RATIONAL RELATIONS OF BINARY TREES

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By extending the idea behind transductions by Elgot & Mezei [1965] (or equivalently, K-transductions by Nivat [1968], a-transducers by Ginsburg & Greibach [1969], or rational relations by Eilenberg [1974]), we define a class of tree transformations called rational relations of binary trees. They may be considered intuitively as the relations between input and output trees recognized simultaneously by "two-tape" tree automata that process in each step multi-level (possibly zero-level) branches. The formal definition is given by using recognizable sets (of binary trees) and a certain type of tree functions called tree-morphisms. The tree-morphisms here are meant to be a tree-version of homomorphisms, characterized by the feature of preserving subtree construction (possibly with permutation) as well as that of sending "null tree" to itself. (They do not duplicate branching edges.)

A result obtained is the closure property under composition of the class of rational relations. As a corollary to this, we get the fact that the recognizable sets (of binary trees) are preserved not only by tree-morphisms but also by their inverses.

In section 1, we give basic definitions of trees, tree-morphisms, rational relations, and other related concepts. In section 2, we develop preliminary results, and in section 3 present our main results.

Throughout the paper we restrict our attention to binary trees in the sense of Knuth [1968].

1. Definitions

1.1 (Trees and indexed trees)

Let Σ be a finite alphabet. The set $\mathcal{T}_\Sigma$ of (<u>binary</u>) <u>trees</u> over Σ is defined as the smallest set satisfying;

$$\# \in \mathcal{T}_\Sigma$$

$$\text{if } \sigma \in \Sigma \text{ and } t_1, t_2 \in \mathcal{T}_\Sigma \text{ then } \sigma\langle t_1, t_2\rangle \in \mathcal{T}_\Sigma .$$

The #, as a member of $\mathcal{T}_\Sigma$, may be termed as <u>null tree</u>.

Objects like trees but some frontier nodes may be labeled by symbols other than # are called <u>indexed trees</u>. Formally, the set $\mathcal{T}_{\Sigma,\Omega}$

of indexed trees over Σ with index alphabet Ω is defined as the smallest set such that

$\Omega \cup \{\#\} \subset \mathcal{T}_{\Sigma,\Omega}$

if $\sigma \in \Sigma$ and $t_1, t_2 \in \mathcal{T}_{\Sigma,\Omega}$ then $\sigma<t_1,t_2> \in \mathcal{T}_{\Sigma,\Omega}$.

With each $t \in \mathcal{T}_{\Sigma,\Omega}$ we associate a subset $D(t)$ of $\{1,2\}^*$ (the set of finite strings over alphabet $\{1,2\}$, including null string ε) such that

if $t \in \Omega \cup \{\#\}$ then $D(t) = \{\varepsilon\}$,

if $t = \sigma<t_1,t_2>$, $\sigma \in \Sigma$ and $t_1, t_2 \in \mathcal{T}_{\Sigma,\Omega}$ then

$$D(t) = \{\varepsilon\} \cup \{1\}\cdot D(t_1) \cup \{2\}\cdot D(t_2) .$$

Each member d of D(t) specifies the "address" of a node in t, and we write t(d) to mean the symbol ($\in \Sigma \cup \Omega \cup \{\#\}$) at the node. In this respect, $t \in \mathcal{T}_{\Sigma,\Omega}$ is viewed as a mapping from $D(t)$ to $\Sigma \cup \Omega \cup \{\#\}$ such that

if $t \in \Omega \cup \{\#\}$ then $t(\varepsilon) = t$,

if $t = \sigma<t_1,t_2>$, $\sigma \in \Sigma$ and $t_1, t_2 \in \mathcal{T}_{\Sigma,\Omega}$ then

$$t(\varepsilon) = \sigma \text{ and } t(i\cdot d) = t_i(d) \quad (d \in D(t_i),\ i=1,2).$$

Fr(t) of $t \in \mathcal{T}_{\Sigma,\Omega}$ is the set of addresses of frontier nodes of t. I.e., $Fr(t) = \{d \in D(t) \mid \{d\}\{1,2\}^+ \cap D(t) = \emptyset\}$ where $\{1,2\}^+ = \{1,2\}^* - \{\varepsilon\}$.

Yield(t) of $t \in \mathcal{T}_{\Sigma,\Omega}$ stands for the concatenation of symbols at the frontier nodes of t (in order). I.e., Yield(t) = t if $t \in \Omega \cup \{\#\}$; $Yield(\sigma<t_1,t_2>) = Yield(t_1)\cdot Yield(t_2)$ if $\sigma \in \Sigma$ and $t_1, t_2 \in \mathcal{T}_{\Sigma,\Omega}$.

Suppose $t \in \mathcal{T}_{\Sigma,\Omega}$ and $d \in D(t)$. The <u>subtree</u> of t at d, denote by t/d, is the member t' of $\mathcal{T}_{\Sigma,\Omega}$ such that

$D(t') = \{d' \mid dd' \in D(t)\}$,

$t'(d') = t(dd')$ for each $d' \in D(t')$.

Replace(t,d←t') stands for the result obtained from t by replacing its subtree at d by t'. I.e., Replace(t,d←t') = t" where

$D(t'') = (D(t) - \{d\}\{1,2\}^*) \cup \{d\}\cdot D(t')$,

$t''(d'') = t(d'')$ for each $d'' \in D(t) - \{d\}\{1,2\}^*$,

$t''(dd') = t'(d')$ for each $d' \in D(t')$.

We also write Replace $(t, d_1 \leftarrow t_1, d_2 \leftarrow t_2)$ to mean the result of double replacement $Replace(Replace(t, d_1 \leftarrow t_1), d_2 \leftarrow t_2)$.

Now we take two specific symbols X_1 and X_2 ($\notin \Sigma \cup \{\#\}$), let $\Omega = \{X_1, X_2\}$, and define subsets of $\mathcal{T}_{\Sigma,\Omega}$ as follows;

$\mathcal{A}_\Sigma = \{\tau \in \mathcal{T}_{\Sigma,\Omega} \mid Yield(\tau) \in \{\#\}^*\{X_1\}\{\#\}^*\{X_2\}\{\#\}^*\}$,

$\bar{\mathcal{A}}_\Sigma = \{\tau \in \mathcal{T}_{\Sigma,\Omega} \mid Yield(\tau) \in \{\#\}^*\{X_2\}\{\#\}^*\{X_1\}\{\#\}^*\}$,

$\mathcal{B}_\Sigma = \mathcal{A}_\Sigma \cup \bar{\mathcal{A}}_\Sigma$.

Each member τ of $\mathcal{B}_\Sigma$ can be taken as a binary function $\tau: \mathcal{T}_\Sigma \times \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma$ such that

$$\tau(t_1,t_2) = \text{Replace}(\tau,d_1\leftarrow t_1,d_2\leftarrow t_2) \qquad (t_1,t_2 \in \mathcal{T}_\Sigma)$$

where d_1 and d_2 indicate the addresses of index symbols X_1 and X_2 (resp.) in τ. That is, d_j is the (unique) element of $D(\tau)$ satisfying $\tau(d_j) = X_j$ $(j=1,2)$. The pair (d_1,d_2) will be denoted by Index(τ).

