Optimal control with minimum problems and variational inequalities

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We go to consider here problems of parameter optimization in abstract process relations such as minimum problems and mixed variational inequalities in Banach spaces. Besides we obtain existence results and projected and iterated approximation methods for the solution of these problems which cover, for instance, problems of optimal control, of identification and inverse problems for partial differential equations.

1. Minimum problems

Let V be a real Banach space with norm $\|.\|$, V^* its adjoint space and $(g,u) = g(u)$ for $g \in V^*$, $u \in V$. By the symbols $\rightarrow$, $\rightharpoonup$ we denote strong and weak convergence in V, respectively. The symbol $M[(.)]$ will be used to denote the set of all solutions of the problem represented by the formula $(\cdot)$.

Let $U \subset V$, $U \neq \emptyset$ and $f,h \in (U \rightarrow R^1)$. We consider the problem (1) - (2):

(1) $\quad f(u) = \inf f(v) \;, \quad v \in M[(2)]$,

(2) $\quad h(v) = \inf h(u) \;, \quad u \in U.$

Existence theorems for (1)-(2) can be obtained from the generalized Weierstraß theorems.

Let $U_n \subset V$, $U_n \neq \emptyset$, $e_n > 0$, $e_n \rightarrow 0$, f_n, $h_n \in (U_n \rightarrow R^1)$, $n=1,2,\ldots.$ As an approximation method for (1)-(2) we consider

(3n) $\quad j_n(w_n) = \inf j_n(u) \;, \; u \in U_n, \; j_n = h_n + e_n \cdot f_n, \; n=1,2,\ldots.$

Let the following assumptions be fulfilled:

I. $w - \overline{\text{Lim}}\, U_n \subset U$.

II. $(f_n,U_n) \rightarrow (f,U)$ upper semi-continuously ($v_n \in U_n, v_n \rightarrow v \in U$ implies $\overline{\lim}\, f_n(v_n) \leq f(v)$) and weakly lower semi-continuously ($v_n \in U_n$, $v_n \rightharpoonup v \in U$ implies $\underline{\lim}\, f_n(v_n) \geq f(v)$).

III. $(h_n,U_n) \rightarrow (h,U)$ upper semi-continuously (u.s.c.) and weakly lower semi-continuously (w.l.s.c.).

Definition 1. The notation "$(f_n,U_n) \rightarrow (f,U)$ with the property (F_+)" means: if $v_n \in U_n$, $v_n \rightharpoonup v \in U$, $\overline{\lim}\, f_n(v_n) \leq f(v)$ then $\|v_n - v\| \rightarrow 0$.

Theorem 1. Let (3n) have at least one solution w_n for every n and let the following assumptions be fulfilled:

1. (1)-(2) has at least one solution w such that there exists a sequence $\{v_n\}$ with the properties $v_n \in U_n$ for $n \geq n_o$, $\|v_n - w\| \to o$ and $\overline{\lim}\,[(h_n(v_n) - h_n(w_n))/e_n] \leq 0$.
2. One of the following conditions (i),(ii) is fulfilled:
   (i) $U_n \subset E \subset V$, n=1,2,...; E is weakly compact.
   (ii) B is a reflexive space and $\overline{\lim}\, f_n(u_n) = +\infty$ if $u_n \in U_n$ and $\|u_n\| \to +\infty$.

Then w - $\underline{\mathrm{Lim}}\, w_n \neq \emptyset$ and w - $\overline{\mathrm{Lim}}\, w_n \subset M\,[(1)-(2)]$. If additionally

3. (f_n, U_n) (f,U) with property (F_+)

is fulfilled, then every weakly convergent subsequences of $\{w_n\}$ is also strongly convergent.

Proof: cf. [8]. For special cases see [5-7].

Remarks. 1. Based on the weak* (B_1 -) convergence in $B = B_1^*$ an analogous theorem on the convergence of $\{w_n\}$ can be derived [8]. 2. If under the assumptions of Theorem 1 problem (1)-(2) has a unique solution then $w_n \rightharpoonup w$ or $\|w_n - w\| \to o$ holds.

Let be V a Hilbert space H, P(K) the operator of projection from B onto the convex and closed set $K \subset H$, $\{H_n\}$ a sequence of subspaces and $P_n = P(H_n)$. Let for $p > o$ be $\lceil p \rceil$ the smallest integer greater than or equal to p.

