Analytical Study of a Case of the Homicidal Chauffeur Game Problem

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ABSTRACT

This paper is a shorter presentation of the Ref 11 presented at the IFIP conference.

The analytical study of the homicidal chauffeur game problem is developped with the help of the Pontryagin's theory of optimization and its generalization to deterministic two-player zero-sum differential games.

Different phenomena are encountered : universal line, disperal line, barrier, equivocal line etc... and their equations or at least their differential equations are given. The study is not complete because its end is very complicated but is shows at least how a systematic construction of the "separatrix" can solve the problem and help to understand it.

INTRODUCTION

Many people (Ref. 1-7) have studied the homicidal chauffeur game problem as a simple and easily understandable example of pursuit-evasion game. However their results are surprisingly complicated and a systematic analytical study using the generalization of the optimality theory of Pontryagin to deterministic two-player zero-sum differential games may help to understand these results.

1 - THE HOMICIDAL CHAUFFEUR GAME

The evader and the pursuer are on an unlimited plane, the evader has no inertia but a velocity limited to E, the limit velocity of the pursuer is C but he also has a limit radius of curvature equal to R, both players are considered as points and the capture distance is called L.

The performance index is the time of capture t_f and it is assumed that the pursuer only goes at maximum speed in its forward direction.

The simplest set of axes is "the set of axes of the pursuer" in which the pursuer is at the origin and the y axis is in its forward direction. In this set of axes, if x and y are the coordinates of the evader, the equations of motion are :

$$\begin{aligned} dx/dt &= v \cos\Psi - y\Omega \\ dy/dt &= v \sin\Psi - C + x\Omega \end{aligned} \qquad (1)$$

with : v = velocity of the evader : $0 \leqslant v \leqslant E$

v, Ψ = control parameters of the evader

C = velocity of the pursuer

Ω = rate of rotation of the pursuer $|\Omega| \leqslant C/R$

Ω : control parameter of the pursuer

The initial conditions are x_o, y_o, t_o; the playing space is defined by $x^2+y^2 > L^2$ and the "terminal surface" is the cylinder of equation $x^2+y^2 = L^2$ in the x, y, t space.

2 - PRELIMINARY CONSIDERATIONS

2.1 - If $E \geqslant C$ the evader can always escape and the problem has no interest hence we will always assume $E < C$.

2.2 - The problem has an olvious symmetry (+x into −x) and scale considerations on the units of length time show that the problem only depends on the two ratios E/C and L/R and, since $E < C$, we will put : A = Arc sin E/C.

3 - THE PONTRYAGIN'S CONDITIONS

Let us call H the Hamiltonian of the problem and p_x and p_y the adjoint parameters to x and y :

$$H = p_x\left(v\cos\Psi - y\Omega\right) + p_y\left(v\sin\Psi - C + x\Omega\right) \qquad (2)$$

The Pontryagin's conditions lead to :

3.1 - Optimal controls and the maxi-minimum principle

If p_x and/or $p_y \neq 0$ the maxi-minimization gives for the evader :

$$v = E \quad ; \quad \cos\Psi = p_x / \left[p_x^2 + p_y^2\right]^{1/2} \quad ; \quad \sin\Psi = p_y / \left[p_x^2 + p_y^2\right]^{1/2} \qquad (3)$$

and for the pursuer :

$$\Omega = \frac{C}{R}\ \mathrm{sign}\left(p_x y - p_y x\right) \qquad (4)$$

3.2 - Final or "transversality" conditions

We will only study a given value of the performance index : $t_f = t_1$ the other results being deduced from this case by a simple translation.

The "terminal surface" $\mathcal{C}$ is then divided into :

$$\left.\begin{array}{llll} \text{The half-cylinder} & \mathcal{C}_+ : & x^2 + y^2 = L^2 & ; \ t > t_1 \\ \text{The circle} & \mathcal{C}_o : & x^2 + y^2 = L^2 & ; \ t = t_1 \\ \text{The half-cylinder} & \mathcal{C}_- : & x^2 + y^2 = L^2 & ; \ t < t_1 \end{array}\right\} \quad (5)$$

Extremal trajectories of the problem of interest cannot end at $\mathcal{C}_+$: in the vicinity of these trajectories we have necessarily $t_f > t_1$ at least as soon as $t > t_1$