1.2 (Expansive tree-morphisms)

Let Γ and Σ be alphabets not containing symbols #, X_1 and X_2, and τ_γ for each $\gamma \in \Gamma$ be a binary function: $\mathcal{T}_\Sigma \times \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma$ that belongs to $\mathcal{B}_\Sigma$. Let f: $\mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ be the function defined inductively as

$$f(\#) = \#,$$

$$f(\gamma<t_1,t_2>) = \tau_\gamma(f(t_1),f(t_2)) \qquad (\gamma \in \Gamma,\ \ t_1,t_2 \in \mathcal{T}_\Gamma).$$

We call such function f an <u>expansive tree-morphism</u> (<u>etm</u> for short), and call the mapping $\underline{\underline{f}}:\Gamma \to \mathcal{B}_\Sigma$ such that $\underline{\underline{f}}(\gamma) = \tau_\gamma$ $(\gamma \in \Gamma)$ the <u>basis</u> of f. In particular when $\underline{\underline{f}}(\Gamma) \subset \mathcal{A}_\Sigma$ (i.e., Yield$(\tau_\gamma) \in \{\#\}^*\{X_1\}\{\#\}^*\{X_2\}\{\#\}^*$ for each $\gamma \in \Gamma$) we say the etm f is <u>simple</u>. Moreover if $\underline{\underline{f}}(\Gamma) \subset \{\sigma<X_1,X_2> \mid \sigma \in \Sigma\}$ we call the etm a <u>projection</u>. In the latter case, instead of writing $\underline{\underline{f}}(\gamma) = \sigma<X_1,X_2>$ we may simply write $\underline{\underline{f}}(\gamma) = \sigma$. The basis $\underline{\underline{f}}$ of projection f is then seen to be a mapping from Γ to Σ.

In literature, "linear nondeleting homomorphisms" by Engelfriet [1975] or equivalently "homomorphismes linéaires complets stricts" by Arnold & Dauchet [1976], when both restricted to binary trees, are coextensive with our etm. The notion of simple etm coincides with the restriction to binary trees of the notion of "simple pure transformations" by Thatcher [1969].

It should be noticed that by application of these functions trees never get "smaller". In this respect the etm (expansive tree-morphisms) are compared to ε-free homomorphisms of strings. Then what would be the right notion of "tree-morphisms" which can really be compared to general, not necessarily ε-free, homomorphisms? Hopefully the notion should enjoy nice mathematical properties such as the preservation of recognizable sets, the same property of their inverses, composability, etc., besides laying down a reasonable base for theory of tree transformations. To answer the question, we remind the case of strings; there we have the null string, i.e., the unit element of the underlying algebraic structure, and by allowing the basis functions to send symbols to the unit element we can obtain homomorphisms not necessarily expansive. In the case of trees the situation seems more complicated; there is no "unit" element in $\mathcal{T}_\Sigma$ to begin with. However we find it possible to bring in certain entities to the universe of trees and let them behave somehow like null string. "Dummy node" is the name given for the constituents of such entities.

1.3 (Trees with dummy nodes)

Suppose λ is a distinct symbol (to mark the dummy nodes), and let $\Lambda = \{\lambda\}$. We assume that $\lambda \notin \Sigma \cup \{\#, X_1, X_2\}$. Let $\mathcal{U} = \mathcal{T}_\Lambda$, and $\mathcal{U}_\Sigma$ be the smallest subset of $\mathcal{T}_{\Sigma\cup\Lambda}$ such that

(1) $\# \in \mathcal{U}_\Sigma$

(2) if $\sigma \in \Sigma$ and $u_1, u_2 \in \mathcal{U}_\Sigma$ then $\sigma\langle u_1, u_2\rangle \in \mathcal{U}_\Sigma$

(3) if $u \in \mathcal{U}$ and $u' \in \mathcal{U}_\Sigma$ then $\lambda\langle u, u'\rangle \in \mathcal{U}_\Sigma$ and $\lambda\langle u', u\rangle \in \mathcal{U}_\Sigma$.

For example, $u_1 = a\langle\lambda\langle\lambda\langle\#,\#\rangle, b\langle\#, c\langle\#,\#\rangle\rangle\rangle, \lambda\langle\#,\#\rangle\rangle$ (Fig.1) is a member of $\mathcal{U}_{\{a,b,c\}}$, but $u_2 = a\langle\lambda\langle b\langle\#,\#\rangle, c\langle\#,\#\rangle\rangle, \#\rangle$ is not. Note that $\mathcal{T}_\Sigma \subset \mathcal{U}_\Sigma$ (by rules (1) and (2)), and $\mathcal{U} \subset \mathcal{U}_\Sigma$ (by rules (1) and (3)).

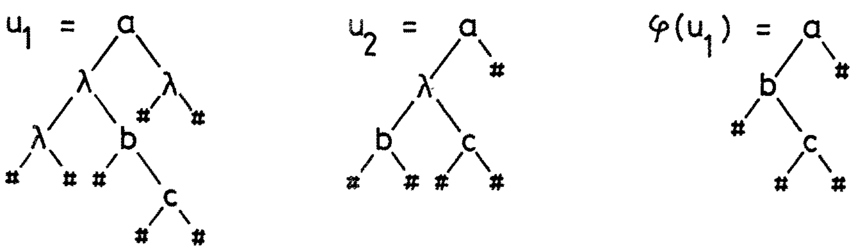


Fig. 1. Examples of an element and a non-element of $\mathcal{U}_\Sigma$, and an application of φ (see below).

Let $\Rightarrow$ be the weakest partial ordering in $\mathcal{T}_{\Sigma\cup\Lambda}$ such that

if $t \in \mathcal{T}_{\Sigma\cup\Lambda}$ and $u \in \mathcal{U}$,
then $\lambda\langle t, u\rangle \Rightarrow t$ and $\lambda\langle u, t\rangle \Rightarrow t$,

if $\alpha \in \Sigma \cup \Lambda$, $t_1, t_2, t_1', t_2' \in \mathcal{T}_{\Sigma\cup\Lambda}$ and $t_i \Rightarrow t_i'$ $(i=1,2)$,
then $\alpha\langle t_1, t_2\rangle \Rightarrow \alpha\langle t_1', t_2'\rangle$.

By means of relation $\Rightarrow$, the subsets $\mathcal{U}$ and $\mathcal{U}_\Sigma$ of $\mathcal{T}_{\Sigma\cup\Lambda}$ can be characterized as

$$\mathcal{U} = \{u \in \mathcal{T}_{\Sigma\cup\Lambda} \mid u \Rightarrow \#\},$$

$$\mathcal{U}_\Sigma = \{u \in \mathcal{T}_{\Sigma\cup\Lambda} \mid u \Rightarrow t \text{ for some } t \in \mathcal{T}_\Sigma\}.$$

Since in $\mathcal{T}_\Sigma$ ($\subset \mathcal{T}_{\Sigma\cup\Lambda}$) the relation $\Rightarrow$ is nothing but the identity, for each $u \in \mathcal{U}_\Sigma$ the tree $t \in \mathcal{T}_\Sigma$ such that $u \Rightarrow t$ is unique, which we will denote by $\varphi(u)$. For $u \in \mathcal{T}_{\Sigma\cup\Lambda} - \mathcal{U}_\Sigma$ we leave $\varphi(u)$ undefined. Thus $\varphi : \mathcal{T}_{\Sigma\cup\Lambda} \to \mathcal{T}_\Sigma$ is set to be a partial function with domain $\mathcal{U}_\Sigma$. Intuitively an application of φ means to contract trees by cutting off dummy branches. For example, by contracting the u_1 above one gets

$\varphi(u_1)$ = a<b<#,c<#,#>>,#> (Fig. 1).

