence symbol of F_1 does not appear on the right hand side of any production.

Let the resulting grammar form be denoted by F. Clearly $\mathcal{L}_s(F_1,\Rightarrow) \subseteq \mathcal{L}_s(F,\Rightarrow) \subsetneq \mathcal{L}(REG)$.

Now let d, $d_1,\ldots,d_{n+1}$ be new terminal symbols and let $\Sigma = \{d_1,\ldots,d_{n+1}\}$. Define the finite forms D_1 and D_2 by:

$L(D_1,\Rightarrow) = \Sigma^{k_1} \cup \ldots \cup \Sigma^{k_n} - \{d_i^{k_j}: 1 \le i \le n+1, 1 \le j \le n\}$ and

$L(D_2,\Rightarrow) = \{d^{k_j}: 1 \le j \le n\}$.

For finite forms it is easy to show that the language of the form completely specifies the form as far as its language family is concerned. Consider $F \oplus D_1$ and $F \oplus D_2$. Clearly $\mathcal{L}_s(F,\Rightarrow) \subseteq \mathcal{L}_s(F \oplus D_1,\Rightarrow) \subseteq \mathcal{L}_s(F \oplus D_2,\Rightarrow) \subseteq \mathcal{L}(REG)$, since $F \trianglelefteq_s D_i$, $i = 1, 2$ and $D_1 \trianglelefteq_s D_2$. All these containments are proper if we assume that at least one k_i is different from one. For considering each of them in turn we have:

(a) $L(D_1,\Rightarrow)$ is not in $\mathcal{L}_s(F,\Rightarrow)$. For assume otherwise, in which case there is an $F' \trianglelefteq_s F(\mu)$, for some μ such that $L(F',\Rightarrow) = L(D_1,\Rightarrow)$. However this means μ maps $\{a_1,\ldots,a_n\}$ onto $\{d_1,\ldots,d_{n+1}\}$, that is there exists an a_i with $\#\mu(a_i) \ge 2$. Without loss of generality assume that d_1 and d_2 are in $\mu(a_i)$. Now there is a word x in $\{d_1,d_2\}^* \cap L(D_1,\Rightarrow)$ such that $|x| = k_i$ and hence $\mu^{-1}(x) = a_i^{k_i}$ is in $L(F,\Rightarrow)$. This is a contradiction, hence $\mathcal{L}_s(F,\Rightarrow) \subsetneq \mathcal{L}_s(F \oplus D_1,\Rightarrow)$.

(b) Observe that $L(D_2,\Rightarrow)$ is not in $\mathcal{L}_s(F \oplus D_1,\Rightarrow)$, since it is not in $\mathcal{L}_s(F,\Rightarrow)$ and it is not in $\mathcal{L}_s(D_1,\Rightarrow)$. Therefore $\mathcal{L}_s(F \oplus D_1,\Rightarrow) \subsetneq \mathcal{L}_s(F \oplus D_2,\Rightarrow)$.

(c) $\mathcal{L}_s(F \oplus D_2,\Rightarrow) \subsetneq \mathcal{L}(REG)$ since a^+ is neither in $\mathcal{L}_s(F,\Rightarrow)$ nor in $\mathcal{L}_s(D_2,\Rightarrow)$.

In the case that $k_1 = k_2 = \ldots = k_n = 1$ we have $L(F,\Rightarrow) = \{a_1,\ldots,a_n\}^* - \{a_1,\ldots,a_n\}$, $L(D_1,\Rightarrow) = \emptyset$ and $L(D_2,\Rightarrow) = \{d\}$. Thus $\mathcal{L}_s(F,\Rightarrow) = \mathcal{L}_s(F \oplus D_1,\Rightarrow)$ but $\mathcal{L}_s(F,\Rightarrow) = \mathcal{L}_s(F \oplus D_1,\Rightarrow) \subsetneqq \mathcal{L}_s(F \oplus D_2,\Rightarrow) \subsetneqq \mathcal{L}(\text{REG})$.

However in the following construction it is of no consequence whether or not $\mathcal{L}_s(F,\Rightarrow)$ is properly contained in $\mathcal{L}_s(F \oplus D_1,\Rightarrow)$. We proceed by defining two finite forms H_1 and H_2 obtained by taking all words of length $\le k$ from $L(F \oplus D_1,\Rightarrow)$ and $L(F \oplus D_2,\Rightarrow)$ respectively. We have the proper inclusion:

$$\mathcal{L}_s(H_1,\Rightarrow) \subsetneqq \mathcal{L}_s(H_2,\Rightarrow),$$

since $L(F \oplus D_1,\Rightarrow)$ and $L(F \oplus D_2,\Rightarrow)$ agree for all words of length $> k$. Letting $F^{(k)}$ be the finite form obtained from F by taking all words of length $\le k$ from $L(F,\Rightarrow)$, then $H_i = F^{(k)} \oplus D_i$, $i = 1, 2$. Note that proper inclusion always holds; even for the exceptional case. We now have the following preliminary lemma.

Lemma 6.3

If the pair $(\mathcal{L}_s(H_1,\Rightarrow), \mathcal{L}_s(H_2,\Rightarrow)$ is not dense then the pair $(\mathcal{L}_s(F \oplus D_1,\Rightarrow), \mathcal{L}_s(F \oplus D_2,\Rightarrow))$ is not dense.

Proof: By the assumption of the lemma there are two forms E_1 and E_2 such that

$$\mathcal{L}_s(H_1,\Rightarrow) \subseteq \mathcal{L}_s(E_1,\Rightarrow) \subsetneqq \mathcal{L}_s(E_2,\Rightarrow) \subseteq \mathcal{L}_s(H_2,\Rightarrow)$$

and moreover E_2 is a successor of E_1.

We claim that $E_i = F^{(k)} \oplus E_i'$, $i = 1, 2$, where $\mathcal{L}_s(D_1,\Rightarrow) \subseteq \mathcal{L}_s(E_1',\Rightarrow) \subsetneqq \mathcal{L}_s(E_2',\Rightarrow) \subseteq \mathcal{L}_s(D_2,\Rightarrow)$.

Consider $L(E_2,\Rightarrow)$. It can be written as

$$L(E_2,\Rightarrow) = L(\bar{F}^{(k)},\Rightarrow) \uplus L(E_2',\Rightarrow),$$

where $\bar{F}^{(k)} \trianglelefteq_s F^{(k)}$ and $E_2' \trianglelefteq_s D_2$. But we also have:

$$F^{(k)} \oplus D_1 \trianglelefteq_s \bar{F}^{(k)} \oplus E_2'.$$

Now $L(D_1,\Rightarrow)$ is not in $\mathcal{L}_s(F,\Rightarrow)$ unless $L(D_1,\Rightarrow) = \emptyset$. In either case we have $D_1 \trianglelefteq_s E_2'$. Since $L(F^{(k)},\Rightarrow) = \{a_1,\ldots,a_n\}^{\le k} - \{a_i^{k_i}: 1 \le i \le n\}$, is not in $\mathcal{L}_s(D_2,\Rightarrow)$ and hence not in $\mathcal{L}_s(E_2',\Rightarrow)$. Moreover $L(F^{(k)},\Rightarrow)$ is clearly coherent, therefore $L(F^{(k)},\Rightarrow)$ is in $\mathcal{L}_s(\bar{F}^{(k)},\Rightarrow)$. But this implies $F^{(k)} \trianglelefteq_s \bar{F}^{(k)} \trianglelefteq_s F^{(k)}$. Thus we have demonstrated that $E_2 = F^{(k)} \oplus E_2'$, where $D_1 \trianglelefteq_s E_2' \trianglelefteq_s D_2$.

