

THE EFFECT ON OPTIMAL CONSUMPTION OF INCREASED UNCERTAINTY
IN LABOR INCOME IN THE MULTIPERIOD CASE*

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ABSTRACT

We consider a multiperiod, additive utility, optimal consumption model with a riskless investment and a stochastic labor income. The main result is that for utility functions belonging to the set F , consumption decreases when we go from any sequence of distribution functions representing labor income to a more risky sequence. It is shown that a concave utility function belongs to F if and only if its first derivative exists everywhere and is convex.

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I. Introduction

The impact on consumption of increased uncertainty in future labor or capital income has been examined by a number of authors in the last ten years. As illustrated by Sandmo [23], the answers one gets are different in the two cases of uncertain labor income and uncertain capital income. Therefore, in order to separate these effects, the models with random labor income generally have one non-risky investment opportunity, and those with random investment opportunities have a deterministic (or zero) labor income. The model in this paper conforms to the above dichotomy. The only exception to that rule seems to be in Section 8 of Merton [14] which treats the case of a nondecreasing Poisson income stream, an exponential utility function, and two investment opportunities, one riskless and the other described by Brownian motion.

Three relatively early papers which examine the random labor income case are the two-period models of Leland [11], Sandmo [23] and Dr  ze and Modigliani [5]. Their problem is: given the first period labor income y_1 and the distribution function Y_2 of the labor income in period two, choose consumption c_1 in period 1, $0 \leq c_1 \leq y_1$, so as to maximize $EU(c_1, (1+r)(y_1 - c_1) + Y_2)$.

Assuming the utilities are additive, $(U = u_1 + u_2)$, Leland [11, eq. (25)] concludes that concavity and a positive third derivative imply that there is a decrease in consumption when going from the deterministic income case to the random income case with the same mean and an infinitesimal random element (such that a second-order Taylor approximation is valid). Sandmo compares parameterized versions of Y_2 of the form

$\alpha Y_2 + (1-\alpha)E(Y_2)$, $0 \leq \alpha \leq k$, where k is such that the income remains nonnegative, and he demonstrates that c_1 is a decreasing function of α (and of risk) when U has decreasing temporal risk aversion. His results imply that in the case of a concave additive utility function, c_1 is a decreasing function of α when the third derivative is positive. In [5] Drèze and Modigliani look at the income and substitution effects of increased risk in labor income.

The model that we will be working with is an infinite horizon additive utility model which the author used in [15]. There the main qualitative result was that for isoelastic utility functions, consumption decreases when we go from the deterministic labor income case to the random labor income case with the same mean. In this paper we will show that for utility functions belonging to the set F (defined in Section 3), consumption decreases when we go from any sequence of distribution functions representing labor income to a more risky sequence where we are using increased risk in the sense of Rothschild and Stiglitz [20,22]. In Section 3 we show that a concave utility function belongs to F if and only if its first derivative exists everywhere and is convex. Therefore if a concave utility function is thrice differentiable, then it belongs to F if and only if its third derivative is nonnegative. It is an easy exercise to verify that the isoelastic utility functions, $\frac{1}{\gamma} c^\gamma$, $\gamma < 1$, $\gamma \neq 0$, belong to F .

In the random capital income case with an isoelastic utility function, the effect on consumption of increased risk in the return on capital is different depending on whether $\gamma < 0$ or $\gamma > 0$ (Rothschild and Stiglitz [21, Section 3]). Therefore we get the differing conclusions in the

random labor income and random capital income models with an infinite horizon that Sandmo observed in his two-period model. In [4], p. 354, Diamond and Stiglitz have further analyzed and clarified the effect of increased risk in the random capital income case with an isoelastic utility function using the concept of mean utility preserving increase in risk.

In view of the importance of a nonnegative third derivative as exhibited in Leland [11] and Mirman [16, Appendix], it is not surprising that the third derivative is also the key condition in the model considered here. Its import is made all the more plausible when we recall the certainty-equivalence results of Theil [25], Simon [24], and recently Duchan [6]. Essentially, their results state that with a quadratic utility function (third derivative zero) and linear state equation with an additive random disturbance, the decisions are unaltered if the random elements are replaced by their means.

In Section 2 we consider the special case of an n period model with a quadratic utility function. In addition to showing that the higher derivatives of the optimal return function may not exist, this model serves as an example where the qualitative results of a two-period model do not hold for a multiperiod model. The general model is introduced in Section 3, and, using some earlier results of Miller [15], the main result is established.