Usual transversality conditions give :

$$\text{At } \mathcal{C}_o : \quad t_f = t_1 \quad ; \quad p_{xf}/x_f = p_{yf}/y_f > 0 \quad ; \quad H_f \leqslant 0 \qquad (6)$$

$$\text{At } \mathcal{C}_- : \quad t_f < t_1 \quad ; \quad p_{xf}/x_f = p_{yf}/y_f > 0 \quad ; \quad H_f = 0 \qquad (7)$$

Conditions (2), (3), (4) give at the final instant :

$$H_f = E\left[p_{xf}^2 + p_{yf}^2\right]^{1/2} - p_{yf}\,C \qquad (8)$$

Since $p_{xf}/x_f = p_{yf}/y_f > 0$ and $x_f^2 + y_f^2 = L^2$, the condition $H_f \leqslant 0$ is equivalent to :

$$E\,L - y_f\,C \leqslant 0 \qquad (9)$$

that is :

$$y_f \geqslant L\,E/C = L \sin A \qquad (10)$$

Hence the final conditions of extremal trajectories of the problem are (fig. 1) :

Either $t_f = t_1$ and (x_f , y_f) is on the arc ABA' of the figure 1, or $t_f < t_1$ and (x_f , y_f) is at A or at A' (it thus define the "useable part of the terminal surface").

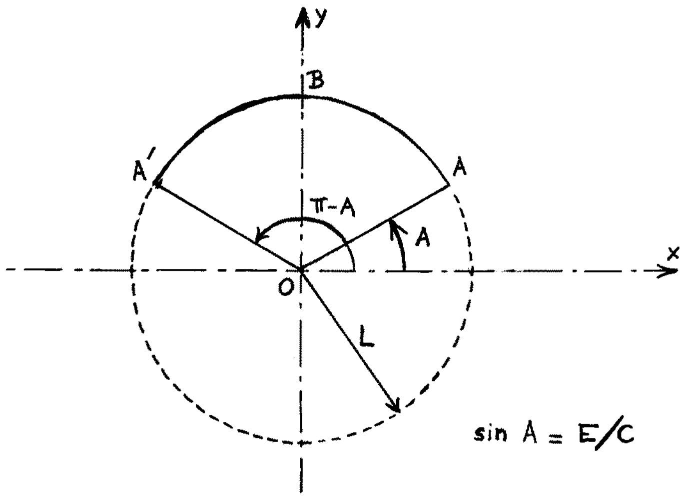

Figure 1 - The final conditions

3.3 - Final "ordinary trajectories of Pontryagin"

Let us integrate back the equations of Pontryagin from the final conditions ; these equations are :

$$dp_x/dt = -\partial H/\partial x = -\Omega\, p_x \qquad (11)$$

$$dp_y/dt = -\partial H/\partial y = \Omega\, p_y \qquad (12)$$

$$dH/dt = \partial H/\partial t = 0 \qquad (13)$$

$$dx/dt = E p_x \Big/ \left[p_x^2 + p_y^2\right]^{1/2} - y\Omega \qquad (14)$$

$$dy/dt = E p_y \Big/ \left[p_x^2 + p_y^2\right]^{1/2} - C + x\Omega \qquad (15)$$

(2), (3), (4) and (13) give :

$$H_f \equiv H = E\left(p_x^2 + p_y^2\right) - C p_y - \frac{C}{R}\left|p_x y - p_y x\right| \qquad (16)$$

on the other hand $\Omega = \frac{C}{R}$ sign ($p_x y - p_y x$), let us then study $p_x y - p_y x$:

$$d\left(p_x y - p_y x\right)/dt = -C p_x \qquad (17)$$

since $p_{xf} y_f - p_{yf} x_f = 0$ and since $p_{xf}/x_f > 0$, Ω and $p_x y - p_y x$ are positive in the vicinity of the arc AB of figure 1 and negative near A'B the two cases being symmetrical.

Let us study the first case and let us start from the final point $x_f = L\cos B$, $y_f = L \sin B$, $t_f \leq t_1$ with $A \leq B \leq \pi/2$ if $t_f = t_1$ and $B = A$ if $t_f < t_1$.