1.4 (Tree-morphisms)

Given etm $f': \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma\cup\Lambda}$ (where $\Sigma \cap \Lambda = \emptyset$), let $f: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ be a partial function such that

$$f(t) = \begin{cases} \varphi(f'(t)) & \text{if } f'(t) \in \mathcal{U}_\Sigma \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Such partial function f, written as $\varphi f'$, is called a <u>tree-morphism</u>, and the function f' is called its <u>ground</u>. The domain of f, Domain(f), is equal to $(f')^{-1}(\mathcal{U}_\Sigma)$. Tree-morphisms which have simple grounds (i.e., grounds which being simple etm) are called <u>simple tree-morphisms</u>.

Tree-morphisms preserve subtree construction in a sense that if t is in the domain of tree-morphism f and t' is a subtree of t, then f(t') is also defined, is included in f(t) as a subtree, and in fact unless f(t') = # the subtree image f(t') is uniquely traceable in f(t). If f is a simple tree-morphism, it keeps left-to-right relation between non-overlapping subtrees, provided their images are not null.

1.5 (Rational relations)

Suppose T is a recognizable set in $\mathcal{T}_\Gamma$, $\Sigma \cap \Lambda = \emptyset$, and $f_1, f_2: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ are tree-morphisms such that $T \subset \text{Domain}(f_1) \cap \text{Domain}(f_2)$. Then the relation R such that

$$R = \{(f_1(t), f_2(t)) \mid t \in T\}$$

is called a <u>rational relation (of binary trees)</u>.

In connection with the rational relation R, we often need in later sections to refer to a relation of form

$$R' = \{(f_1'(t), f_2'(t)) \mid t \in T\}$$

where $f_i': \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma\cup\Lambda}$ is a ground of tree-morphism f_i (i=1,2). We will call such relation R' a <u>ground</u> of R. In case where both f_1' and f_2' are projections, R' is named a <u>fine ground</u>. When both f_1' and f_2' are simple, R' is said to be <u>simple</u>. <u>Simple rational relations</u> are rational relations which have simple grounds.

Though we have defined rational relations by using a novel idea of tree-morphisms, it turns out that the class of rational relations has a concise characterization in terms of some familiar concepts: The class is identical with the restriction to binary trees of class $\text{LHOM}^{-1}\circ\text{FTA}\circ\text{LHOM}$ (= $\text{LB-FST}^{-1}\circ\text{LB-FST}$), in the terminology of Engelfriet [1975]; or with the restriction of B(HL,HL), in the notation of Arnold & Dauchet [1976]. The concept of "linear homomorphisms" (denoted here by LHOM in the former case and by HL in the latter) has been so common in literature that the characterization of rational relations thereby may be more intelligible. However for our purposes in theoretical development tree-morphisms are convenient, and we will stay with our formulation. The proof of the characterization just mentioned is given in Appendix.

1.6 (Miscellaneous)

Suppose (t_1,t_2) be a pair of trees in $\mathcal{T}_\Sigma$ such that $D(t_1) = D(t_2)$. Then we write $\pi(t_1,t_2)$ for their <u>product</u> t ($\in \mathcal{T}_{\Sigma\times\Sigma}$) which is defined by

$$D(t) = D(t_1)$$
$$t(d) = \begin{cases} (t_1(d),t_2(d)) & \text{for } d \in D(t) - Fr(t), \\ \# & \text{for } d \in Fr(t). \end{cases}$$

The mapping in the opposite direction, sending $\pi(t_1,t_2)$ to t_1(or to t_2), is denoted by π_1 (or π_2, resp.). In other words π_i: $\mathcal{T}_{\Sigma\times\Sigma} \to \mathcal{T}_\Sigma$ is the projection such that $\underline{\pi}_i(\sigma_1,\sigma_2) = \sigma_i$ $(i=1,2)$ for each $\sigma_1,\sigma_2 \in \Sigma$.

As for the products of elements of $\mathcal{B}_\Sigma$, providing a pair $(\tau_1,\tau_2) \in \mathcal{B}_\Sigma \times \mathcal{B}_\Sigma$ satisfies both $D(\tau_1) = D(\tau_2)$ and $\mathrm{Yield}(\tau_1) = \mathrm{Yield}(\tau_2)$, we will write $\pi(\tau_1,\tau_2)$ for the element τ of $\mathcal{B}_{\Sigma\times\Sigma}$ such that

$$D(\tau) = D(\tau_1),$$
$$\tau(d) = \begin{cases} (\tau_1(d),\tau_2(d)) & \text{for } d \in D(\tau) - Fr(\tau) \\ \tau_1(d)\ (= \tau_2(d)) & \text{for } d \in Fr(\tau). \end{cases}$$

In this situation we also write $\pi_i(\tau) = \tau_i$ $(i=1,2)$.

The relation $\Rightarrow$ defined in §1.3 for trees in $\mathcal{T}_{\Sigma\cup\Lambda}$ is now extended to the members of $\mathcal{B}_{\Sigma\cup\Lambda}$ as follows;

$$\tau \Rightarrow \tau' \quad \text{iff} \quad \text{for any } t_1,t_2 \in \mathcal{T}_{\Sigma\cup\Lambda} \quad \tau(t_1,t_2) \Rightarrow \tau'(t_1,t_2).$$

For binary relation R, we write

$$\mathrm{Domain}(R) = \{x \mid (x,y) \in R \ \text{ for some } y\},$$
$$\mathrm{Range}(R) = \{y \mid (x,y) \in R \ \text{ for some } x\}.$$

The <u>composition</u> R_2R_1 of binary relations R_1 and R_2 is defined by

$$R_2R_1 = \{(x,z) \mid (x,y) \in R_1, \ (y,z) \in R_2 \ \text{ for some } y\}.$$

(The right component is applied first.)

In what follows unless otherwise specified we exclude symbols #, X_1 and X_2 from alphabets that we use (such as $\Sigma,\Gamma,\Delta,\Sigma\times\Sigma$, ...). In addition, whenever relation $\Rightarrow$ and applications of partial function φ are involved, symbol λ is also excluded from our alphabets, except Λ which is constantly set to $\{\lambda\}$.

2. Preliminary Results

2.1 PROPOSITION If T is a recognizable set in $\mathcal{T}_{\Sigma\cup\Lambda}$ such that $T \subset \mathcal{U}_\Sigma$ then $\varphi(T)$ is a recognizable set in $\mathcal{T}_\Sigma$.

(Proof) Recognizable sets are identical with the projective images of local sets (of trees). (A local set in $\mathcal{T}_\Gamma$ is the set of trees generated by a tree-grammar $G = (\Gamma \cup \{\#\}, P, I)$ where $\Gamma \cup \{\#\}$ specifies the vocabulary of G, P ($\subset \{\gamma<\gamma_1,\gamma_2> \mid \gamma \in \Gamma,\ \gamma_1,\gamma_2 \in \Gamma \cup \{\#\}\} \cup \{\#\}$) is the set of branches, and I ($\subset \Gamma \cup \{\#\}$) is the set of root symbols. We write $\mathcal{T}G$ for the local set generated by G. For more details consult with Takahashi [1975].)