By a similar argument we obtain

$$E_1 = F^{(k)} \oplus E_1',$$

where $D_1 \underset{s}{\lhd} E_1' \underset{s}{\lhd} D_2$. Since $E_1 \underset{s}{\lhd} E_2$ we also have $E_1' \underset{s}{\lhd} E_2'$ and hence

$$\mathcal{L}_s(D_1,\Rightarrow) \subseteq \mathcal{L}_s(E_1',\Rightarrow) \subseteq \mathcal{L}_s(E_2',\Rightarrow) \subseteq \mathcal{L}_s(D_2,\Rightarrow).$$

Since $\mathcal{L}_s(F^{(k)} \oplus E_1',\Rightarrow) \subsetneqq \mathcal{L}_s(F^{(k)} \oplus E_2',\Rightarrow)$ we conclude that $\mathcal{L}_s(E_1',\Rightarrow) \subsetneqq \mathcal{L}_s(E_2',\Rightarrow)$.

Moreover we claim that E_2' is a successor of E_1'. For if this is not the case there is a form E_3' such that $\mathcal{L}_s(E_1',\Rightarrow) \subsetneqq \mathcal{L}_s(E_3',\Rightarrow) \subsetneqq \mathcal{L}_s(E_2',\Rightarrow)$ and this implies that the form $F^{(k)} \oplus E_3'$ lies properly in between $\mathcal{L}_s(E_1,\Rightarrow)$ and $\mathcal{L}_s(E_2,\Rightarrow)$, a contradiction.

Having established the claim we now proceed with the lemma by defining:

$$G_i = F \oplus E_i',\ i = 1,\ 2.$$

This gives

$$\mathcal{L}_s(F \oplus D_1,\Rightarrow) \subseteq \mathcal{L}_s(G_1,\Rightarrow) \subsetneqq \mathcal{L}_s(G_2,\Rightarrow) \subseteq \mathcal{L}_s(F \oplus D_2,\Rightarrow).$$

That G_2 is a successor of G_1 follows by establishing the following claim.

Claim: Let G be a grammar form satisfying

$$\mathcal{L}_s(G_1,\Rightarrow) \subsetneqq \mathcal{L}_s(G,\Rightarrow) \subsetneqq \mathcal{L}_s(G_2,\Rightarrow).$$

Then there is a grammar form $G^{(k)}$ satisfying

$$\mathcal{L}_s(F^{(k)} \oplus E_1',\Rightarrow) \subsetneqq \mathcal{L}_s(G^{(k)},\Rightarrow) \subseteq \mathcal{L}_s(F^{(k)} \oplus E_2',\Rightarrow).$$

Proof of Claim: It is convenient to introduce some auxiliary notation. For a language L and a positive integer t, we denote by $L_{\le t}$ $(L_{>t})$ the subset of L consisting of all words of length $\le t$ $(>t)$.

For a grammar form H, the families $\mathcal{L}_s(H,\Rightarrow)_{\le t}$ and $\mathcal{L}_s(H,\Rightarrow)_{>t}$ are defined by:

$$\mathcal{L}_s(H,\Rightarrow)_{\le t} = \{L_{\le t}\colon L \text{ in } \mathcal{L}_s(H,\Rightarrow)\}$$

and

$$\mathcal{L}_s(H,\Rightarrow)_{>t} = \{L_{>t}\colon L \text{ in } \mathcal{L}_s(H,\Rightarrow)\}$$

For the G_1, G and G_2 of the claim we have:

$$\mathcal{L}_s(G_1,\Rightarrow)_{>k} = \mathcal{L}_s(G,\Rightarrow)_{>k} = \mathcal{L}_s(G_2,\Rightarrow)_{>k}.$$

It is also clear that:

$$\mathcal{L}_s(G_1,\Rightarrow)_{\le k} \subseteq \mathcal{L}_s(G,\Rightarrow)_{\le k} \subseteq \mathcal{L}_s(G_2,\Rightarrow)_{\le k}.$$

Now assume the claim is false, in other words either

$$\mathcal{L}_s(G_1,\Rightarrow)_{\le k} = \mathcal{L}_s(G,\Rightarrow)_{\le k} \text{ or } \mathcal{L}_s(G,\Rightarrow)_{\le k} = \mathcal{L}_s(G_2,\Rightarrow)_{\le k}.$$

We will only consider the first alternative since the second can be dealt with analogously.

Thus $\mathcal{L}_s(G,\Rightarrow)_{\le k} = \mathcal{L}_s(G_1,\Rightarrow)_{\le k} = \mathcal{L}_s(F^{(k)} \oplus E_1',\Rightarrow)$ and because we have both $\mathcal{L}_s(G_1,\Rightarrow)_{>k} = \mathcal{L}_s(G,\Rightarrow)_{>k}$ and $\mathcal{L}_s(G_1,\Rightarrow)_{\leq k} = \mathcal{L}_s(G,\Rightarrow)_{\leq k}$ the inclusion $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$ can be proper only if some combination of the "high" and "low" languages is possible in the family $\mathcal{L}_s(G,\Rightarrow)$, which is not possible in the family $\mathcal{L}_s(G_1,\Rightarrow)$. However, this contradicts the definition of G and the choice of k, because of the direct sum no new dependencies can be created in $\mathcal{L}_s(G,\Rightarrow)$. That is $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}_s(G_1,\Rightarrow)$, a contradiction. Thus the claim has been established. □

The lemma now follows since G_2 is a successor of G_1 and hence the pair $(\mathcal{L}_s(F,\Rightarrow), \mathcal{L}(REG))$ is not dense. □

This lemma immediately yields the following "reduction" theorem.

Theorem 6.4

The pair $(\mathcal{L}_s(H_1,\Rightarrow), \mathcal{L}_s(H_2,\Rightarrow))$ is dense iff the pair $(\mathcal{L}_s(F \oplus D_1,\Rightarrow), \mathcal{L}_s(F \oplus D_2,\Rightarrow))$ is dense.

Proof: if: Assume the pair $(\mathcal{L}_s(H_1,\Rightarrow), \mathcal{L}_s(H_2,\Rightarrow))$ is dense, then for all H_3 and H_4 such that

$$\mathcal{L}_s(H_1,\Rightarrow) \subseteq \mathcal{L}_s(H_3,\Rightarrow) \subsetneq \mathcal{L}_s(H_4,\Rightarrow) \subseteq \mathcal{L}_s(H_2,\Rightarrow)$$

there is an H_5 with

$$\mathcal{L}_s(H_3,\Rightarrow) \subsetneq \mathcal{L}_s(H_5,\Rightarrow) \subsetneq \mathcal{L}_s(H_4,\Rightarrow).$$

Now since $H_i = F^{(k)} \oplus D_i$, $i = 1, 2$, we must have

$$H_i = F^{(k)} \oplus C_i,\ i = 3, 4, 5$$

by similar arguments to those used in the proof of Lemma 6.3, where $D_1 \trianglelefteq_s C_i \trianglelefteq_s D_2$, $i = 1, 2, 3$ and $C_1 \trianglelefteq_s C_3 \trianglelefteq_s C_2$. Thus we have

$$\mathcal{L}_s(F \oplus D_1,\Rightarrow) \subseteq \mathcal{L}_s(F \oplus C_1,\Rightarrow) \subseteq \mathcal{L}_s(F \oplus C_3,\Rightarrow) \subseteq \mathcal{L}_s(F \oplus C_2,\Rightarrow) \subseteq \mathcal{L}_s(F \oplus D_2,\Rightarrow).$$