Let us choose $p_{xf}^2 + p_{yf}^2 = 1$, we obtain :

$$p_{xf} = \cos B \quad ; \quad p_{yf} = \sin B \qquad (18)$$

and from (11) and (12) :

$$p_x = \cos\left[B + C(t-t_f)/R\right] \quad ; \; p_y = \sin\left[B + C(t-t_f)/R\right] \qquad (19)$$

$$dx/dt = E\cos\left[B + C(t-t_f)/R\right] - C\,y/R \qquad (20)$$

$$dy/dt = E\sin\left[B + C(t-t_f)/R\right] - C + C\,x/R \qquad (21)$$

with $i = \sqrt{-1}$, (20) and (21) are integrated into :

$$x + iy = R\left[1 - \exp\{iC(t-t_f)/R\}\right] + \left[L + E(t-t_f)\right].\exp\{i\left[B + C(t-t_f)/R\right]\} \qquad (22)$$

these equations are valid only as long as $p_x y - p_y x$ remains positive (wich implies $\Omega = C/R$), i.e. only for :

$$t_f - R(\pi + 2B)/C \leq t \leq t_f \qquad (23)$$

In the x, y plane the different trajectories defined by (22) have a common point D (see fig. 2) for $L + E(t-t_f) = 0$ at $x + iy = R - R\exp\{-iLC/RE\}$; on the otherhand the trajectories corresponding to B = A and $t_f \leq t_1$ have the same projection ADin the x, y plane (projection which is an evolute of circle) and this phenomenum corresponds to the phenomenum of barrier discovered by Isaacs (Ref. 1).

3.4 - Singular trajectory

There exists a singular trajectory corresponding to $p_x y - p_y x \equiv 0$, it is easy to study that case :

(17) implies : $\quad p_x \equiv 0 \quad ; \; p_{x_f} = 0$

the final point of the singular trajectory is B (fig. 1 and 2) and then $t_f = t_1$.

(11) and (12) imply then : $\Omega \equiv 0 \quad ; \; p_y \equiv p_{y_f} > 0 \qquad (24)$

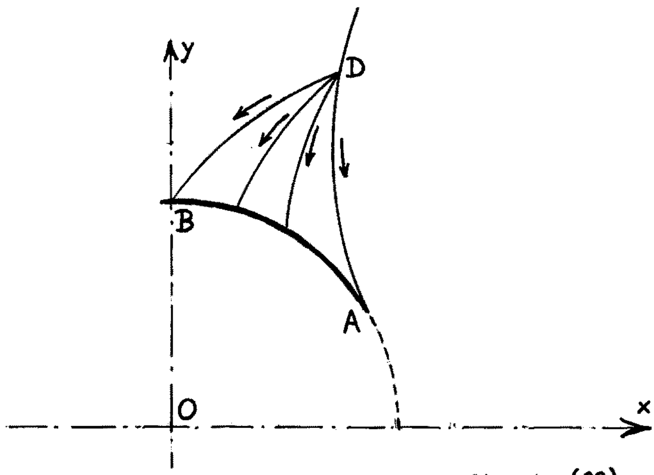

Fig. 2 - Trajectories corresponding to (22)

and from (14) and (15)

$$x \equiv 0 \quad ; \quad y = L + (C-E)(t_1 - t) \qquad (25)$$

The singular trajectory corresponds to a straight race along the y - axis, with its tributaries of the next section it will be a "universal surface" of Isaacs (Ref. 1).

3.5 - The tributaries of the singular trajectory

Let us start from an arbitrary point of the singular trajectory corresponding to $t = t_2 \leq t_1$, let us choose $\Omega = C/R$ (or symmetrically $\Omega = -C/R$) and let us integrate the equations (11) to (15) for $t \leq t_2$, we obtain (if we choose $p_{y_f} = 1$) :

$$p_x = \sin\left[(t_2 - t)C/R\right] \quad ; \quad p_y = \cos\left[(t_2 - t)C/R\right] \qquad (26)$$

$$x + iy = R + \left[-R + i\{L + C(t_1 - t_2) + E(t - t_1)\}\right].\exp\{-i(t_2 - t)C/R\} \qquad (27)$$

3.6 - Thus the Pontryagin's conditions and the symmetry + x into - x lead to four types of extremal trajectories of the problem of interest :

A) The two types the equation of which is written in (22), either for $t_f = t_1$ and $A \leq B \leq \frac{\pi}{2}$ or for $t_f \leq t_1$ and $B = A$, this second type corresponding to the "barriers"

B) The singular trajectory along the y - axis, the equation of which is written in (25).