With the characterization of recognizable sets in mind, what is to be shown is as follows; given tree-grammar $G = (\Gamma \cup \{\#\}, P, I)$ and projection $p: \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma\cup\Lambda}$ such that $T = p(\mathcal{T}G) \subset \mathcal{U}_\Sigma$, one can find a tree-grammar $G' = (\Gamma' \cup \{\#\}, P', I')$ and a projection $p': \mathcal{T}_{\Gamma'} \to \mathcal{T}_\Sigma$ such that $p'(\mathcal{T}G') = \varphi(T) = \varphi(p(\mathcal{T}G))$. Let $\Gamma_0 = \{\gamma \in \Gamma \mid \underline{p}(\gamma) = \lambda\}$ and $\Gamma_1 = \Gamma - \Gamma_0$ (where $\underline{p}: \Gamma \to \Sigma \cup \Lambda$ is the basis of projection p). With each $\gamma \in \Gamma_0$ we associate a context-free grammar $G_\gamma = (\Gamma_0, \Gamma_1 \cup \{\#\}, R, \gamma)$ where Γ_0 is the set of nonterminal symbols of G_γ, $\Gamma_1 \cup \{\#\}$ is that of terminal symbols, the set R of production rules is defined by

$$R = \{\gamma_0 \to \gamma_1\gamma_2 \mid \gamma_0<\gamma_1,\gamma_2> \in P,\ \gamma_0 \in \Gamma_0,\ \gamma_1,\gamma_2 \in \Gamma \cup \{\#\}\},$$

and γ is the initial symbol of G_γ. Note that $L(G_\gamma) \subset \{\#\}^*\Gamma_1\{\#\}^* \cup \{\#\}^+$ for each $\gamma \in \Gamma_0$, unless γ is useless (where $L(G_\gamma)$ is the context-free language generated by G_γ). Let

$$A_\gamma = \{\alpha \in \Gamma_1 \cup \{\#\} \mid L(G_\gamma) \cap \{\#\}^*\{\alpha\}\{\#\}^* \neq \emptyset\}$$

for each $\gamma \in \Gamma_0$. As for symbols γ in $\Gamma_1 \cup \{\#\}$ we simply set $A_\gamma = \{\gamma\}$. Finally by setting

$$P' = \{\gamma<\gamma_1,\gamma_2> \mid \gamma<\gamma',\gamma''> \in P,\ \gamma \in \Gamma_1,\ \gamma_1 \in A_{\gamma'},\ \gamma_2 \in A_{\gamma''}\} \cup \{\#\},$$

$$I' = \bigcup_{\gamma\in I} A_\gamma,$$

we define tree-grammar $G' = (\Gamma_1 \cup \{\#\}, P', I')$. Then applying the same projection p as above we can get $p(\mathcal{T}G') = \varphi(p(\mathcal{T}G)) = \varphi(T)$. ::
(Remark: In the proof each A_γ can be obtained effectively from given G, and hence so is G'.)

2.2 COROLLARY If f is a tree-morphism and T is a recognizable set in Domain(f), then f(T) is also recognizable. ::
(Proof) It is known that an etm preserves recognizable sets (Thatcher [1969]). Combine this with proposition 2.1. ::

2.3 COROLLARY For rational relation R, Domain(R) and Range(R) are recognizable sets. ::

2.4 PROPOSITION A rational relation is simple if and only if it has a fine ground.

(Proof) "If" part is obvious by definition. Verifying "only if" part amounts to show that given simple rational relation R one can find a recognizable set $T \subset \mathcal{T}_\Delta$ and projections $p_1, p_2: \mathcal{T}_\Delta \to \mathcal{T}_{\Sigma\cup\Lambda}$ such that $p_1(T) \cup p_2(T) \subset \mathcal{U}_\Sigma$ and $R = \{(\varphi p_1(t), \varphi p_2(t)) \mid t \in T\}$.

Suppose $R = \{(\varphi f_1(t), \varphi f_2(t)) \mid t \in T_0\}$, where T_0 is a recognizable set in $\mathcal{T}_\Gamma$ and $f_1, f_2: \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma\cup\Lambda}$ are simple etm satisfying $f_1(T_0) \cup f_2(T_0) \subset \mathcal{U}_\Sigma$.

For each $\gamma \in \Gamma$, let $\tau_{\gamma,i} = \underline{\underline{f}}_i(\gamma)$ (i=1,2). Since f_1 and f_2 are simple, each $\tau_{\gamma,i}$ belongs to $\mathcal{A}_\Sigma$. The pair $(\tau_{\gamma,1}, \tau_{\gamma,2})$ then can be converted to a pair $(\theta_{\gamma,1}, \theta_{\gamma,2})$ of members of $\mathcal{A}_{\Sigma\cup\Lambda}$ satisfying conditions;

(1) $\theta_{\gamma,i} \Rightarrow \tau_{\gamma,i}$ (i=1,2),

(2) $D(\theta_{\gamma,1}) = D(\theta_{\gamma,2})$,

(3) $\mathrm{Yield}(\theta_{\gamma,1}) = \mathrm{Yield}(\theta_{\gamma,2})$.

(This fact is verified in next subsection.)

Obtain the product $\pi(\theta_{\gamma,1}, \theta_{\gamma,2})$ of binary functions $\theta_{\gamma,1}$ and $\theta_{\gamma,2}$ and name it θ_γ. θ_γ is then a binary function in $\mathcal{A}_{\Sigma'\times\Sigma'}$ where $\Sigma' = \Sigma \cup \Lambda$. Let $g: \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma'\times\Sigma'}$ be the simple etm with basis $\underline{\underline{g}}(\gamma) = \theta_\gamma$ ($\gamma \in \Gamma$), and $g_i: \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma'}$ be the simple etm with basis $\underline{\underline{g}}_i(\gamma) = \theta_{\gamma,i}$ ($\gamma \in \Gamma$, i=1,2). From the construction, clearly $g_i(t) = \pi_i g(t)$ for each $t \in \mathcal{T}_\Gamma$ (i=1,2). (π_i is the projection : $\mathcal{T}_{\Sigma'\times\Sigma'} \to \mathcal{T}_{\Sigma'}$ to take the i-th component of product trees.) We also know from condition (1) (which is equivalent to; $\theta_{\gamma,i}(t_1,t_2) \Rightarrow \tau_{\gamma,i}(t_1',t_2')$ holds for each $t_1, t_2, t_1', t_2' \in \mathcal{T}_{\Sigma\cup\Lambda}$ provided $t_i \Rightarrow t_i'$ (i=1,2)) that $g_i(t) \Rightarrow f_i(t)$ and hence $g_i(t) \in \mathcal{U}_\Sigma$ hold for each $t \in T_0$ (i=1,2). This means that $\varphi g_i(t)$ is defined and is equal to $\varphi f_i(t)$ for each $t \in T_0$ (i=1,2). These observations yield

$$\begin{aligned} R &= \{(\varphi f_1(t), \varphi f_2(t)) \mid t \in T_0\} \\ &= \{(\varphi g_1(t), \varphi g_2(t)) \mid t \in T_0\} \\ &= \{(\varphi \pi_1 g(t), \varphi \pi_2 g(t)) \mid t \in T_0\} \\ &= \{(\varphi \pi_1(t'), \varphi \pi_2(t')) \mid t' \in T\} \end{aligned}$$

where $T = g(T_0)$. The set T is recognizable by proposition 2.2. ::

2.5 LEMMA Given τ_1 and τ_2 in $\mathcal{A}_{\Sigma\cup\Lambda}$, one can find θ_1 and θ_2 in $\mathcal{A}_{\Sigma\cup\Lambda}$ satisfying