II. A Quadratic Utility Multiperiod Model

As described in the introduction, the effect of uncertainty on consumption has been investigated in both two-period models and multiperiod

models. The difference between analyzing the two-period and the multi-period models is that the multiperiod models require an induction step on some property of the optimal return functions of dynamic programming. This induction step has been carried out in the uncertain capital income case by Fama [7] with the property of concavity (he does not assume additive utilities), by Neave [18] for decreasing absolute risk aversion, and by both Hakansson [8] and Mossin [17] for isoelastic functions. However, Hakansson [9] has given examples where the induction property does not hold and this section provides another. At the same time we show that the third derivative of the optimal return function may not exist even if that of the utility function does. This motivates the definition of the set of functions F in Section 3.

In order to describe the multiperiod quadratic utility model we define:

- x_j : the nonnegative amount of capital at the beginning of the period when j periods remain.
- R_j : the nonnegative labor income received at the end of the period when j periods remain. We assume that the R_j are bounded by K , and allow no borrowing against future labor income. We let \bar{R}_j stand for the mean of R_j .
- $u(c_j)$: the utility from consumption of c_j when j periods remain. We assume $u(c_j) = c_j + bc_j^2$ where $b < 0$. We also assume that the point where u is decreasing, $-1/2b$, is very large relative to K and the problem parameters.
- $r-1$: the rate of interest. In this model we set $r = 1$.

Our objective is to determine the decision rule which maximizes

$$E\left(\sum_{j=1}^N u(c_j)\right).$$

Here, as elsewhere, E stands for expected value.

If we let $V(x, j)$ be the optimal return function when capital equals x and j periods remain, then the terminal condition is $V(x, 0) = 0$.

Since we consume all the capital when there is one period to go as long as u is increasing, $V(x, 1) = x + bx^2$, $0 \leq x \leq -1/2b$.

For $0 \leq x \leq -1/2b - K$, and $j = 2$,

$$\begin{aligned} V(x, 2) &= \max_{0 \leq c \leq x} E(c + bc^2 + V(x - c + R_2, 1)) \\ &= \max_{0 \leq c \leq x} E(c + bc^2 + (x - c + R_2) + b(x - c + R_2)^2). \end{aligned}$$

By concavity, the optimality condition is

$$E(1 + 2bc - 1 - 2b(x - c + R_2)) = 0,$$

$$\text{or} \quad c = \frac{x + \bar{R}_2}{2}, \quad (1)$$

subject to $0 \leq c \leq x$. Since $x, \bar{R}_2 \geq 0$, the inequality $c \geq 0$ is always satisfied. If $x \geq \bar{R}_2$, then the optimal decision is given by (1). If $x < \bar{R}_2$ the best we can do is set $c = x$.

Therefore for the two-period additive model we have the following result: the optimal decision depends only on labor income through its mean.

In order to evaluate $V(x, 2)$ we substitute the optimal decision to obtain

$$\begin{aligned} V(x, 2) &= K_0 + a_0 x + b_0 x^2 & 0 \leq x < \bar{R}_2 \\ &= K_1 + a_1 x + b_1 x^2 & \bar{R}_2 \leq x \leq -1/2b - K \end{aligned}$$

where $a_0 = 1$, $b_0 = b$, $a_1 = 1 + b\bar{R}_2$, $b_1 = b/2$.

Thus $V(\cdot, 2)$ is described by two quadratic functions. Its third derivative is zero except at $x = \bar{R}_2$ where it is undefined and where the second derivative is discontinuous.

For $0 \leq x \leq -1/2b - 2K$, and $j = 3$,

$$V(x, 3) = \max_{0 \leq c \leq x} E(c + bc^2 + V(x - c + R_3, 2))$$

so that the optimality condition is

$$1 + 2bc - \int_{y: x-c+y < \bar{R}_2} (a_0 + 2b_0(x-c+y)) dF(y) - \int_{y: x-c+y \geq \bar{R}_2} (a_1 + 2b_1(x-c+y)) dF(y) = 0$$

where F is the distribution function of R_3 . Let $x = 10$, $\bar{R}_2 = 10.01$, $R_3 = 10$ with probability one in case (a), and $R_3 = 6, 14$ with probability one half each in case (b). In case (a), $c = (10 + 10)/2 = 10$. In case (b), $c = (30 + 12 + 14 + 10.01)/7 = 9.43$. Therefore for the multiperiod model it is no longer true that the optimal decision only depends on labor income through its mean.