Moreover $\mathcal{L}_s(F \oplus C_1,\Rightarrow) \subsetneq \mathcal{L}_s(F \oplus C_3,\Rightarrow) \subsetneq \mathcal{L}_s(F \oplus C_2,\Rightarrow)$, since $\mathcal{L}_s(F^{(k)} \oplus C_1,\Rightarrow) \subsetneq \mathcal{L}_s(F^{(k)} \oplus C_3,\Rightarrow) \subsetneq \mathcal{L}_s(F \oplus C_2,\Rightarrow)$. In other words the pair $(\mathcal{L}_s(F \oplus D_1,\Rightarrow), \mathcal{L}_s(F \oplus D_2,\Rightarrow))$ is dense.

only if: Assume the pair $(\mathcal{L}_s(F \oplus D_1,\Rightarrow), \mathcal{L}_s(F \oplus D_2,\Rightarrow))$ is dense and the pair $(\mathcal{L}_s(H_1,\Rightarrow), \mathcal{L}_s(H_2,\Rightarrow))$ is not dense. Then Lemma 6.3 provides a contradiction.

This completes the theorem. □

Now the pair ($\mathcal{L}$(REG), $\mathcal{L}$(CF)) is maximal dense iff there is no sub-regular grammatical family $\mathcal{L}$ such that the pair ($\mathcal{L}$, $\mathcal{L}$(REG)) is dense. Although we have not settled this question we have reduced it via the above theorem to the problem of the denseness of a pair of specific finite forms.

The techniques developed in this section to establish the denseness of the pair ($\mathcal{L}$(REG), $\mathcal{L}$ (CF)) are inapplicable if we restrict our attention to the collection of two-symbol s-grammatical families. Thus Section 4 is devoted to establishing the denseness of this latter collection by use of alternative techniques.

II.6.3 MSW Spaces

Theorem 6.2 can be considerably generalized by considering arbitrary collections of language families satisfying certain basic properties necessary for proving a result akin to it; such collections are termed MSW spaces. After defining an MSW space we will prove the analogue of Theorem 6.2 and then show how an arbitrary collection can be turned into an MSW space in a particularly simple manner. This demonstrates not only that such spaces are easily obtained, but also that the abstraction is meaningful in that "most" MSW spaces are not generated by grammar forms.

A collection of $\mathcal{M}$ of language families is an MSW-space if it satisfies the following three conditions:

(i) Each $\mathcal{L}$ in $\mathcal{M}$ is closed under superdisjoint union and breaking.

(ii) $\mathcal{M}$ is closed under superdisjoint wedge.

(iii) For each infinite language L occurring in some language family of $\mathcal{M}$ there exist subsets L_i of L for i = 1, 2,... such that (a) and (b) hold:

(a) L is in a language family $\mathcal{L}$ of $\mathcal{M}$ iff L_i is in $\mathcal{L}$ for all i with $L_i \neq L$,

(b) If L belongs to $\mathcal{L}$ in $\mathcal{M}$, then for every i with $L_i \neq L$ there exists an $\mathcal{L}_i$ in $\mathcal{M}$ such that $\mathcal{L}_i \subseteq \mathcal{L}$, L_i is in $\mathcal{L}_i$ and L is not in $\mathcal{L}_i$.

Corollary 6.5

$\mathcal{M}$, the collection of s-grammatical families is an MSW space.

Proof: Conditions (i) and (ii) are contained in Lemma 6.1. Consider condition (iii). Let L_i = L(i) for i ≥ 1, then essentially condition (ii,a) has been proved under Claim 2 of Theorem 6.2. Similarly $\mathcal{L}_i$

of condition (iii,b) is defined by H_i in Claim 3 of the proof of Theorem 6.2. Assuming $H \triangleleft_s G_2$ is a grammar form with $L(H,\Rightarrow) = L$ and H_i is the grammar form fulfilling $L(H_i,\Rightarrow) = L \cap (\Delta^*-\Delta^i)$ when $L \subseteq \Delta^*$, then $\mathcal{L}_s(H_i,\Rightarrow)$ is the extraction of $\mathcal{L}_s(H,\Rightarrow)$ with respect to i. Clearly L is not in $\mathcal{L}_s(H_i,\Rightarrow)$ if $L \neq L(i)$, since words of length i do not appear in languages in $\mathcal{L}_s(H_i,\Rightarrow)$. □

It is now possible to generalize Theorem 6.2 considerably, namely:

Theorem 6.6

Let $\mathcal{M}$ be an MSW space and let $\mathcal{F}$ be the collection of all finite languages occurring in language families of $\mathcal{M}$. If $\mathcal{L}$ is any family of $\mathcal{M}$ containing $\mathcal{F}$, then $\mathcal{L}$ is density forcing.

Proof: This is left to the reader. □

The notion of an MSW space together with Theorem 6.6 enables many "dense" families to be exhibited. Letting $\mathcal{M}$ be a collection of language families, denote by $\mathcal{M}(i)$ for $i \geq 1$, the collection $\{\mathcal{L}(i)\colon\ \mathcal{L} \text{ is in } \mathcal{M}\}$.

Let $\mathcal{F}$ denote the collection of all finite languages, $\mathcal{M}_1$ be the collection of all context-free s-grammatical families, $\mathcal{M}_2$ the collection of all s-grammatical families (see Section IV.1), $\mathcal{M}_3$ the collection of all linear s-grammatical families and $\mathcal{M}_4$ the collection of all synchro-EOL grammatical families (see Section III.2). The following corollary can easily be shown.

Corollary 6.7

For all i, $1 \geq i \geq 4$, for all $j \geq 1$, if $\mathcal{L}$ is a language family in $\mathcal{M}_i(j)$ containing $\mathcal{F}(j)$ then $\mathcal{L}$ is density forcing with respect to $\mathcal{M}_i(j)$.

We now turn to the problem of "constructing" MSW spaces. First we establish an "invariance" theorem concerning closure under superdisjoint wedge and extraction.

Theorem 6.8

Let $\mathcal{M}$ be a collection of language families such that each family $\mathcal{L}$ of $\mathcal{M}$ is closed under superdisjoint union, intersection with regular sets and covering. Let $\mathcal{N}$ be the closure of $\mathcal{M}$ under superdisjoint wedge and extraction. Then each $\mathcal{L}$ in $\mathcal{N}$ is closed under superdisjoint union, intersection with regular sets and covering and $\mathcal{N}$ is in MSW space.

Proof: We first show that each $\mathcal{L}$ in $\mathcal{N}$ has the specified closure properties. Consider two arbitrary language families $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{M}$. By the assumptions of the theorem both $\mathcal{L}_1$ and $\mathcal{L}_2$ have the desired closure properties. We first establish the effect of closure under superdisjoint wedge by demonstrating that $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \mathcal{L}_2$ has the desired closure properties, that is these closure properties are invariant under superdisjoint wedge.

Now each L in $\mathcal{L}$ can be expressed as $L = L_1 \ddot{\cup} L_2$ by the definition of superdisjoint wedge (recall that L_1 and L_2 are over disjoint alphabets).