(1) $\theta_i \Rightarrow \tau_i$ (i=1,2),

(2) $D(\theta_1) = D(\theta_2)$,

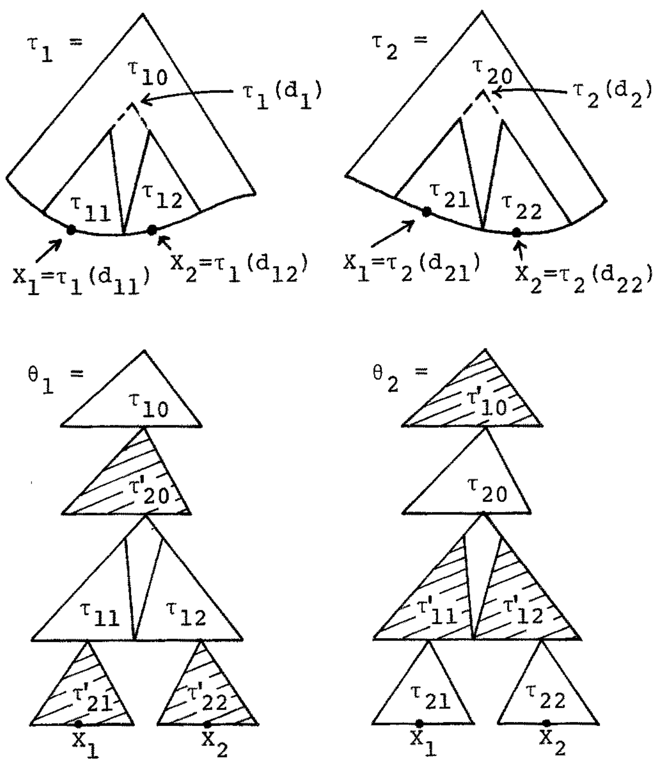
(3) $\mathrm{Yield}(\theta_1) = \mathrm{Yield}(\theta_2)$.

(Proof) The basic idea is very simple: Divide both τ_i's into three parts τ_{i0}, τ_{i1} and τ_{i2} as illustrated in Fig.2(a) where τ_{i1} is the largest subtree of τ_i containing symbol X_1 but not X_2, τ_{i2} is its dual with respect to symbols X_1 and X_2, and τ_{i0} is the rest (i=1,2). Then obtain

θ_1 from τ_1 by attaching at d_{11} (the address at which X_1 resides in τ_1) a copy of τ_{21} with labels changed to λ, and similarly at d_{12} (where X_2 resides) a copy of τ_{22} with labels also changed to λ, and then at d_1 (the address of the node under which two subtrees τ_{11} and τ_{12} start) insert a copy of τ_{20} with labels also changed to λ. Likewise, θ_2 is constructed from τ_2 by inserting copies of three parts $\tau_{10},\tau_{11},\tau_{12}$ of τ_1 at appropriate places (cf. Fig.2(b)). Technical details which are dropped from the informal description are now presented.

Suppose Index(τ_i) = (d_{i1},d_{i2}), and d_i be the longest common initial segment of d_{i1} and d_{i2} (i=1,2). Then $d_{ij} = d_i e_{ij}$ for some $e_{ij} \in \{j\}\{1,2\}^*$ (i j=1,2). We define $\theta_1,\theta_2 \in \mathcal{A}_{\Sigma\cup\Lambda}$ so that they satisfy conditions (1) - (3), as follows:

$$\begin{aligned} D(\theta_1) = D(\theta_2) = {} & (D(\tau_1) - \{d_1\}\{1,2\}^*) \\ & \cup \{d_1\}(D(\tau_2) - \{d_2\}\{1,2\}^*) \\ & \cup \{d_1 d_2 d \mid d_1 d \in D(\tau_1)\} \\ & \cup \{d_1 d_2 e_{11} d \mid d_2 d \in D(\tau_1),\quad d \in \{1\}\{1,2\}^*\} \\ & \cup \{d_1 d_2 e_{12} d \mid d_2 d \in D(\tau_2),\quad d \in \{2\}\{1,2\}^*\}. \end{aligned}$$



(a) Division of τ_i into three parts τ_{i0},τ_{i1} and τ_{i2}.

(b) Construction of θ_i. The shaded area τ'_{ij} shows a copy of τ_{ij} with labels changed to λ.

Fig. 2. Conversion of (τ_1,τ_2) to (θ_1,θ_2).

The labels $\theta_i(d)$ $(d \in D(\theta_i),\ i=1,2)$ are set to λ except

$$\theta_1(d) = \tau_1(d) \quad \text{if } d \in D(\tau_1) - \{d_1\}\{1,2\}^*,$$
$$\theta_1(d_1d_2d) = \tau_1(d_1d) \quad \text{if } d_1d \in D(\tau_1) - \{d_{11}, d_{12}\},$$
$$\theta_1(d_1d_2e_{1j}e_{2j}) = X_j \quad (j=1,2)$$
$$\theta_2(d_1d) = \tau_2(d) \quad \text{if } d \in D(\tau_2) - \{d_2\}\{1,2\}^+,$$
$$\theta_2(d_1d_2e_{1j}d) = \tau_2(d_2d)$$
$$\text{if } d_2d \in D(\tau_2) \text{ and } d \in \{j\}\{1,2\}^* \quad (j=1,2),$$
$$\theta_i(d) = \# \quad \text{if } d \in Fr(\theta_i) - \{d_1d_2e_{1j}e_{2j} \mid j=1,2\} \quad (i=1,2). \quad ::$$

2.6 PROPOSITION If T is a recognizable set in $\mathcal{T}_\Sigma$ then so is $\varphi^{-1}(T)$ $(\subset \mathcal{T}_{\Sigma\cup\Lambda})$.

(Proof) Let $T = p(\mathcal{T}G)$ where $p: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ is a projection and $G = (\Gamma \cup \{\#\}, P, I)$ is a tree-grammar. Then we can construct a tree-grammar $G' = (\Gamma' \cup \{\#\}, P', I')$ and a projection $p': \mathcal{T}_{\Gamma'} \to \mathcal{T}_{\Sigma\cup\Lambda}$ satisfying $p'(\mathcal{T}G') = \varphi^{-1}(T) = \varphi^{-1}(p(\mathcal{T}G))$, as follows:

$\Gamma' = \{\gamma, \gamma' \mid \gamma \in \Gamma\} \cup \{\#'\}$
(where we assume that symbols added differ each other and differ from those already in $\Gamma \cup \{\#\}$),

$P' = \{\gamma<\gamma_1,\gamma_2>,\ \gamma<\gamma_1',\gamma_2>,\ \gamma<\gamma_1,\gamma_2'>,\ \gamma<\gamma_1',\gamma_2'>$
$\mid \gamma<\gamma_1,\gamma_2> \in P,\ \ \gamma \in \Gamma,\ \ \gamma_1,\gamma_2 \in \Gamma \cup \{\#\}\}$
$\cup \{\gamma'<\gamma',\#'>,\ \gamma'<\gamma,\#'>,\ \gamma'<\#',\gamma'>,\ \gamma'<\#',\gamma>,$
$\gamma'<\gamma',\#>,\ \gamma'<\gamma,\#>,\ \gamma'<\#,\gamma'>,\ \gamma'<\#,\gamma> \mid \gamma \in \Gamma\}$
$\cup \{\#'<\#',\#'>,\ \#'<\#,\#'>,\ \#'<\#',\#>,\ \#'<\#,\#>,\ \#\}$

$I' = \{\gamma, \gamma' \mid \gamma \in I\}$,

$\underline{\underline{p}}'(\gamma) = \underline{\underline{p}}(\gamma)$ for each $\gamma \in \Gamma$,

$\underline{\underline{p}}'(\gamma') = \lambda$ for each $\gamma \in \Gamma \cup \{\#\}$.