Theorem 2. Let (1)-(2) have a unique solution w, (3n) have a solution w_n for every n, $U_n \subset H_n$ be convex and closed; j_n be defined on H and strongly convex with convexity constant $c(j_n)$; the assumptions 1-3 of Theorem 1 be fulfilled. Besides assume that for every j_n the gradient j_n' exists and is Lipschitz-continuous with Lipschitz constant $L(j_n')$. Then the projection-iteration method

$$(4)\qquad a_{n+1} = \{P(U_n)P_n\,[I - t_n j_n']\}^{i_n} a_n\ ,\ n=1,2,\ldots,\ a_1 \in H,\ \text{with}$$
$$o < t_n < 16c(j_n)/L(j_n')^2\ ,\ i_n = \lceil e_o/(1-L_n)\rceil,\ e_o > o,$$
$$L_n = \sqrt{1 - 16\,t_n c(j_n) + t_n^2 L(j_n')^2}$$

converges strongly to w in B.

Proof: cf [8]. For special cases see [5-7].

Theorem 3. Let (1) ($M[2)] = U$) have a unique solution w; $U_n \subset H_n$, n=1,2,..., be convex and closed; U = s - $\underline{\mathrm{Lim}}\, U_n$; f_n be defined on H, strongly convex with $c(f_n) \geq c > o$ and have the gradient f_n' with $L(f_n') \leq L$. If $\bigcup_n U_n$ is unbounded, then let $\|f_n'(o)\| \leq q$ and $|f_n(o)| \leq r$.

$$(5n)\qquad f_n(w_n) = \inf f_n(u)\quad ,\quad u \in U_n\ ,$$

hase a unique solution w_n for every n and the sequence $\{w_n\}$ as well as the sequence $\{a_n\}$ generated by the projection-iteration method.

(6) $\quad a_{n+1} = P(U_n)P_n\left[I - t_n \cdot f_n'\right] a_n$, $n=1,2,\dots$, $a_1 \in H$,

with $0 < \varepsilon_1 \leq t_n \leq \frac{16\ c(f_n)}{L^2} - \varepsilon_2$, $\varepsilon_2 > o$, converge strongly to w in H.

Proof: cf. [8]. For special cases see [5-7].

Remark. In [8] also the methods (4) and (6) for locally Lipschitz-continuous gradients j_n' and f_n', respectively, are given (see [10]).

2. Generalized trace funtionals

Let also Y be a real Banach space, $X \subset Y$, $F \in (X \times U \to R^1)$, $S \in (U \to X)$. We consider problem (1) in the form

(7) $\quad F(Sw,w) = \inf F(Su,u)$, $u \in U$.

If F is w.l.s.c. and S is weakly continuous then $F(S.,.)$ is w.l.s.c. The same is true, if F is (strongly, weakly) - lower semi-continuous (i.e. $x_n \to x$ and $u_n \rightharpoonup u$ implies $\underline{\lim}\ F(x_n,u_n) \geq F(x,u)$) and S is increased continuous (i.e. $u_n \rightharpoonup u$ implies $Su_n \to Su$). In both cases the generalized Weierstraß theorems may be applied. Further existence theorems can be obtained for weakly or increased closed S, for multi-valued $S \in (U \to 2^X)$ and on the base of weak* convergence (cf. [8]).

Let besides be $X_n \subset Y$, $F_n \in (X_n \times U_n \to R^1)$, $S_n \in (U_n \to X_n)$. As an approximation method for (7) we consider (3n) in the form

(8) $\quad F_n\ (S_n\ w_n,\ w_n) = \inf F_n(S_n u,u)$, $u \in U_n$.

If $(F_n, X \times U_n) \to (F, X \times U)$ u.s.c. and (strongly, weakly) - l.s.c. [w.l.s.c.] and $(S_n,U_n) \to (S,U)$ increased continuously [continuously and weakly continuously] then $(F_n(S_n.,.), U_n) \to (F(S.,.),U)$ u.s.c. and w.l.s.c. If, in addition, for $x_n \in X_n$ and $x_n \to x \in X$ [$x_n \rightharpoonup x \in X$] $(F_n(x_n,.)U_n) \to (F(x,.),U)$ with property (F_+) then also $(F_n(S_n.,.),U_n) \to (F(S.,.),U)$ with property (F_+) holds.

So we can apply Theorem 1 (with $f_n = F_n(S_n\ .,.)$, $h_n \equiv o$) to (8n).

Let be Y a Hilbert space, $\{Y_n\}$ a sequence of subspaces and $Q_n = P(Y_n)$. Assume $F_n(S_n\ .,.) = F_1(S_n.) + F_2(.)$, $F_1 \in (Y \to R^1)$ convex, $F_2 \in (V \to R^1)$ strongly convex, U_n, X_n convex, $S_n = \bar{S}_n + \bar{x}_n$, $\bar{S}_n$ linear and bounded, $\bar{x}_n \in X_n$. Then the functional $f_n = F_n(S_n.,.)$ are strongly convex with $c(f_n) = c(F_2)$ and

$$f_n' = F_2' + \bar{S}_n^* Q_n\ F_1'\ (S_n.)$$

holds. Under appropriate assumptions the Theorems 2 and 3 may be applied (cf. [8]).