(i) Closure under $\ddot{\cup}$: Consider arbitrary languages L and M in $\mathcal{L}$, where $L = L_1 \ddot{\cup} L_2$ and $M = M_1 \ddot{\cup} M_2$ with L_i and M_i in $\mathcal{L}_i$, i = 1, 2. We need to show that $L \ddot{\cup} M$ is in $\mathcal{L}$. This implies L_1, L_2, M_1 and M_2 are over pairwise disjoint alphabets. Now $L \ddot{\cup} M = (L_1 \ddot{\cup} L_2) \ddot{\cup} (M_1 \ddot{\cup} M_2)$, hence this can be expressed as $(L_1 \ddot{\cup} M_1) \ddot{\cup} (L_2 \ddot{\cup} M_2)$, and because $L_i \ddot{\cup} M_i$ is in $\mathcal{L}_i$, i = 1, 2, then $L \ddot{\cup} M$ is in

(ii) Closure under $\cap R$: Consider an arbitrary regular set R, then $L \cap R = (L_1 \cap R) \ddot{\cup} (L_2 \cap R)$ and since $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed under intersection with regular sets, $L_i \cap R$ is in $\mathcal{L}_i$, i = 1, 2. Hence $L \cap R$ is in $\mathcal{L}$.

(iii) Closure under covering: We need to show that $\mathcal{L}$ is closed under covering in order to demonstrate the invariance of the three closure properties under superdisjoint wedge closure.
Again consider an arbitrary $L \subseteq \Sigma^*$ such that $L(i) \neq L$ is in $\mathcal{L}$ for infinitely many i. We need to show that L is in $\mathcal{L}$. Clearly, for all i, $L(i) = L_{1,i} \ddot{\cup} L_{2,i}$ with $L_{j,1}$ in $\mathcal{L}_j$, j = 1, 2. First observe that there are only a finite number of partitions of Σ into $\Sigma_1 \ddot{\cup} \Sigma_2$. Hence there is one partition $\Sigma = \Sigma_1 \ddot{\cup} \Sigma_2$ say, such that for infinitely many i we have:

$L(i) = L_{1,i} \ddot{\cup} L_{2,i}$ where $L_{1,i} \subseteq \Sigma_1^*$ and $L_{2,i} \subseteq \Sigma_2^*$,

that is where $L_{1,i} = M_1(i)$ with $M_1 = L \cap \Sigma_1^*$ and similarly $L_{2,i} = M_2(i)$ with $M_2 = L \cap \Sigma_2^*$.
Since for infinitely many i, $M_j(i)$ is in $\mathcal{L}_j$, for j = 1, 2 then M_j is in $\mathcal{L}_j$, for j = 1, 2 since $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed under covering. But $L = M_1 \ddot{\cup} M_2$ hence L is in $\mathcal{L}$ as claimed.

In the second part of the proof we consider the effect of extraction closure on the three closure operations. Thus for each $\mathcal{L}$ in $\mathcal{M}$ we need to show that for all $p > 0$, $\mathcal{L}(p)$ satisfies the required closure properties.

(iv) Closure under $\ddot{\cup}$: Consider two languages L and M in $\mathcal{L}(p)$. Now $L = L'(p)$ and $M = M'(p)$ for some L' and M' in $\mathcal{L}$. Since we wish to form $L \ddot{\cup} M$ we may assume that L and M are over disjoint alphabets, that is L'(p) and M'(p) are over disjoint alphabets. Since $\mathcal{L}$ is closed under intersection with regular sets, then not only are L' and M' in $\mathcal{L}$ but also L'(p) and M'(p) are in $\mathcal{L}$. Thus $L'(p) \ddot{\cup} M'(p)$ is in $\mathcal{L}$ and hence in $\mathcal{L}(p)$ as desired.

(v) Closure under $\cap$ R: Consider a language L in $\mathcal{L}(p)$ and a regular language R. Now $L = L'(p)$ for some L' in $\mathcal{L}$, $L \cap R = (L' \cap R)(p)$ and the result follows.

(vi) Closure under covering: Consider an arbitrary language L such that $L(i) \neq L$ is in $\mathcal{L}(p)$ for infinitely many i. Now $L(i) = L_i(p)$ for some L_i in $\mathcal{L}$, where L_i may or may not contain words of length p. However since $\mathcal{L}$ is closed under intersection with regular sets $L_i(p)$ is also in $\mathcal{L}$, that is L(i) is in $\mathcal{L}$ for infinitely many i. But this implies that L is in $\mathcal{L}$ and hence L(p) is in $\mathcal{L}(p)$ as desired.

Finally we need to show that $\mathcal{N}$ is indeed an MSW space. But this follows immediately from the construction of $\mathcal{N}$. □

Corollary 6.9

Consider an arbitrary family $\mathcal{L}$ of languages and the collection $\mathcal{K} = \{\{L\}: L \text{ is in } \mathcal{L}\}$. Close each language family {L} in $\mathcal{K}$ with respect to the operations of $\ddot{\cup}$, intersection with regular sets and covering yielding a collection $\mathcal{M}$ of language families. Close $\mathcal{M}$ under superdisjoint wedge and extraction to obtain $\mathcal{N}$. Then $\mathcal{N}$ is an MSW space.

Corollary 6.10

Let $\mathcal{N}$ be the collection of all language families consisting of context-free languages, such that each $\mathcal{L}$ in $\mathcal{N}$ is closed under $\ddot{\cup}$, intersection with regular sets and covering.

Then $\mathcal{N}$ is an MSW space.

Proof: The closure of $\mathcal{N}$ under $\dot{\vee}$ and extraction is $\mathcal{N}$ itself. □

Corollary 6.11

Let $\mathcal{S}$ be an arbitrary family of languages closed under union and intersection with regular sets. Let $\mathcal{N}$ be the collection of all subsets $\mathcal{L}$ of $\mathcal{S}$ which are closed under $\dot{\cup}$, intersection with regular sets and covering.

Then $\mathcal{N}$ is an MSW space.

Proof: By Theorem 6.8 it suffices to show that $\mathcal{N}$ is closed under $\dot{\vee}$ and extraction. Consider two families $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{N}$. Since $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2 = \{L_1 \dot{\cup} L_2 \colon L_i \text{ in } \mathcal{L}_i,\ i = 1, 2\}$, $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathcal{S}$ and $\mathcal{L}_1 \vee \mathcal{L}_2 \subseteq \mathcal{S}$ we have $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2 \subseteq \mathcal{S}$. However since $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed under $\dot{\cup}$, intersection with regular sets and covering, by Theorem 6.8, $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2$ has these closure properties. But by definition $\mathcal{N}$ contains all such subsets of $\mathcal{S}$, hence $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2$ is in $\mathcal{N}$.

By a similar argument we can show that if $\mathcal{L}$ is in $\mathcal{N}$ then $\mathcal{L}(p)$ is in $\mathcal{N}$ for all $p > 0$. Hence $\mathcal{N}$ is an MSW space. □

In Corollary 6.9 beginning with an arbitrary language family $\mathcal{L}$ we first formed its closure under $\dot{\cup}$, intersection with regular sets and covering to give a collection $\mathcal{M}$. Secondly we closed $\mathcal{M}$ under $\dot{\vee}$ and extraction to give a new collection $\mathcal{N}$ which is an MSW space. We now show that these closure operations need be applied only once in the prescribed order: intersection with regular sets, covering, $\dot{\cup}$, extraction and $\dot{\vee}$. This provides us with a simple means for constructing MSW spaces.

Lemma 6.12

Let $\mathcal{S}$ be an arbitrary language family and $\mathcal{L}$ its closure under $\dot{\cup}$. Then $\mathcal{S}$ is closed under intersection with regular sets and covering iff $\mathcal{L}$ is so closed.