(p' is the projection with basis $\underline{\underline{p}}': \Gamma' \to \Sigma \cup \Lambda$ so defined.) ::

2.7 PROPOSITION If $f_1: \mathcal{T}_{\Sigma_1} \to \mathcal{T}_{\Sigma_2}$ and $f_2: \mathcal{T}_{\Sigma_2} \to \mathcal{T}_{\Sigma_3}$ are tree-morphisms, so is the composed partial function $f = f_2f_1: \mathcal{T}_{\Sigma_1} \to \mathcal{T}_{\Sigma_3}$. If f_1 and f_2 are simple tree-morphisms, then f is also simple.

(Proof) Let $f_i': \mathcal{T}_{\Sigma_i} \to \mathcal{T}_{\Sigma_i\cup\Lambda}$ be a ground of f_i $(i=1,2)$. In case of $f_1 = f_1'$ (i.e., f_1 is an etm such that $f_1(\mathcal{T}_{\Sigma_1}) \subset \mathcal{T}_{\Sigma_2}$), the composition $f' = f_2'f_1': \mathcal{T}_{\Sigma_1} \to \mathcal{T}_{\Sigma_3\cup\Lambda}$ is an etm, and can serve as a ground of f. (The closure of etm under composition is easily shown by a constructive proof.) In case where $f_1 \neq f_1'$, extend the domain of f_2' to $\mathcal{T}_{\Sigma_2\cup\Lambda}$ by setting $\underline{\underline{f}}_2'(\lambda) = \lambda<X_1,X_2>$. Then the extended function $f_2': \mathcal{T}_{\Sigma_2\cup\Lambda} \to \mathcal{T}_{\Sigma_3\cup\Lambda}$ is an etm satisfying $\varphi f_2'(u) = \varphi f_2'(\varphi(u))$ for each $u \in \varphi^{-1}(\text{Domain}(f_2))$, because $f_2'(u) \Rightarrow f_2'(\varphi(u)) \Rightarrow \varphi f_2'(\varphi(u))$. Hence whenever $f(t)$ is defined we have $f(t) = f_2(f_1(t)) = \varphi f_2'(\varphi f_1'(t)) = \varphi f_2'f_1'(t)$. This implies that f is a tree-morphism with ground

$f_2'f_1': \mathcal{T}_{\Sigma_1} \to \mathcal{T}_{\Sigma_3\cup\Lambda}$. The preservation of simplicity (of grounds) under composition should be obvious. ::

2.8 COROLLARY If R is a rational relation, and f_1 and f_2 are tree-morphisms such that $R \subset \mathrm{Domain}(f_1) \times \mathrm{Domain}(f_2)$, then complex relation $R' = \{(f_1(t_1), f_2(t_2)) \mid (t_1,t_2) \in R\}$ is rational. In case where R, f_1 and f_2 are simple, so is the complex relation R'. ::

3. Main Results

3.1 THEOREM The class of simple rational relations is closed under composition.

(Proof) Given simple rational relations R_1 and R_2, take recognizable sets T_i ($\subset \mathcal{T}_\Gamma$) and projections $p_{i1}, p_{i2}: \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma\cup\Lambda}$ such that $p_{i1}(T_i) \cup p_{i2}(T_i) \subset \mathcal{U}_\Sigma$ and $R_i = \{(\varphi p_{i1}(t), \varphi p_{i2}(t)) \mid t \in T_i\}$ (i=1,2) (§2.4). To see that the composition

$$R_2R_1 = \{(\varphi p_{11}(t_1), \varphi p_{22}(t_2)) \mid t_i \in T_i \ (i=1,2),\ \varphi p_{12}(t_1) = \varphi p_{21}(t_2)\}$$

is rational it suffices (by corollary 2.8) to verify that relation

$$R = \{(t_1,t_2) \in T_1 \times T_2 \mid \varphi p_1(t_1) = \varphi p_2(t_2)\}$$

is a simple rational relation where $p_1 = p_{12}$ and $p_2 = p_{21}$. For the purpose, first let us extend the domain of projections p_1 and p_2 from $\mathcal{T}_\Gamma$ to $\mathcal{T}_{\Gamma\cup\Lambda}$ by setting $\underline{\underline{p}}_1(\lambda) = \underline{\underline{p}}_2(\lambda) = \lambda$, and let

$$R' = \{(\varphi(u_1), \varphi(u_2)) \mid u_i \in \varphi^{-1}(T_i) \ (i=1,2),\ p_1(u_1) = p_2(u_2)\}.$$

Then we can observe that R equals R'. (To see $R \subset R'$, let T_i' be the set of all subtrees of trees in T_i (i=1,2). Then we have $p_1(T_1') \cup p_2(T_2') \subset \mathcal{U}_\Sigma$ since $p_1(T_1) \cup p_2(T_2) \subset \mathcal{U}_\Sigma$ and $\mathcal{U}_\Sigma$ is "subtree-closed," i.e., subtrees of trees in $\mathcal{U}_\Sigma$ are also contained in $\mathcal{U}_\Sigma$. Now we can verify by induction on the total number of nodes in trees that for each pair $(t_1,t_2) \in T_1' \times T_2'$ such that $\varphi p_1(t_1) = \varphi p_2(t_2)$, there exists a pair $(u_1,u_2) \in \mathcal{U}_\Gamma \times \mathcal{U}_\Gamma$ satisfying

$$(1)\ \begin{cases} u_i \Rightarrow t_i \ (i=1,2), \text{ and} \\ p_1(u_1) = p_2(u_2). \end{cases}$$

The inductive process goes as follows: If $t_i = \gamma_i\langle t_{i1}, t_{i2}\rangle$ ($\gamma_i \in \Gamma$, $t_{ij} \in \mathcal{T}_\Gamma$ (i,j=1,2)) and $\underline{\underline{p}}_1(\gamma_1) \neq \lambda \neq \underline{\underline{p}}_2(\gamma_2)$, then we should have $\underline{\underline{p}}_1(\gamma_1) = \underline{\underline{p}}_2(\gamma_2)$ and $\varphi p_1(t_{1j}) = \varphi p_2(t_{2j})$ (j=1,2). Assuming that we get pairs $(u_{1j}, u_{2j}) \in \mathcal{U}_\Gamma \times \mathcal{U}_\Gamma$ satisfying condition (1) for the pair (t_{1j}, t_{2j}) (j=1,2), set $u_i = \gamma_i\langle u_{i1}, u_{i2}\rangle$ (i=1,2). Then the pair (u_1,u_2) fulfils condition (1) for original pair (t_1,t_2). If $t_1 = \gamma_1\langle t_{11}, t_{12}\rangle$ with $\underline{\underline{p}}_1(\gamma_1) = \lambda$, then we must have either $p_1(t_{11}) \in \mathcal{U}$ or $p_1(t_{12}) \in \mathcal{U}$