One might wonder how this example could be consistent with the certainty-equivalence results of [6], [24] and [25] which apply in the multiperiod case. The answer is that those results require that there be no constraints on the decision variables. It is precisely the constraint $c \leq x$ which causes $V(x, 2)$ to be described by two quadratic functions.

III. The Model and the Main Results

Except for a more general class of utility functions, the model we consider is the same as that presented by Miller in [15], so that we will limit ourselves to the bare essentials and refer the reader to [15] for

discussion of the model. Unlike the example of Section 2, the periods are numbered chronologically and some borrowing is permitted against future labor income. Consider

- (x, j) : the state of the system where x represents the capital at the beginning of period j .
- $r-1$: the rate of interest for both lending and borrowing where $r > 1$.
- Y_j : the nonnegative random income received at the end of period j . It is convenient to divide Y_j into certain and uncertain parts by $Y_j = y_j + R_j$ where $y_j = \sup\{h: F_j(h) = 0\}$ and $F_j(\cdot)$ is the distribution function of Y_j . We also assume that the Y_j are independent, but not identically distributed, that the means of R_j are uniformly bounded, and that $\sum_{i=1}^{\infty} r^{-i} y_i < \infty$. It is significant that we do not assume that the Y_j are identically distributed, for otherwise the optimal decision in period j would depend on the value of x_j and not on j .
- D_j : the amount of debt allowed in period j equals $\sum_{i=1}^{\infty} r^{-i} y_{j+i-1}$. D_j is finite by our assumption above concerning the y_j . Thus we allow the individual to borrow against certain future income and x_j can take on values in $[-D_j, \infty)$.
- c_j : the consumption in period j . We require that $0 \leq c_j \leq x_j + D_j$.
- $U(c_1, c_2, c_3, \dots)$: the utility function for all feasible c_1, c_2, \dots , equals $\sum_{i=1}^{\infty} \alpha^{i-1} u(c_i)$, where α is a discount factor between 0 and 1. We will restrict our attention to $u \in F$ where F is defined below

Definition. A concave function $g: X \rightarrow \mathbb{R}$, with the convex set $X \subset \mathbb{R}$, belongs to F if for every set of $\lambda_i, \Delta_i, i = 1, \dots, n$, satisfying $\lambda_i \geq 0, \sum \lambda_i = 1$, and $\sum \lambda_i \Delta_i = 0$,

$$g(x_1) - \sum_{i=1}^n \lambda_i g(x_1 + \Delta_i) \geq g(x_2) - \sum_{i=1}^n \lambda_i g(x_2 + \Delta_i) \quad (2)$$

where $x_2 \geq x_1$, x_2, x_1 are in the interior of X .

In this paper X will be $[0, \infty)$ or $(0, \infty)$ if g is a utility function, and $[-D_j, \infty)$ if g is related to the optimal return function of period j .

The decision making takes place as follows. In period 1 the individual has x_1 units of capital. He decides to consume c_1 , where $0 \leq c_1 \leq x_1 + D_1$, and he receives a utility $u(c_1)$. The resulting capital (or debt) grows to $r(x_1 - c_1)$ and a random income Y_1 is received at the end of period 1 so that x_2 equals $r(x_1 - c_1) + Y_1$. In general starting from state (x, j) the new state is given by

$$T(x, j) = (r(x - c) + Y_j, j + 1) . \quad (3)$$

By a policy δ we mean a decision rule that specifies the amount $c_j = \delta(x, j)$ that we consume given that we are in state (x, j) . We let $f_\delta(x, j)$ be the expected value of U when using an admissible policy δ and starting from state (x, j) , and define $f(x, j) = \sup_{\delta} f_\delta(x, j)$.

A policy δ^* is said to be optimal if $f_{\delta^*} = f$. The functional equation of dynamic programming is

$$f(x, j) = \sup_{0 \leq c_j \leq x_j + D_j} (u(c_j) + \alpha E f(T(x, j))) . \quad (4)$$

Some useful notation is

$$\begin{aligned} h((x, j), c, v) &= u(c) + \alpha E v(T(x, j)) \\ (Av)(x, j) &= \sup_{0 \leq c \leq x + D_j} h((x, j), c, v) . \end{aligned}$$

Thus equation (4) can be written as

$$f(x,j) = Af(x,j) \quad (5)$$

so that the problem of finding solutions to (4) is then equivalent to the problem of finding fixed points of A .