Proof: Suppose L is in $\mathcal{L}$ and R is an arbitrary regular set. Then $L = L_1 \dot{\cup} \ldots \dot{\cup} L_n$ for some $n \geq 1$ with L_i in $\mathcal{S}$, $1 \leq i \leq n$. Immediately,

$$L \cap R = (L_1 \cap R) \dot{\cup} \ldots \dot{\cup} (L_n \cap R)$$

and since $L_i \cap R$ is in $\mathcal{S}$, $1 \leq i \leq n$, it follows that $L \cap R$ is in $\mathcal{L}$, by construction. Since $\mathcal{S} \subseteq \mathcal{L}$ the converse follows immediately.

Second, suppose for some $L \subseteq \Sigma^*$, $L(i) \neq L$ is in $\mathcal{L}$ for infinitely many i. We need to show that L is in $\mathcal{L}$.

Since Σ has only a finite number of partitions $\Sigma = \Sigma_1 \dot{\cup} \Sigma_2 \dot{\cup} \ldots \dot{\cup} \Sigma_k$ for $k \geq 1$ and $\Sigma_i \neq \emptyset$, $1 \leq i \leq k$, there exists one partition $\Sigma = \Sigma_1 \dot{\cup} \ldots \dot{\cup} \Sigma_k$ say, such that for infinitely many i, $L(i) = L_{1,i} \dot{\cup} \ldots \dot{\cup} L_{k,i}$ with $L_{j,i} \subseteq \Sigma_j^*$ and $L_{j,i}$ in $\mathcal{S}$, $1 \leq j \leq k$. Furthermore for this partitition $L_{j,i} = M_j(i)$ for $M_j = \Sigma_j^* \cap L$, $1 \leq j \leq k$. But for each j, $1 \leq j \leq k$, $M_j(i)$ is in $\mathcal{S}$ for infinitely many i, hence M_j is in $\mathcal{S}$ and therefore $L = M_1 \dot{\cup} \ldots \dot{\cup} M_k$ is in $\mathcal{L}$.

Again the converse is immediate. □

Lemma 6.13

Let $\mathcal{S}$ be an arbitrary language family and $\mathcal{L}$ its closure under covering. If $\mathcal{S}$ is closed under intersection with regular sets then $\mathcal{L}$ is so closed.

Proof: Assume that $\mathcal{S}$ is closed under intersection with regular sets. Given an arbitrary L in $\mathcal{L}$ and an arbitrary regular set R we need to show that $L \cap R$ is in $\mathcal{L}$. Clearly if L is in $\mathcal{S}$ then by assumption $L \cap R$ is in $\mathcal{S}$, therefore consider the case L is not in $\mathcal{S}$. Now $L(i) \neq L$ is in $\mathcal{S}$ for infinitely many i and for these i, $L(i) \cap R = L(i) \cap R(i) = (L \cap R)(i)$ is in $\mathcal{S}$. But by the construction this implies $L \cap R$ is in $\mathcal{L}$ as required. □

Lemma 6.14

Let $\mathcal{M}$ be a collection of language families and $\mathcal{N}$ be its closure under $\dot{\vee}$. If $\mathcal{M}$ is closed under extraction, then so is $\mathcal{N}$.

Proof: Let $\mathcal{L}$ be an arbitrary language family of $\mathcal{N}$. Then $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \ldots \dot{\vee} \mathcal{L}_k$, for some $k > 0$ and $\mathcal{L}_j$ in $\mathcal{M}$, $1 \leq j \leq k$. Clearly $\mathcal{L}(i) = \mathcal{L}_1(i) \dot{\vee} \ldots \dot{\vee} \mathcal{L}_k(i)$, for all $i > 0$ and since $\mathcal{M}$ is closed under extraction, $\mathcal{L}_j(i)$ is in $\mathcal{M}$, $1 \leq j \leq k$ and hence by construction $\mathcal{L}(i)$ is in $\mathcal{N}$. □

We now combine these three technical lemmas into our main theorem, namely:

Theorem 6.15

Let $\mathcal{M}$ be an arbitrary collection of language families. Close each family $\mathcal{L}$ in $\mathcal{M}$ first under intersection with regular sets, second under covering and third under superdisjoint union. Close the resulting collection $\hat{\mathcal{M}}$ under extraction and then under superdisjoint wedge. Then $\mathcal{N}$, the resulting collection, is an MSW space.

Proof: By Lemmas 6.12, 6.13 and 6.14. □

For example, let $\mathcal{M} = \{\{\Sigma^*: \Sigma \text{ an alphabet of } n \text{ symbols}\}\}$, for some $n \geq 1$. Then $\hat{\mathcal{M}} = \{\mathcal{L}\}$ where $\mathcal{L}$ contains all languages of the form:

$R = R_1 \ddot{\cup} \ldots \ddot{\cup} R_k$

for some $k \geq 1$, where the R_i are regular sets over disjoint alphabets Σ_i and $\#\Sigma_i \leq n$. Note that no R in $\mathcal{L}$ contains a word with more than n different letters.

Closing $\hat{\mathcal{M}}$ under extraction gives $\mathcal{M}' = \{\mathcal{L}_{i_1,\ldots,i_q}: 1 \leq i_1 < i_2 < \ldots < i_q, q \geq 0\}$, where $\mathcal{L}_{i_1,\ldots,i_q}$ is defined as $\mathcal{L}_{i_1,\ldots,i_{q-1}}(i_q)$, that is $\mathcal{L}_{i_1,\ldots,i_q}$ is obtained by extracting all words of lengths $i_1,\ldots,i_q$ from $\mathcal{L}$. Finally closing $\mathcal{M}'$ under superdisjoint wedge gives an MSW space $\mathcal{N}$ by Theorem 6.15.

Notice that the remark about the structure of words in a language R in $\mathcal{L}$ also holds for any word in a language of a language family in $\mathcal{N}$, hence $\mathcal{N}$ misses many regular sets.

II.6.4 Two-symbol Grammatical Families

In this section we are concerned with two-symbol grammar forms and their associated families. As we have already seen the families $\mathcal{L}$(REG), $\mathcal{L}$(LIN) and $\mathcal{L}$(CF) are two-symbol grammatical. Indeed the grammar forms G_1, G_2 and G_3 defined by the productions:

G_1: $S \to a$; $S \to aS$,

G_2: $S \to a$; $S \to aS$; $S \to Sa$,

G_3: $S \to a$; $S \to SS$

generate the families $\mathcal{L}$(REG), $\mathcal{L}$(LIN) and $\mathcal{L}$(CF), respectively. Furthermore for an arbitrary two-symbol form G the supernormal theorem tells us that $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(CF)$ iff $L(G,\Rightarrow) = a^+$ and there is a production $S \to \alpha$ in G such that α contains at least two appearances of S. Also $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(REG)$ iff $\mathcal{L}_s(G,\Rightarrow) \supseteq \mathcal{L}(FIN)$ iff $L(G,\Rightarrow) = a^+$.

Dual to the notion of density are the notions of predecessor and successor, that is for two families $\mathcal{L}_1$ and $\mathcal{L}_2$ with $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$, we say $\mathcal{L}_1$ is a <u>predecessor</u> of $\mathcal{L}_2$ or $\mathcal{L}_2$ is a <u>successor</u> of $\mathcal{L}_1$ if there is no family $\mathcal{L}_3$ properly in between.

Within the framework of context-free two-symbol grammatical families, we can observe that $\mathcal{L}(CF)$ is a successor of $\mathcal{L}(LIN)$ and in fact the only successor. For assume that there is a two-symbol form G such that

$$\mathcal{L}(LIN) \subseteq \mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}(CF).$$

Now by the observations above $L(G,\Rightarrow) = a^+$ and either there is a production $S \to \alpha$ in G with α containing at least two appearances of S or there is not. In the latter case we obtain $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(LIN)$, and in the former case we have $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}(CF)$. Hence there is no G with $\mathcal{L}_s(G,\Rightarrow)$ properly in between $\mathcal{L}(LIN)$ and $\mathcal{L}(CF)$.