since $p_1(t_1) \in \mathcal{U}_\Sigma$. If $p_1(t_{11}) = u \in \mathcal{U}$ we set $u_1 = \gamma_1 \langle t_{11}, u_1' \rangle$ and $u_2 = \lambda \langle u, u_2' \rangle$ where $(u_1', u_2') \in \mathcal{U}_\Gamma \times \mathcal{U}_\Gamma$ is a pair satisfying condition (1) for (t_{12}, t_2). Note that in this case we have $\varphi p_1(t_{12}) = \varphi p_1(t_1) = \varphi p_2(t_2)$ and $(t_{12}, t_2) \in T_1' \times T_2'$, and hence we can find such (u_1', u_2') for (t_{12}, t_2) by inductive hypothesis. Other cases (where $p_1(t_{12}) \in \mathcal{U}$ or where $t_2 = \gamma_2 \langle t_{21}, t_{22} \rangle$ with $\underline{\underline{p}}_2(\gamma_2) = \lambda$) can be worked out in the same principle. Finally when $(t_1, t_2) = (\#, \#)$ let $(u_1, u_2) = (\#, \#)$. Since the first part of condition (1) implies that $\varphi(u_i) = t_i$, this completes the proof of $R \subset R'$. The converse is easy; just remark that $\varphi p_i(\varphi(u_i)) = \varphi p_i(u_i)$ for each $u_i \in \varphi^{-1}(T_i) \subset \varphi^{-1} p_i^{-1}(\mathcal{U}_\Sigma)$ $(i=1,2)$.)

Next, to see that R' is rational we consider the subset U of $\mathcal{T}_{\Sigma' \times \Sigma'}$ (where $\Gamma' = \Gamma \cup \Lambda$) defined by

$$U = \{\pi(u_1, u_2) \mid u_i \in \varphi^{-1}(T_i)\ (i=1,2),\ p_1(u_1) = p_2(u_2)\}.$$

(Since p_1 and p_2 are projections, $p_1(u_1) = p_2(u_2)$ implies $D(u_1) = D(u_2)$ and hence $\pi(u_1, u_2)$ is defined.)

Let

$$\Delta = \{(\alpha_1, \alpha_2) \in \Gamma' \times \Gamma' \mid \underline{\underline{p}}_1(\alpha_1) = \underline{\underline{p}}_2(\alpha_2)\}.$$

Then the set U can be written as

$$U = \mathcal{T}_\Delta \cap \pi_1^{-1}(\varphi^{-1}(T_1)) \cap \pi_2^{-1}(\varphi^{-1}(T_2)).$$

Here the sets $\mathcal{T}_\Delta$, T_1 and T_2 are recognizable, and operations $\cap, \pi_1^{-1}$, π_2^{-1} and φ^{-1} are known to preserve recognizable sets (Thatcher & Wright [1968], and proposition 2.6). Therefore U is recognizable.

Finally by noting equality

$$R' = \{(\varphi\pi_1(u), \varphi\pi_2(u)) \mid u \in U\},$$

we can conclude that R' is a simple rational relation. ::

3.2 THEOREM The class of rational relations is closed under composition.

(Proof) Suppose $R_i = \{(f_{i1}(t), f_{i2}(t)) \mid t \in T_i\}$ $(i=1,2)$ are rational relations where T_1, T_2 are recognizable sets and $f_{11}, f_{12}, f_{21}, f_{22}$ are tree-morphisms such that $T_i \subset \mathrm{Domain}(f_{ij})$. We assume here that f_{12} and f_{21} are simple. (Otherwise, flipping suitable branching edges of trees in T_i and exchanging accordingly the symbols X_1 and X_2 shown in the bases $\underline{\underline{f}}_{ij}'$ of the grounds f_{ij}' of f_{ij} one can convert the recognizable sets T_i and tree-morphisms f_{ij} $(i,j=1,2)$ so that they meet the assumption.) Apply theorem 3.1 to simple rational relations $R_1' = \{(t, f_{12}(t)) \mid t \in T_1\}$ and $R_2' = \{(f_{21}(t), t) \mid t \in T_2\}$, and then apply corollary 2.8. ::

3.3 COROLLARY If R is a rational relation and T is a recognizable set, then the image of T by R

$$R(T) = \{y \mid (x,y) \in R \text{ for some } x \in T\}$$

is recognizable.

(Proof) Consider the composition RR_1 of rational relations where $R_1 = \{(x,x) \mid x \in T\}$. Then the image $R(T)$ is equal to Range(RR_1). Apply theorem 3.2 and corollary 2.3. ::

3.4 COROLLARY The inverses of tree-morphisms preserve recoginizability.

(Proof) The inverse of etm $f: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ preserves recognizability, since $R = \{(f(t),t) \mid t \in \mathcal{T}_\Gamma\}$ is a rational relation and $R(T) = f^{-1}(T)$ for each $T \subset \mathcal{T}_\Sigma$. Then the inverse of tree-morphism φf where $f: \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma \cup \Delta}$ is an etm is shown to have the same property because $(\varphi f)^{-1} = f^{-1}\varphi^{-1}$. ::

APPENDIX

A <u>linear</u> <u>homomorphism</u> (<u>of</u> <u>binary</u> <u>trees</u>), abbreviated by <u>lh</u>, is a function $h: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ defined as

$$h(\#) = \underline{\underline{h}}(\#),$$

$$h(\gamma\langle t_1,t_2\rangle) = \underline{\underline{h}}(\gamma)(h(t_1),h(t_2)) \qquad (\gamma \in \Gamma,\ t_1,t_2 \in \mathcal{T}_\Gamma),$$

by giving its basis $\underline{\underline{h}}: \Gamma \cup \{\#\} \to \mathcal{C}_\Sigma$ satisfying $\underline{\underline{h}}(\#) \in \mathcal{T}_\Sigma$. Here

$$\mathcal{C}_\Sigma = \{\tau \in \mathcal{T}_{\Sigma,\{X_1,X_2\}} \mid \text{Yield}(\tau) \text{ contains } X_1 \text{ and } X_2 \text{ at most once, respectively}\},$$

and each τ in $\mathcal{C}_\Sigma$ is viewed as a binary function sending $(t_1,t_2) \in \mathcal{T}_\Sigma \times \mathcal{T}_\Sigma$ to the tree that is obtained from indexed tree τ by replacing its index symbol X_j, if any, by t_j ($j=1,2$). For example, $\sigma\langle X_2,X_1\rangle(t_1,t_2) = \sigma\langle t_2,t_1\rangle$, $X_1(t_1,t_2) = t_1$, $\#(t_1,t_2) = \#$.

If $h_1,h_2: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ are lh, and T is a recognizable set in $\mathcal{T}_\Gamma$, then binary relation $\{(h_1(t),h_2(t)) \mid t \in T\}$ is termed as a <u>linear</u> <u>bimorphism</u> (<u>lb</u>, for short).

Now we prove the equivalence of the notions of lb and of rational relations.

PROPOSITION An lb is a rational relation.