An interpretation that can be given to the function $h((x,j),c,v)$ is that it represents the expected return in a one-period model where the state is (x,j) , the decision is c , and v is the terminal reward function. In turn, $(Av)(x,j)$ represents the expected return in the same situation when an optimal decision is made. Often the v chosen will be the optimal return function.

Let v be fixed, and for a given state (x,j) let $c^*(x,j)$ be the (feasible) value of c which maximizes $h((x,j),c,v)$. If both v and u are concave functions, then it is known (and also very easy to prove) that both

$$c^*(x,j) \text{ and } x - c^*(x,j) \text{ are nondecreasing functions of } x. \quad (6)$$

In the event that there is more than one optimal decision we let $c^*(x,j)$ be the smallest such decision.

In this paper we will assume that we are only considering utility functions such that a unique finite-valued f satisfying (5) exists. In [15] this question was examined in detail for the isoelastic functions. For example, with the log utility function it was shown that a unique finite valued f satisfying (5) exists if we restrict $\varepsilon(x_j + D_j) \leq c_j \leq (1-\varepsilon)(x_j + D_j)$ for any fixed $\varepsilon > 0$. In order to go to the case here of $0 \leq c_j \leq (x_j + D_j)$ one needs to go through the exercise of showing the nonoptimality (with respect to f) of the newly admissible c_j . The difficulty is that the basic papers of discrete dynamic programming, Blackwell

[1] and Denardo [3], require that the reward function be bounded over all admissible states and decisions, an assumption which is not satisfied by any unbounded u . Only recently have techniques been developed which get away from this restriction (Lippman [12,13] and Harrison [10]). Fortunately, there is no difficulty whatsoever in the finite period case, so our results apply without qualification for all $u \in F$.

Theorem 1. A concave function $g: X \rightarrow R$, where the convex set $X \subset R$, belongs to F if and only if the first derivative of g , g' , exists everywhere on the interior of X and is convex. From Rockafeller [19, Theorems 23.1, 24.1, 24.2 and Corollary 24.2.1.] the convexity of g' implies that

- (a) the right hand and left hand derivatives of g' , g'_+ and g'_- , exist everywhere on the interior of X , are increasing, and satisfy $g'_+ \geq g'_-$.
 (b) for any $x, y \in X$,

$$g'(y) - g'(x) = \int_x^y g'_+(t) dt = \int_x^y g'_-(t) dt .$$

Proof. We first establish the "if" part of the theorem by showing that if the first derivative of g exists everywhere on the interior of X and is convex, then g belongs to F . We must show that (2) holds which we rewrite as

$$\sum \lambda_i [g(x_1) - g(x_1 + \Delta_i) - g(x_2) + g(x_2 + \Delta_i)] \geq 0 .$$

For any i ,

$$\begin{aligned} & (g(x_1) - g(x_1 + \Delta_i) - g(x_2) + g(x_2 + \Delta_i)) \\ &= - \int_{x_1}^{x_1 + \Delta_i} g'(y) dy + \int_{x_2}^{x_2 + \Delta_i} g'(y) dy \\ &= - \int_{x_1}^{x_1 + \Delta_i} [g'(x_1) + \int_{x_1}^y g'_+(z) dz] dy \\ &\quad + \int_{x_2}^{x_2 + \Delta_i} [g'(x_2) + \int_{x_2}^y g'_+(z) dz] dy , \end{aligned}$$

using (b) of Theorem 1,

$$\geq -\Delta_i [g'(x_1) - g'(x_2)] \quad (7)$$

since g_+'' is increasing. To see this inequality, observe that $g_+''(z_2) \geq g_+''(z_1)$ where z_2 is the same distance above x_2 that z_1 is above x_1 when $\Delta_i > 0$. If $\Delta_i < 0$ then $y < x_1$ or $y < x_2$ as the case may be, and $g_+''(z_2) \geq g_+''(z_1)$ where z_2 is the same distance below x_2 that z_1 is below x_1 . Therefore

$$\begin{aligned} & \Sigma \lambda_i (g(x_1) - g(x_1 + \Delta_i) - g(x_2) + g(x_2 + \Delta_i)) \\ & \geq -\Sigma \lambda_i \Delta_i [g'(x_1) - g'(x_2)] = 0. \end{aligned}$$

We begin the proof of the "only if" part by establishing that g' exists everywhere on the interior of X . Assume the contrary, that is for some x , the derivative does not exist. Since g is concave, both the right and left hand derivatives exist at x and we must have $g'_+(x) - g'_-(x) = k < 0$, or

$$\lim_{y \downarrow 0} \frac{g(x+y) - g(x) - g(x) + g(x-y)}{y} = k < 0.$$

Since $g \in F$ we observe from (2) that $(-g(x) + g(x+y))/2 + g(x-y)/2$ is an increasing function of x . Consequently the derivative cannot exist for any $x' < x$ which is inconsistent with the concavity of g .