Note that on the other hand $\mathcal{L}(LIN)$ is not the only predecessor of $\mathcal{L}(CF)$. For example, G defined by the productions:

$S \to aa$; $S \to aaa$; $S \to SS$

has $\mathcal{L}_s(G,\Rightarrow) \subsetneqq \mathcal{L}(CF)$ and there is no other family (two-symbol family, that is) in between.

Second observe that if G is a two-symbol form such that $\mathcal{L}_s(G,\Rightarrow) \subsetneqq \mathcal{L}(REG)$, then $\mathcal{L}_s(G,\Rightarrow)$ and $\mathcal{L}(REG)$ do not form a dense pair. This follows almost immediately, since $L(G,\Rightarrow) \neq a^+$, by the remarks above. Let $i \geq 1$ be the smallest integer such that a^i is not in $L(G,\Rightarrow)$. Define a new two-symbol form G_1 by:

$S \to a;\ldots;S \to a^{i-1}$; $S \to a^iS$; $S \to a^{2i}$

then $L(G_1,\Rightarrow) = a^+ - \{a^i\}$, hence

$$\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(G_1,\Rightarrow) \subsetneqq \mathcal{L}(REG)$$

and $\mathcal{L}(REG)$ is a successor of $\mathcal{L}_s(G_1,\Rightarrow)$.

These preliminary observations lead us to consider the pair $\mathcal{L}(REG)$ and $\mathcal{L}(LIN)$ of two-symbol families. We will demonstrate that these form a dense pair and as a by-product of the proof we will also show that it is decidable whether two linear two-symbol forms are form equivalent.

Since we are only dealing with linear two-symbol forms in the sequel we can assume that we only have two types of productions:

(1) $S \to a^i$, $i \geq 0$

and

(2) $S \to a^iSa^j$, $i, j \geq 0$.

Those of type (1) are <u>terminating</u> productions and those of type (2) are <u>nonterminating</u> productions. Since a type (2) production is uniquely determined by the exponents i and j, we will often speak of the production (i,j). We can always assume that $i + j > 0$ without loss of generality, that is the production $S \to S$ can always be omitted, without changing the language family.

If G is a linear two-symbol form and p and q are non-negative integers with $p + q > 0$, we say that the pair (p,q) is <u>generated by G</u> if for some $n > 0$,

$$S \Rightarrow^* a^{np} S a^{nq}$$

is a derivation according to G.

To establish the denseness of the pair ($\mathcal{L}$(REG), $\mathcal{L}$(LIN)) for the two-symbol grammatical families we first need three technical lemmas, which demonstrate some basic properties of pairs (p,q).

Lemma 6.16

Let G be a linear two-symbol form whose nonterminating productions are $(i_1,j_1),\ldots,(i_t,j_t)$ where the numbering is chosen so that

$$\frac{i_1}{j_1} \le \ldots \le \frac{i_t}{j_t}.$$

(For $j_v = 0$, $\frac{i_v}{j_v}$ is considered to be ∞).

Then a pair (p,q) of nonnegative integers with $p + q > 0$ is generated by G iff there is a v such that

$$(3) \quad \frac{i_v}{j_v} \le \frac{p}{p} \le \frac{i_{v+1}}{j_{v+1}}.$$

Proof: <u>if</u>: Assume there is a v satisfying (3). Clearly if one of the inequalities in (3) is not strict then (p,q) is generated by G. Moreover if t=1 this is the only case to consider. Hence consider the case that $t > 1$ and both inequalities in (3) are strict. In this case the determinants:

$$c_{v+1} = \begin{vmatrix} p & q \\ i_v & j_v \end{vmatrix}, \quad c_v = \begin{vmatrix} i_{v+1} & j_{v+1} \\ p & q \end{vmatrix}, \quad n = \begin{vmatrix} i_{v+1} & j_{v+1} \\ i_v & j_v \end{vmatrix}$$

are all positive integers. Using the identities

$$c_v u_v + c_{v+1} i_{v+1} = np$$

and

$$c_v j_v + c_{v+1} j_{v+1} = nq$$

we are able to construct a derivation

$S \Rightarrow^* a^{np} S a^{nq}$

in G by applying c_v times the production (i_v, j_v) and c_{v+1} times the production (i_{v+1}, j_{v+1}). Thus (p,q) is generated by G.

only if: Assume there is no index v satisfying (3) then either

$$\frac{p}{q} < \frac{i_1}{j_1} \text{ or } \frac{i_t}{j_t} < \frac{p}{q}.$$

With loss of generality we assume $\frac{p}{q} < \frac{i_1}{j_1}$, since the other alternative can be dealt with by interchanging the roles of p and q.

If the pair (p,q) is generated by G, there are nonnegative integers c_v, $v = 1,\dots,t$ and a positive integer n satisfying

$$\sum_{v=1}^{t} c_v i_v = np \text{ and } \sum_{v=1}^{t} c_v j_v = nq.$$

But this is possible only if $\frac{p}{q} \geq \frac{i_1}{j_1}$, a contradiction.

Therefore (p,q) is not generated by G. □

Lemma 6.17

Assume that F and G are linear two-symbol forms and (p,q) is a pair of non-negative integers with $p + q > 0$. If F does not generate (p,q), G possesses a production (p,q) and also generates a non-regular language, then $\mathcal{L}_s(G,\Rightarrow)$ is not contained in $\mathcal{L}_s(F,\Rightarrow)$.

Proof: Let the nonterminating productions of F be $(i_1,j_1),\dots,(i_t,j_t)$ ordered so that

$$\frac{i_1}{j_1} \leq \frac{i_2}{j_2} \leq \dots \leq \frac{i_t}{j_t}.$$

The assumptions for G imply that there is an $m > 0$ such that $S \Rightarrow^* a^m$ in G. By Corollary II.2.10 we may assume that $S \to a^m$ is a production of G.

Now if F has no nonterminating productions, that is t=0, then $\mathcal{L}_s(F,\Rightarrow)$ only contains finite languages. Since $\mathcal{L}_s(G,\Rightarrow)$ contains a non-regular language we have $\mathcal{L}_s(G,\Rightarrow) \not\subseteq \mathcal{L}_s(F,\Rightarrow)$ as claimed. Assume $t \geq 1$. By Lemma 6.16 since (p,q) is not generated by F we may assume $\frac{p}{q} < \frac{i_1}{j_1}$ (the other alternative can be treated symmetrically), in which case $i_1 > 0$.

Since G generates a non-regular language, there are positive integers r and s such that

$$S \Rightarrow^* a^r S a^s$$

in G. If in the given pair (p,q) we have p=0, then we have a derivation

$$S \Rightarrow^* a^r S a^{s+qv}$$

in G for all v ≥ 0. Now choose v large enough so that

$$\frac{r}{s+qv} \le \frac{i_1}{j_1},$$

this must be possible since $i_1 > 0$. By another application of Corollary II.2.10 we may assume that the production $S \rightarrow a^r S a^{s+qv}$ is in G, and moreover we can replace (p,q) by (r,s+qv) both of whose components are positive. Again if we have q=0 in the original pair (p,q) we could proceed in a similar manner. Thus we may assume by the above argument that both p and q are positive.