(Proof) Given lb R, one can find a tree-grammar $G = (\Gamma \cup \{\#\},P,I)$ and lh $h_1,h_2: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ such that $R = \{(h_1(t),h_2(t)) \mid t \in \mathcal{T}G\}$. Assuming that no symbols in Γ are useless (i.e., $\mathcal{T}G \not\subset \mathcal{T}_{\Gamma'}$ for any proper subset Γ' of Γ), we define tree-grammar $G' = (\Delta \cup \{\#\},P',I')$ as follows;

$$\Delta' = (\Gamma \cup \{\#,*\}) \times (\Gamma \cup \{\#,*\}) \text{ where } * \notin \Gamma \cup \{\#\},$$

$$I' = \{(\gamma,\gamma) \mid \gamma \in I\},$$

$$P' = \{(\gamma,\gamma)<([\gamma_1]_{\gamma,1},[\gamma_1]_{\gamma,2}),([\gamma_2]_{\gamma,1},[\gamma_2]_{\gamma,2})>,$$
$$(\gamma,*)<([\gamma_1]_{\gamma,1},\ *\ ),([\gamma_2]_{\gamma,1},\ *\ )>,$$
$$(*,\gamma)<(\ *\ ,[\gamma_1]_{\gamma,2}),(\ *\ ,[\gamma_2]_{\gamma,2})>$$
$$\mid \gamma<\gamma_1,\gamma_2> \in P\} \cup \{(\#,\#)<(*,*),(*,*)>,$$
$$(\#,*)<(*,*),(*,*)>,\ \ (*,\#)<(*,*),(*,*)>,$$
$$(*,*)<\#,\#>,\ \ \#\}$$

where $[\gamma_j]_{\gamma,i}$ stands for either γ_j or $*$ depending on whether $\underline{\underline{h}}_i(\gamma)$ contains X_j or not, respectively. Next we define etm f_i: $\mathcal{T}_\Delta \to \mathcal{T}_{\Sigma\cup\Lambda}$ (i=1,2), as follows;

$$\underline{\underline{f}}_i(\gamma_1,\gamma_2) = \begin{cases} \underline{\underline{h}}_i(\gamma_i) & \text{if } \gamma_i \in \Gamma \text{ and } \underline{\underline{h}}_i(\gamma_i) \text{ contains both } X_1 \text{ and } X_2, \\ \lambda<\underline{\underline{h}}_i(\gamma_i),X_2> & \text{if } \gamma_i \in \Gamma \text{ and } \underline{\underline{h}}_i(\gamma_i) \text{ contains } X_1 \text{ but not } X_2, \\ \lambda<\underline{\underline{h}}_i(\gamma_i),X_1> & \text{if } \gamma_i \in \Gamma \text{ and } \underline{\underline{h}}_i(\gamma_i) \text{ contains } X_2 \text{ but not } X_1, \\ \lambda<\underline{\underline{h}}_i(\gamma_i),\lambda<X_1,X_2>> & \text{if } \gamma_i \in \Gamma \cup \{\#\} \text{ and } \underline{\underline{h}}_i(\gamma_i) \text{ contains neither } X_1 \text{ nor } X_2, \\ \lambda<X_1,X_2> & \text{if } \gamma_i = *. \end{cases}$$

Then it can be verified that $f_i(\mathcal{T}G') \subset \mathcal{U}_\Sigma$ (i=1,2) and $R = \{(\varphi f_1(t), \varphi f_2(t)) \mid t \in \mathcal{T}G'\}$. ::

PROPOSITION A rational relation is an lb.

(Proof) It is observed that any etm $f: \mathcal{T}_\Gamma \to \mathcal{T}_\Sigma$ can be decomposed into the form $f = f''f'$ where $f': \mathcal{T}_\Gamma \to \mathcal{T}_\Delta$ is a simple etm, and $f'': \mathcal{T}_\Delta \to \mathcal{T}_\Sigma$ is an etm satisfying $\underline{\underline{f}}''(\Delta) \subset \{\sigma<X_1,X_2>,\sigma<X_2,X_1> \mid \sigma \in \Sigma\}$. From this combined with proposition 2.4, given rational relation R one can find tree-grammar $G = (\Gamma \cup \{\#\},P,I)$ and etm $f_1,f_2: \mathcal{T}_\Gamma \to \mathcal{T}_{\Sigma\cup\Lambda}$ satisfying

$$\underline{\underline{f}}_i(\Gamma) \subset \{\alpha<X_1,X_2>,\alpha<X_2,X_1> \mid \alpha \in \Sigma \cup \Lambda\} \quad (i=1,2),$$
$$f_i(\mathcal{T}G) \subset \mathcal{U}_\Sigma \quad (i=1,2),$$
$$R = \{(\varphi f_1(t),\varphi f_2(t)) \mid t \in \mathcal{T}G\}.$$

Assuming that no symbols in Γ are useless, let $G_\gamma = (\Gamma \cup \{\#\},P,\{\gamma\})$ for each $\gamma \in \Gamma$, and $\Gamma_i = \{\#\} \cup \{\gamma \in \Gamma \mid \varphi f_i(\mathcal{T}G_\gamma) = \{\#\}\}$ (i=1,2). Then if $\underline{\underline{f}}_i(\gamma) \in \{\lambda<X_1,X_2>,\lambda<X_2,X_1>\}$ and $\gamma<\gamma',\gamma''> \in P$, we have either $\gamma' \in \Gamma_i$ or $\gamma'' \in \Gamma_i$, because $f_i(\mathcal{T}G) \subset \mathcal{U}_\Sigma$ (i=1,2). Based on the observation we now define tree-grammar $G' = (\Delta \cup \{\#\},P',I')$ and lh $h_1,h_2: \mathcal{T}_\Delta \to \mathcal{T}_\Sigma$, so that they satisfy $R = \{(h_1(t),h_2(t)) \mid t \in \mathcal{T}G'\}$, as follows;

$$\Delta = \{[\gamma,\gamma',\gamma''] \mid \gamma<\gamma',\gamma''> \in P\},$$
$$P' = \{[\gamma,\gamma',\gamma'']<[\alpha,\alpha',\alpha''],[\beta,\beta',\beta'']> \mid \gamma' = \alpha,\ \gamma'' = \beta\}$$
$$\cup \{[\gamma,\gamma',\gamma'']<[\alpha,\alpha',\alpha''],\#> \mid \gamma' = \alpha,\ \gamma'' = \#\}$$
$$\cup \{[\gamma,\gamma',\gamma'']<\#,[\beta,\beta',\beta'']> \mid \gamma' = \#,\ \gamma'' = \beta\}$$
$$\cup \{[\gamma,\gamma',\gamma'']<\#,\#> \mid \gamma' = \gamma'' = \#\} \cup \{\#\}$$

where $[\gamma,\gamma',\gamma'']$, $[\alpha,\alpha',\alpha'']$ and $[\beta,\beta',\beta'']$ run over Δ,

$$I' = \{[\gamma,\gamma',\gamma''] \in \Delta \mid \gamma \in I\} \cup (I \cap \{\#\}),$$

$$\underline{\underline{h}}_i([\gamma,\gamma',\gamma'']) = \begin{cases} \underline{\underline{f}}_i(\gamma) & \text{if } \underline{\underline{f}}_i(\gamma) \notin \Phi, \\ X_1 & \text{if } \underline{\underline{f}}_i(\gamma) \in \Phi, \quad \gamma' \notin \Gamma_i \text{ and } \gamma'' \in \Gamma_i, \\ X_2 & \text{if } \underline{\underline{f}}_i(\gamma) \in \Phi, \quad \gamma' \in \Gamma_i \text{ and } \gamma'' \notin \Gamma_i, \\ \# & \text{if } \underline{\underline{f}}_i(\gamma) \in \Phi \text{ and } \gamma',\gamma'' \in \Gamma_i, \end{cases}$$

where $\Phi = \{\lambda\langle X_1,X_2\rangle, \lambda\langle X_2,X_1\rangle\}$, $[\gamma,\gamma',\gamma''] \in \Delta$, $i=1,2$,

$\underline{\underline{h}}_1(\#) = \underline{\underline{h}}_2(\#) = \#$. ::

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