It remains to show that for any $x_2 > x_1$ in the interior of X and $0 < \lambda < 1$,

$$-g'(\lambda x_1 + (1-\lambda)x_2) + \lambda g'(x_1) + (1-\lambda)g'(x_2) \geq 0.$$

The derivatives equal the right hand derivatives so that the left hand side equals

$$\lim_{y \rightarrow 0} \frac{1}{y} [-g(\lambda x_1 + (1-\lambda)x_2 + y) + g(\lambda x_1 + (1-\lambda)x_2) + \lambda g(x_1 + y) - \lambda g(x_1) + (1-\lambda)g(x_2 + y) - (1-\lambda)g(x_2)] .$$

The term in brackets is nonnegative, since by (2)

$$g(\lambda x_1 + (1-\lambda)x_2) - \lambda g(x_1) - (1-\lambda)g(x_2) \geq g(\lambda x_1 + (1-\lambda)x_2 + y) - \lambda g(x_1 + y) - (1-\lambda)g(x_2 + y) .$$

Q.E.D.

We note that F is large enough to include utility functions u whose absolute risk aversion, $-u''/u'$, is decreasing. This is true since u' is decreasing by concavity and therefore we must have $-u''$ decreasing.

We also want the result that if $g \in F$ and Z is any random variable with zero expectation such that $Eg(x_1 + Z)$ and $Eg(x_2 + Z)$ exist (are finite), then

$$E[g(x_1) - g(x_1 + Z) - g(x_2) + g(x_2 + Z)] \geq 0, \text{ when } x_2 > x_1 . \quad (8)$$

This follows from the definition of F if Z is a simple function. In order to go from simple functions to random variables we apply the same method of proof as that in Chung [2], Theorem 9.1.4. The result is also true if we replace x_1 and x_2 by $x_1 + X$ and $x_2 + X$ where X is a random variable, and if Z is a random variable such that $E(Z|X = x) = 0$ for all x . Again assuming that all expectations are defined we get (by conditioning on $X = x$)

$$E[g(x_1 + X) - g(x_1 + X + Z) - g(x_2 + X) + g(x_2 + X + Z)] \geq 0 . \quad (9)$$

If X is a nonnegative random variable with a finite mean, then $v(x) = Eg(x + X)$ also exists and is concave. If we let Z be the discrete random variable $P(Z = \Delta_i) = \lambda_i$, where $\Delta_i, \lambda_i, i = 1, \dots, n$, have the properties of those same terms in the definition of F , and be independent of X , then (9) shows that $v \in F$.

Next, we examine the idea of increasing risk as defined by Rothschild and Stiglitz [20,22]. There they establish the equivalence of three measures of risk when comparing two random variables. The definition most useful for our purposes is that Y is more risky than X if and only if

$$Y \underset{d}{=} X + Z$$

where $\underset{d}{=}$ means "has the same distribution as" and Z is a random variable such that $E(Z|X = x) = 0$ for all x . Clearly from (9) we have that if Y is more risky than X (and all expectations exist) then $g \in F$ and $x_2 > x_1$ imply that

$$E[g(x_1+X) - g(x_1+Y) - g(x_2+X) + g(x_2+Y)] \geq 0. \quad (10)$$

The proof of Theorem 2 starts with a lemma which establishes the induction step of the kind mentioned at the beginning of Section 2 for property of belonging to the set F .

Lemma 1. If $u \in F$, then $f \in F$.

Proof. Our starting point is (5) which states that f is a fixed point of A . We have assumed that we are only considering utility functions such that a unique finite valued f satisfying (5) exists. It remains to show that the range of A is contained in F , since this will imply that the fixed point of A is in F .