Finally, consider the language $L = \{a^{ip} b^m a^{iq} : i \ge 0\}$. Clearly L is in $\mathcal{L}_s(G,\Rightarrow)$. We claim it is not in $\mathcal{L}_s(F,\Rightarrow)$.

Assume that L is in $\mathcal{L}_s(F,\Rightarrow)$, then there is an $F' \triangleleft_s F$ such that $L = L(F',\Rightarrow)$. Now F' is a linear grammar, since F is linear. Because L is infinite there must be at least one "looping" nonterminal A in F', that is there are derivations:

$$S \Rightarrow^* uAv,\ A \Rightarrow^+ xAy \text{ and } A \Rightarrow^* w$$

in F', where u, v, x, y and w are terminal words and xy is not the empty word.

Now for all n, $ux^n wy^n v$ is in L and since F' is an interpretation of F we must have:

$$\frac{i_1}{j_1} \le \frac{|x|}{|y|} \le \frac{i_t}{j_t}.$$

However, xy must be in a^+ and since $ux^n wy^n v$ is in L for all n ≥ 0, then $\frac{|x|}{|y|} = \frac{p}{q}$. Thus we have obtained a contradiction since we assumed $\frac{p}{q} < \frac{i_1}{j_1}$. Hence L is not $\mathcal{L}_s(F,\Rightarrow)$ and therefore $\mathcal{L}_s(G,\Rightarrow)$ is not contained in $\mathcal{L}_s(F,\Rightarrow)$. □

Lemma 6.18

Let F be a linear two-symbol form and $S \rightarrow a^p S a^q$ be a production in F, where p + q > 0. Construct G from F by taking into G:

(i) all productions in F apart from $S \to a^p S a^q$,

(ii) the production $S \to a^{pn} S a^{qn}$, for some $n \geq 2$, and

(iii) sufficient productions of the form $S \to a^i$, $i > 0$ such that $L(F,\Rightarrow) = L(G,\Rightarrow)$.

Then F and G are form equivalent.

Proof: Clearly $\mathcal{L}_s(G,\Rightarrow) \subseteq \mathcal{L}_s(F,\Rightarrow)$, by the simulation lemma. Hence we only need consider the reverse inclusion $\mathcal{L}_s(F,\Rightarrow) \subseteq \mathcal{L}_s(G,\Rightarrow)$. Let $F' = (V,\Sigma,P',S')$ be an arbitrary interpretation of F, $F' \trianglelefteq_s F(\mu)$. We demonstrate that there is an interpretation $G' \trianglelefteq_s G$ with $L(G',\Rightarrow) = L(F',\Rightarrow)$ to complete the proof.

In the case that both $S \to a^i S$ and $S \to S a^j$ are in F for some $i, j > 0$, then by Theorem 4.24 we have $\mathcal{L}_s(F,\Rightarrow)$ equals the family of linear languages whose length set is contained in $LS(F,\Rightarrow)$. This same observation holds for G also, since either both p and q are non-zero in which case $S \to a^i S$ and $S \to S a^j$ are in G or one of p and q is zero, in which case either $S \to a^i S$ and $S \to S a^{nj}$ or $S \to a^{ni} S$ and $S \to S a^j$ are in G. In all cases $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}_s(F,\Rightarrow)$.

Returning to F' we next observe that if P' contains no interpretation of $S \to a^p S a^q$ then it is trivially an interpretation of G and therefore $L(F',\Rightarrow)$ is in $\mathcal{L}_s(G,\Rightarrow)$. So we assume that F' does indeed contain some interpretations of $S \to a^p S a^q$.

We will for the purposes of this proof consider a production $A \to uBv$ an (i,j) production if it is an image of a production $S \to a^i S a^j$.

The idea behind the construction of G' is as follows. For any derivation D in F' the corresponding derivation D' in G' will correspond exactly until the first appearance of a (p,q) production in D, $A_0 \to u_1 A_1 v_1$, say, since G' cannot contain a (p,q) production. So we examine the production applied to A_1 in D. There are now three possibilities:

(i) The production applied to A_1 is terminating, $A_1 \to w$ say. In this case D also terminates and in D' we apply the production $A_0 \to u_1 w v_1$ to A_0. We may assume that this production is in G', since the derivation $S \Rightarrow^* a^i a^j a^k$ is in F where $i = |u_1|$, $j = |w|$ and $k = |v_1|$ and moreover F and G generate the same language.

(ii) The production applied to A_1 is nonterminating and is not a (p,q) production. Let it be $A_1 \to u_2 A_2 v_2$, then in D' we apply the production

$A_0 \to x_1[x_2 A_2 y_2]y_1$

to A_0, where $|x_1| = |u_2|$, $|y_1| = |v_2|$, $|x_2| = p$, $|y_2| = q$, $x_1x_2 = u_1u_2$, $y_2y_1 = v_2v_1$, and $[x_2A_2y_2]$ is a new nonterminal. Note that this production is an interpretation of the same production as $A_1 \to u_2A_2v_2$. The new nonterminals carry along terminal words to be deposited as soon as possible. One situation in which they can be deposited is when a terminating production is met, the other situation is when terminal words of sufficient length have been accumulated in the nonterminals.

(iii) The production applied to A_1 is nonterminating and is also a (p,q) production, $A_1 \to u_2A_2v_2$ say. In this case we examine the production applied to A_2 in D. Again we have three cases to consider, case (i) is the same as above and case (ii) has a minor modification namely $|x_2| = 2p$, $|y_2| = 2q$. In case (iii) we proceed in the same manner, however when we find n (p,q) productions $A_0 \to u_1A_1v_1, \ldots, A_{n-1} \to u_nA_nv_n$, then we replace them by $A_0 \to u_1 \ldots u_nA_nv_n \ldots v_1$ in D'. This we can do since $|u_1 \ldots u_n| = np$, $|v_n \ldots v_1| = nq$ and G has an (np,nq) production.

It should be clear how to modify the above procedure when dealing with one of the new nonterminals [xAy] in D', namely, the case analysis is the same except that: in case (i) x and y must also be deposited; in case (ii) a new nonterminal will carry terminal information of lengths $|x| + p$ and $|y| + q$ (unless $|x| + p = np$ in which case introduce an (np,nq) production to deposit the terminal words before continuing); in case (iii) terminal information will be accumulated unless its length equals np and nq, when it is deposited. This informal description of the construction of G' we now formalize.

Let the nonterminals of G' be triples of the type

$$[xAy]$$

where x and y are in Σ^*, $|x| = rp$, $|y| = rq$ for some r, $0 \le r < n$ and A is in $V-\Sigma$. Let [S'] be the sentence symbol of G'. Essentially a nonterminal [A] corresponds exactly to a nonterminal A in F' and a nonterminal [xAy], $xy \neq \lambda$ contains the accumulated terminal information as mentioned above.

For each nonterminal A_0 in F' and each nonterminal $[xA_0y]$ in G', where $|x| = rp$ and $|y| = rq$ for some r, $0 \le r < n$, define the following sets of derivations:

$C_r(A_0) = \{A_0 \Rightarrow u_1A_1v_1 \Rightarrow \ldots \Rightarrow u_1\ldots u_sA_sv_s\ldots v_1$ in F': $s \geq 1$, $A_i \to u_{i+1}A_{i+1}v_{i+1}$ is a (p,q) production, $0 \leq i < s$ and $s + r = n\}$,

$D_r(A_0) = \{A_0 \Rightarrow u_1A_1v_1 \Rightarrow \ldots \Rightarrow u_1 \ldots u_{s+1}A_{s+1}v_{s+1} \ldots v_1$ in F': $s \geq 0$, $A_i \to u_{i+1}A_{i+1}v_{i+1}$ is a (p,q) production, $0 \leq i < s$, $s + r < n$ and $A_s \to u_{s+1}A_{s+1}v_{s+1}$ is not a (p,q) production$\}$,

$T_r(A_0) = \{A_0 \Rightarrow u_1A_1v_1 \Rightarrow \ldots \Rightarrow u_1 \ldots u_sA_sv_s \ldots v_1 \Rightarrow u_1 \ldots u_swv_s \ldots v_1$ in F': $s \geq 0$, $A_i \to u_{i+1}A_{i+1}v_{i+1}$ is a (p,q) production, $0 \leq i < s$ and $r + s < n\}$.