C) The tributaries of this singular trajectory depending on the parameter t_2 (with $t_2 \leq t_1$) and the equation of which, for $t \leq t_2$, is written in (27)

Let us note that the trajectories corresponding to the "barriers" goes into the "playing space " in the vicinity of t_f if and only if $E^2/C^2 + L^2/R^2 \leq 1$ if not there remain only three types of trajectories given by the ordinary Pontryagin's conditions.

These trajectories can be used to built the "separatrix", i.e. the surface lying between the zone of the playing space $\mathcal{E}$ where $t_f < t_1$ and the zone where $t_f > t_1$, however these trajectories are insufficient, they cannot go for instance inside the loop of the barriers and they have to be completed with the help of the generalization of the optimality theory of Pontryagin to deterministic two-player zero-sum differential games (Ref. 8).

4 - CONSTRUCTION OF THE "SEPARATRIX"

We will only study the case when $\cos A + A \sin A - 1 < L/R \leqslant \inf\left[\cos A ; (\pi + 2A)\sin A\right]$, for instance, for $C = 2E$ and thus $A = \pi/6$ it gives $0.128... < L/R \leqslant 0.866...$ That case is the case when the two symmetrical barriers exist, don't meet in front of the pursuer (if not it is the "toreador case") and are prolongated by an equivocal line.

The construction of the separatrix (fig. 3) can be done with only Pontryagins trajectories until the point G of the barrier corresponding to $t_f - t = (\pi + 2A) R/C$, that is until the instant t_3 defined by the intersection of the barrier at $(t_f - t) = (\pi + 2A) R/C$ and the proper tributary of the universal line BF. Hence t_3 is the largest instant definded by:

$$t_1 > t_2 > t_3 \tag{28}$$

$$R + R\exp\{-2iA\} + \left[(\pi+2A)R\sin A - L\right]\exp\{-iA\} = R + \left[-R + i\{L + C(t_1 - t_2) + E(t_3 - t_1)\}\right]\exp\{iC(t_3 - t_2)/R\} \tag{29}$$

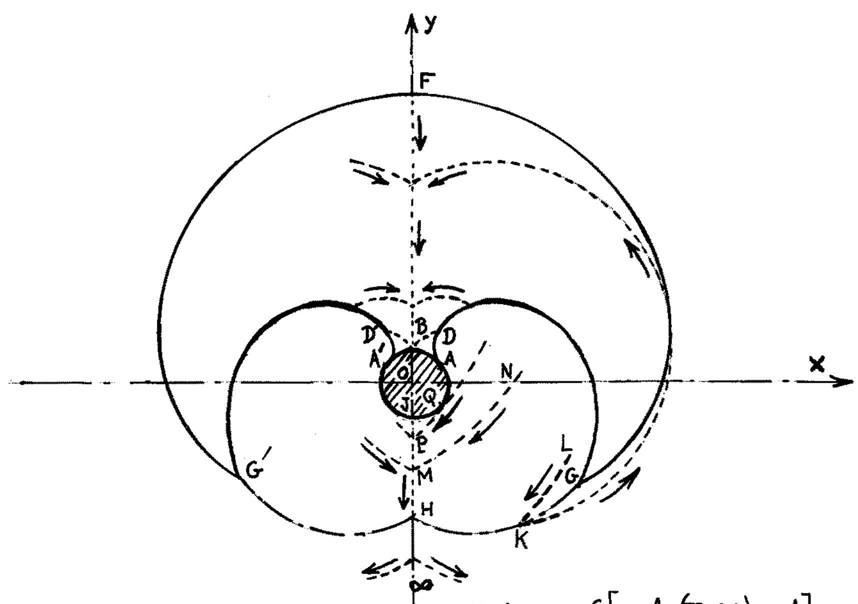

Fig. 3 - Case when : $\cos A + A\sin A - 1 < L/R \leqslant \inf\left[\cos A ; (\pi + 2A)\sin A\right]$

The symmetrical line ADGF G'D'A' (fig. 3) is the intersection of the separatrix with the x, y plane at the instant t_3.