It is known (for a proof in this particular case, see Miller [15]) that if g is concave then Ag is concave. We need to show that if $g(x,j) \in F$ (and hence is concave) then the concave function Ag satisfies

$$Ag(x_1, j) - \sum_{i=1}^n \lambda_i Ag(x_1 + \Delta_i, j) - Ag(x_2, j) + \sum_{i=1}^n \lambda_i Ag(x_2 + \Delta_i, j) \geq 0. \quad (11)$$

Let c_1^i , $i = 1, \dots, n$, be the optimal decisions (with respect to g) for the states $x_1 + \Delta_i$ and c_2 and c_1 be the optimal decisions for the states x_2 and x_1 respectively. From (6) we know that $c_2 \geq c_1$ and $x_2 - c_2 \geq x_1 - c_1$. Let c_2^i , $i = 1, \dots, n$, be the decisions associated with the state $x_2 + \Delta_i$, and be given by $c_2^i = c_2 - c_1 + c_1^i$. They are feasible ($0 \leq c_2^i \leq x_2 + D_j$) since the c_1^i are feasible, $c_2 \geq c_1$, and $x_2 - x_1 \geq c_2 - c_1$.

We have that

$$Ag(x_1 + \Delta_i, j) = u(c_1^i) + \alpha Eg(x_1 + \Delta_i - c_1^i + Y_j, j+1),$$

and similar equations hold for x_1 and x_2 . Since the c_2^i may not be optimal

$$Ag(x_2 + \Delta_i, j) \geq u(c_2^i) + \alpha Eg(x_2 + \Delta_i - c_2^i + Y_j, j+1).$$

Let $\Delta_1^i = c_1^i - c_1 = c_2^i - c_2$. By (7)

$$u(c_1) - u(c_1^i) - u(c_2) + u(c_2^i) \geq -\Delta_1^i (u'(c_1) - u'(c_2)),$$

since $c_2 \geq c_1$ and $u \in F$. By the development after equation (9), the function $v(x) = \alpha Eg(x + Y_j, j+1)$ belongs to F . Therefore by (7)

$$\begin{aligned} & (v(x_1 - c_1) - v(x_1 + \Delta_i - c_1^i) - v(x_2 - c_2) + v(x_2 + \Delta_i - c_2^i)) \\ & \geq -(\Delta_i - \Delta_1^i) [v'(x_1 - c_1) - v'(x_2 - c_2)], \end{aligned}$$

since $v \in F$ and $x_2 - c_2 \geq x_1 - c_1$, and $(x_1 + \Delta_i - c_1^i) - (x_1 - c_1) = (x_2 + \Delta_i - c_2^i) - (x_2 - c_2) = \Delta_i - \Delta_1^i$. Combining the above equalities and inequalities we have that the left hand side of (11) is greater than or equal to

$$-\Sigma \lambda_i [\Delta_1^i (u'(c_1) - u'(c_2)) + (\Delta_i - \Delta_1^i) (v'(x_1 - c_1) - v'(x_2 - c_2))] . \quad (12)$$

If both c_1 and c_2 are interior points of their respective constraint sets, $0 \leq c_1 \leq x_1 + D_j$ and $0 \leq c_2 \leq x_2 + D_j$, then $u'(c_1) = v'(x_1 - c_1)$ and $u'(c_2) = v'(x_2 - c_2)$, by the optimality of c_1 and c_2 and the fact that the derivatives of u and v exist everywhere in the interior. In this case (12) equals $-\Sigma \lambda_i \Delta_i [u'(c_1) - u'(c_2)] = 0$.

We will consider the boundary cases of $c_1 = 0$ or $c_1 = x_1 + D_j$ and $c_2 = 0$ or $c_2 = x_2 + D_j$ by giving the proofs for the cases $c_1 = 0$ and $c_2 = 0$ only. A similar situation arises in the proof of Theorem 2, and there we give the proofs of the cases $c_1 = x_1 + D_j$ and $c_2 = x_2 + D_j$ only.