These three sets exhaust the possible derivations from A_0 which involve an initial, possible empty, sequence of (p,q) productions when the terminal words x and y accumulated so far are of length rp and rq respectively. We can now define the productions of G' as follows: For all nonterminals $[xA_0y]$ in G' where $|x| = rp$ and $|y| = rq$ for some r, $0 \leq r < n$ we include the following productions:

(i) $[xA_0y] \to xu_1\ldots u_s[A_s]v_s\ldots v_1y$, where the derivation $A_0 \Rightarrow^+ u_1\ldots u_sA_sv_s\ldots v_1$ is in $C_r(A_0)$,

(ii) $[xA_0y] \to w[x'A_{s+1}y']z$, where the derivation $A_0 \Rightarrow^+ u_1\ldots u_{s+1}A_{s+1}v_{s+1}\ldots v_1$ is in $D_r(A_0)$, $wx' = xu_1\ldots u_{s+1}$, $y' = v_{s+1}\ldots v_1y$, $|w| = |u_{s+1}|$ and $|z| = |v_{s+1}|$,

(iii) $[xA_0y] \to xu_1\ldots u_swv_s\ldots v_1y$, where the derivation $A_0 \Rightarrow^+ u_1\ldots u_swv_s\ldots v_1$ is in $T_r(A_0)$.

Note that productions of type (i) are possible to obtain from G since G has the production $S \to a^{np}Sa^{nq}$, those of type (ii) since $A_s \to u_{s+1}A_{s+1}v_{s+1}$ is not a (p,q) production and those of type (iii) since the language of G equals that of F.

Hence we have constructed a $G' \mathrel{\triangleleft_s} G$ and the motivation given above for the construction implies that $L(G',\Rightarrow) = L(F',\Rightarrow)$. A detailed proof of this is left to the reader.

Therefore we have shown that $\mathcal{L}_s(G,\Rightarrow) = \mathcal{L}_s(F,\Rightarrow)$ as desired. □

We are now able to establish the required theorem, namely:

Theorem 6.19

The pair $(\mathcal{L}(\mathrm{REG}), \mathcal{L}(\mathrm{LIN}))$ is dense.

Proof: Assume that G_1 and G_2 are two-symbol forms such that $\mathcal{L}(\mathrm{REG}) \subseteq \mathcal{L}_s(G_1,\Rightarrow) \subsetneq \mathcal{L}_s(G_2,\Rightarrow) \subseteq \mathcal{L}(\mathrm{LIN})$.
Note that $L(G_1,\Rightarrow) = L(G_2,\Rightarrow) = a^+$ must hold. Let $(i_1,j_1),\ldots,(i_t,j_t)$ be the nonterminating productions of G_1, ordered by increasing ratios as above. By the previous lemma, Lemma 6.18, and by the assumptions for G_1 and G_2 we must have a (p,q) production in G_2 such that (p,q) is not generated by G_1 (otherwise G_1 and G_2 are form equivalent, a contradiction).

By Lemma 6.16 we either have

$$\frac{p}{q} < \frac{i_1}{j_1} \quad \text{or} \quad \frac{i_t}{j_t} < \frac{p}{q}.$$

As before we assume the former holds since the latter can be treated symmetrically.

Now each of (i_k,j_k), $1 \le k \le t$ are generated by G_2 (Lemma 6.17). Therefore for some $n > 0$

$$S \Rightarrow^* a^{ni_1}Sa^{nj_1}$$

is a derivation in G_2.

Now let G_3 be the two-symbol form obtained from G_1 by adding the production

$$S \rightarrow a^{ni_1+p}Sa^{nj_1+q}$$

Clearly $\mathcal{L}_s(G_1,\Rightarrow) \subseteq \mathcal{L}_s(G_3,\Rightarrow)$ and since $\frac{p}{q} < \frac{i_1}{j_1}$ we also have $\frac{ni_1+p}{nj_1+q} < \frac{i_1}{j_1}$, thus the inclusion is proper. On the other hand
$S \Rightarrow^* a^{ni_1}Sa^{nj_1} \Rightarrow a^{ni_1+p}Sa^{nj_1+q}$ is a derivation in G_2, hence $\mathcal{L}_s(G_3,\Rightarrow) \subseteq \mathcal{L}_s(G_2,\Rightarrow)$. Now $\frac{p}{q} < \frac{ni_1+p}{nj_1+q} < \frac{i_1}{j_1}$, thus (p,q) is not generated by G_3 and the inclusion is once again a proper one.

To summarize: we have constructed a G_3 such that

$$\mathcal{L}_s(G_1,\Rightarrow) \subsetneq \mathcal{L}_s(G_3,\ ) \subsetneq \mathcal{L}_s(G_2,\Rightarrow)$$

for arbitrary G_1 and G_2 satisfying the conditions of the theorem. Therefore $(\mathcal{L}(\mathrm{REG}),\mathcal{L}(\mathrm{LIN}))$ is dense. □

We now use the techniques developed above to give a surprising decidability result.

Theorem 6.20

Given two-symbol forms G_1 and G_2, where G_1 is a linear two-symbol form. Then it is decidable whether or not $\mathcal{L}_s(G_1,\Rightarrow) = \mathcal{L}_s(G_2,\Rightarrow)$.

Proof: Without loss of generality we may assume both G_1 and G_2 contain a terminating production. First decide whether or not $L(G_1,\Rightarrow) = L(G_2,\Rightarrow)$. Since $L(G_i,\Rightarrow)$ is a regular language, $i = 1, 2$, this can be accomplished. If $L(G_1,\Rightarrow) \neq L(G_2,\Rightarrow)$ then G_1 and G_2 cannot be form equivalent. On the other hand if equality holds we can consider the following possibilities.

(i) G_2 is nonlinear. Clearly G_1 and G_2 cannot be form equivalent.

(ii) G_1 and G_2 are either left or right linear. In this case both generate all regular languages whose length sets are contained in those of G_1 and hence of G_2. Therefore they are form equivalent in this case.

(iii) Neither (i) nor (ii) hold. Hence both G_1 and G_2 are linear. Moreover (a) both are finite, (b) both are properly linear or (c) neither (a) nor (b) hold. In case (a) G_1 and G_2 are form equivalent and in (c) they cannot be form equivalent, since either one is finite and the other infinite or one is subregular and the other is properly linear. This only leaves case (b), but by the previous results G_1 and G_2 are form equivalent iff for each production (p,q) in G_1 (p,q) is generated by G_2 and vice versa. This test is effective by the previous lemmas and therefore form equivalence is decidable. □

In Theorem 6.20 the restriction that G_1 be linear cannot be removed at the present time. However the following conjecture, which we strongly believe to hold, would enable its removal to be accomplished.

Conjecture:

Assume G is a non-linear two-symbol form. Then every context-free language whose length set is contained in $LS(G,\Rightarrow)$ is in $\mathcal{L}_s(G,\Rightarrow)$.