At later instants (for $t_3 < t \leqslant t_1 - L/E$) the tributaries of the universal line BF and the trajectories of the barrier meet at points of the symmetrical curves DG and D'G' in conditions of a dispersal line, but at G and $t = t_3$

the conditions of the begining of an "equivocal line" are satisfied (Ref. 8) because the conditions (23) end there to be satisfied and the Pontryagin's trajectory of the barrier would continue with $\Omega = -C/R$ instead of $\Omega = +C/R$.

Let us try to obtain the differential equations of this equivocal line GKH (fig. 3) ; along that line the generalized conditions of Pontryagin (Ref. 8) must be satisfied, that is x, y, p_x, p_y and the Hamiltonian H must be such that :

$$\left.\begin{aligned}
&dx/dt = E\,p_x/[p_x^2+p_y^2]^{1/2} - y\Omega \\
&dy/dt = E\,p_y/[p_x^2+p_y^2]^{1/2} - C + x\Omega \\
&\Omega = \frac{C}{R}\,\mathrm{sign}\,(p_x y - p_y x)\ \begin{cases} p_x y - p_y x > 0 \Rightarrow \Omega = C/R \\ p_x y - p_y x = 0 \Rightarrow |\Omega| \leq C/R \\ p_x y - p_y x < 0 \Rightarrow \Omega = -C/R \end{cases} \\
&dp_x/dt = -\partial H/\partial x + \lambda(p_x - p_{2x}) = -\Omega p_y + \lambda(p_x - p_{2x}) \\
&dp_y/dt = -\partial H/\partial y + \lambda(p_y - p_{2y}) = \Omega p_x + \lambda(p_y - p_{2y}) \\
&dH/dt = \partial H/\partial t + \lambda(H - H_2) = \lambda(H - H_2) \\
&\lambda \geq 0
\end{aligned}\right\} \quad (30)$$

p_{2x}, p_{2y} and H_2 being the components of the adjoint vector $\vec{P}$ and the value of the Hamiltonian H along the tributary of the universal line BF at the point x, y, t of interest (which implies of course that the equivocal line remains on the surface defined by the tributaries in the x, y, t space).

Since the equivocal line remains on the surface of the tributaries let us describe it with the help of equation (27) of these tributaries, t_2 being now considered as a function of t and not as a constant as along a given tributary.

The only solution of the system (30) is obtained for $p_x/x = p_y/y < 0$ and thus $p_x y - p_y x = 0$ and it gives with (27) :

$$\left.\begin{aligned}
&d(x+iy)/dt = -E\,(x+iy)/(x^2+y^2)^{1/2} - iC - i\,\Omega\,(x+iy) = \\
&= iC\left(1 - \frac{dt_2}{dt}\right)(x+iy-R)/R \quad + i\left(E - C\,\frac{dt_2}{dt}\right).\exp\{iC(t-t_2)/R\}
\end{aligned}\right\} \quad (31)$$

The two real quantities Ω and dt_2/dt can be deduced from the first degree and complex equation (31) which is thus the differential equation of the equivocal line GKH.

Along the equivocal line the evader can choose between 2 strategies, either the strategy corresponding to (26) and to the local tributary, or the strategy corresponding to $p_x/x = p_y/y < 0$: the evader runs into the direction of the pursuer and follows the equivocal line until he chooses the other strategy and the pursuer must adapt his own strategy to these modifications.

In the x, y plane the equivocal line reaches either the down part of the circle $x^2+y^2=L^2$ or the negative part of the y axis (as in fig. 3), $H\infty$ is then an ordinary dispersal line and JH is a universal line (where the evader chase the pursuer !) with tributaries such as MN or PQ. There is also tributaries of the equivocal line such as KL starting at K with $p_x/x = p_y/y < 0$.

The analysis can be continued inside the region ADGHJ according to the same principles and leads to many singularities such as the "safe contact motion" (above Q, along the circle of capture), the switch envelopes and the focal lines of Ref. 6 and 7 ; the results are very complicated and numerical computations cannot be avoided.

CONCLUSION

Because of its simplicity the homicidal chauffeur game problem can be studied analytically very far. The notion of "separatrix" (i.e. the surface which, in the playing space, separate the points corresponding to performance indices larger than and smaller than a given value) allows a systematic study of the problem and helps to understant it, for instance the "barriers" correspond to cylindrical parts of the separatrix the generatrices of which are parallel to the t axis.

On the other hand the complexity of the results of this apparently very simple problem emphasizes the need of a systematic way of research and solution of differential game problems.

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