One possibility at the boundary is c_1 at a boundary, say $c_1 = 0$, but c_2 is not. In order to apply (12) where c_1 is a left end point we need to verify that

$$u(x) - u(x + \Delta_1) \geq - \int_x^{x + \Delta_1} [u'_+(x) + \int_x^y u''_+(z)] dz \quad (13)$$

for $x = c_1$. Since u is nondecreasing $u(c_1) \leq \lim_{x \rightarrow c_1} u(x)$. If $u(c_1) < \lim_{x \rightarrow c_1} u(x)$, then $u'_+(c_1) = +\infty$ and (13) holds. If $u(c_1) = \lim_{x \rightarrow c_1} u(x)$, then (13) holds for $x = c_1$ since $u'_+(c_1) \geq \lim_{x \rightarrow c_1} u'_+(x) = \lim_{x \rightarrow c_1} u'(x)$ (Rockafeller [19, Theorem 24.1]), and (13) holds (with equality) for all $x > c_1$. Returning to the main argument, the optimality of c_1 implies that $u'_+(c_1) \leq v'(x_1 - c_1)$ and clearly $\Delta_1^i \geq 0$ for all i . Therefore (12) is greater than or equal to

$$-\Sigma \lambda_i [\Delta_1^i v'(x_1 - c_1) + (\Delta_i - \Delta_1^i) v'(x_1 - c_1)] - \Sigma \lambda_i \Delta_i (-u'(c_2)) = 0.$$

The other possibility at the boundary is c_2 at a boundary, say $c_2 = 0$. By (6) $c_1 = 0$, and therefore $c_1^i = c_2^i$. Then (11) becomes

$$v(x_1) - \Sigma \lambda_i v(x_1 + \Delta_i - c_1^i) - v(x_2) + \Sigma \lambda_i v(x_2 + \Delta_i - c_2^i) = 0.$$

This quantity is nonnegative since $v(x_1) - \sum \lambda_i v(x_1 + \Delta_i) - v(x_2) + \sum \lambda_i v(x_2 + \Delta_i) \geq 0$ by (2) since $v \in F$, and $v(x_1 + \Delta_i) - v(x_1 + \Delta_i - c_1^i) - v(x_2 + \Delta_i) + v(x_2 + \Delta_i - c_2^i) \geq 0$ for all i since v is concave. Q.E.D.

Theorem 2. Let $u \in F$ and X_1, X_2, \dots be a sequence of random variables describing labor income (case a), and Y_1, Y_2, \dots be a second sequence of random variables describing labor income (case b). If for each i , Y_i is riskier than X_i , then the optimal amount to consume as a function of the state (x, j) in case a is greater than the optimal amount to consume in case b.

Verifying the hypothesis of the following lemma leads directly to a proof of Theorem 2.

Lemma 2. Let f_x be the optimal return function in case a and f_y be the optimal return function in case b. If $d(x, j) = f_y(x, j) - f_x(x, j)$ is a nondecreasing function of x then the conclusion of Theorem 2 holds.

Proof. By Lemma 1 we know that $f_x, f_y \in F$. Let c^* be the optimal decision for state (x, j) with the optimal return function f_x . For $c > c^*$,

$$\begin{aligned} h((x, j), c^*, f_y) - h((x, j), c, f_y) &= u(c^*) \\ &+ \alpha E f_x(r(x - c^*) + Y_j, j+1) + \alpha E d(r(x - c^*) + Y_j, j+1) \\ &- u(c) - \alpha E f_x(r(x - c) + Y_j, j+1) - \alpha E d(r(x - c) + Y_j, j+1). \end{aligned}$$

Since d is nondecreasing and $c > c^*$ we have that the right hand side is

$$\begin{aligned} &\geq u(c^*) + \alpha E f_x(r(x - c^*) + Y_j, j+1) \\ &- u(c) - \alpha E f_x(r(x - c) + Y_j, j+1) \\ &\geq u(c^*) + \alpha E f_x(r(x - c^*) + X_j, j+1) \\ &- u(c) - \alpha E f_x(r(x - c) + X_j, j+1) \end{aligned}$$

by (10) since $r(x-c^*) > r(x-c)$ and $f_x \in F$,

$$= h((x,j), c^*, f_x) - h((x,j), c, f_x) \geq 0, \text{ since } c^* \text{ is optimal.}$$

Therefore the optimal amount to consume in case b is less than or equal to c^* . Recall that in case of ties (which could not happen with strict concavity) we pick the smallest c .

The following lemma from [15] is needed to establish that d is non-decreasing.

Lemma 3. Consider the model in the case where Y_1, Y_2, \dots are the random variables describing labor income (case b). Let $v \in F$ and suppose that v satisfies Condition A below. Then $f_Y(x, j) - v(x, j)$ is a nondecreasing function of x .

Condition A. Given any two states x_1, j and x_2, j , $x_2 > x_1$, and decision c_1 for (x_1, j) , there is a feasible decision c_2 for (x_2, j) such that

- (a) $x_2 - c_2 \geq x_1 - c_1$
- (b) $(v(x_1, j) - h((x_1, j), c_1, v) - v(x_2, j) + h((x_2, j), c_2, v)) \geq 0$.

Equation (b) by itself is a necessary and sufficient condition that $Av - v$ be a nondecreasing function. The proof of the lemma consists of verifying an induction hypothesis in order to show that $A^n v - v$ is non-decreasing, and using the fact that $f_Y = \lim_{n \rightarrow \infty} A^n v$. By $A^n v$ we mean A applied n times to v . $A^2 v = A(Av)$.

Proof of Theorem 2. By Lemma 2 and Lemma 3 we need to show that Condition A holds where we let $v = f_x$. Let c_1^* and c_2^* be the optimal decisions for states (x_1, j) and (x_2, j) with the optimal return function f_x . Given a c_1 we set $c_2 = c_1 + c_2^* - c_1^*$.

Since c_1^* and c_2^* satisfy (6), $c_2^* \geq c_1^*$, and $x_2 - x_1 \geq c_2^* - c_1^*$. Thus c_2 is feasible since c_1 is feasible, and $c_2 - c_1 = c_2^* - c_1^* \leq x_2 - x_1$ and (a) of Condition A holds.

Recalling that $f_X = Af_X$, the left hand side of (b) in Lemma 3 equals

$$\begin{aligned}
 & (u(c_1^*) + \alpha \text{Ef}_X(x_1 - c_1^* + X_j, j+1) - u(c_1) - \alpha \text{Ef}_X(x_1 - c_1 + Y_j, j+1)) \\
 & - u(c_2^*) - \alpha \text{Ef}_X(x_2 - c_2^* + X_j, j+1) + u(c_2) + \alpha \text{Ef}_X(x_2 - c_2 + Y_j, j+1)) \\
 & \geq u(c_1^*) + \alpha \text{Ef}_X(x_1 - c_1^* + X_j, j+1) - u(c_1) - \alpha \text{Ef}_X(x_1 - c_1 + X_j, j+1) \\
 & - u(c_2^*) - \alpha \text{Ef}_X(x_2 - c_2^* + X_j, j+1) + u(c_2) + \text{Ef}_X(x_2 - c_2 + X_j, j+1)
 \end{aligned} \tag{14}$$

by (10) since $x_2 - c_2 \geq x_1 - c_1$ and $f_X \in F$.

Now let $\Delta = c_2 - c_2^* = c_1 - c_1^*$, and we will show that (14) is nonnegative. By (7) $(u(c_1^*) - u(c_1) - u(c_2^*) + u(c_2)) \geq -\Delta(u'(c_1^*) - u'(c_2^*))$, since $u \in F$ and $c_2^* \geq c_1^*$. As in the proof of Lemma 1 we let $v(x) = \alpha \text{Eg}(x + X_j, j)$. Then $v \in F$ and $(v(x_1 - c_1^*) - v(x_1 - c_1) - v(x_2 - c_2^*) + v(x_2 - c_2)) \geq \Delta(v'(x_1 - c_1^*) - v'(x_2 - c_2^*))$. Therefore (14) is greater than or equal to

$$\Delta(-u'(c_1^*) + v'(x_1 - c_1^*) + u'(c_2^*) - v'(x_2 - c_2^*)) . \tag{15}$$

If both c_1^* and c_2^* are interior points of their constraint sets, then $u'(c_1^*) = v'(x_1 - c_1^*)$, $u'(c_2^*) = v'(x_2 - c_2^*)$ and (15) equals zero. If c_1^* is at a boundary, say $c_1^* = x_1 + D_j$, and c_2^* is not, then $u'(c_1^*) \geq v'(x_1 - c_1^*)$ and $u'(c_2^*) = v'(x_2 - c_2^*)$. Since Δ must be nonpositive in this case, (15) will be nonnegative. Here we use (13) as applied to v . If c_2^* is at a boundary, say $c_2^* = x_2 + D_j$, then c_1^* must equal $x_1 + D_j$, and $v(x_2 - c_2^*) = v(x_1 - c_1^*)$ and $v(x_2 - c_2) = v(x_1 - c_1)$. Then (14) becomes $(u(c_1^*) - u(c_1) - u(c_2^*) + u(c_2))$ which is nonnegative by the concavity of u . Q.E.D.

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