3. Optimization with minimum problems

We consider (7) where $S \in (U \to X)$ is the solution operator of the following problem (9)-(10):

(9) $k(x,u) = \inf k(y,u)$, $y \in M[(10)]$,

(10) $l(y,u) = \inf l(z,u)$, $z \in X$.

Theorem 1 can then be used to investigate the weak or increased continuity of S(cf.[9]). As an approximation method for (7),(9),(10) we consider (8n) with the solution operator S_n of

(11n) $m_n(x,u) = \inf m_n(y,u)$, $y \in X_n$, $m_n(y,u) = l_n(y,u)+e_n k_n(y,u)$.

In (8n),(11n) as special cases combined Ritz-Ritz (Ritz-projected penalty, Ritz-projected regularization) methods are contained. Theorem 1 gives results on the continuous, weakly continuous or increased continuous convergence of $(S_n,U_n) \to (S,U)$. Using these results and, once more, Theorem 1 (cf. Part 2) the convergence of solutions of (8n),(11n) to solutions of (7),(9),(10) can be proved (cf. [9]). In some cases also iteration methods of the form (4) and (6) can be derived ([9]). For applications of the results in problems with partial differential equations see [11] .

4. Mixed variational inequalities

Let be Y a reflexive space; $T,S \in (Y \to Y^*)$ and $k,l \in (Y \to R^1)$. We consider the mixed variational inequality

(12) $(Tz,y-z) \geq k(z) - k(y)$, $y \in M[(13)]$,

(13) $(Sy,x-y) \geq k(y) - k(x)$, $x \in X$.

Existence theorems for (12)-(13) can be obtained from the results of BREZIS [1] , BROWDER [2] , KLUGE-BRUCKNER [10] and LIONS [13] on variational inequalities.

Let be $X_n \neq \phi$, convex and closed; T_n, $S_n \in (Y \to Y^*)$ be monotone and hemicontinuous operators, $R_n = S_n + e_n \cdot T_n$; k_n, $l_n \in (Y \to R^1)$ be convex and lower semicontinuous functionals and $m_n = l_n + e_n k_n, n=1,2,\ldots$.

As an approximation method for (12)-(13) we consider

(14n) $(R_n z_n, x - z_n) \geq m_n(z_n) - m_n(x)$, $x \in X_n$, $n = 1,2,\ldots$.

Let the following assumptions be fulfilled:

IV. $w - \overline{\text{Lim}}\, X_n \subset X$.

V. $X \subset s - \underline{\text{Lim}}\, X_n$ if $S \neq 0$ and $y \in s\text{-}\underline{\text{Lim}}\, X_n$ for any $y \in M[(13)]$ if $S \neq 0$.

VI. $(T_n,X_n) \to (T,X)$ and $(S_n,X_n) \to (S,X)$ continuously;

$(k_n,X_n) \to (k,X)$ and $(l_n,X_n) \to (l,X)$ u.s.c. and w.l.s.c.

Definition 2. The notation "$(T_n,X_n) \to (T,X)$ with property (S_+)" means: if $y_n \in X_n, y_n \rightharpoonup y \in X$ and $\overline{\lim}\,(T_n y_n, y_n - y) \leq 0$ then $\|y_n - y\| \to 0$.

Theorem 4. Let (14n) have at least one solution z_n for every n and let the following assumptions be fulfilled:

1. (12)-(13) has at least one solution z such that there exists a sequence $\{y_n\}$ with the properties: $y_n \in X_n$ for $n \geq n_o$, $\|y_n - z\| \to 0$,
$\overline{\lim}_{i \to +\infty} \{[(-S_{n_i} y_{n_i}, z_{n_i} - y_{n_i}) + l_{n_i}(y_{n_i}) - l_{n_i}(z_{n_i})]/e_{n_i}\|z_{n_i}\|\} \leq 0$
if $\|z_{n_i}\| \to \infty$ and
$\overline{\lim}_{n \to \infty} \{[-S_n y_n, z_n - y_n) + l_n(y_n) - l_n(z_n)]/e_n\} \leq 0$
if $\{z_n\}$ is bounded.
2. One of the following conditions (i),(ii) is fulfilled:
   (i) $X_n \subset F \subset Y$, $n = 1,2,\ldots$; F is bounded.
   (ii) $\overline{\lim}_{i \to \infty} \{[(T_{n_i} z_{n_i}, z_{n_i} - y_{n_i}) + f_{n_i}(z_{n_i})]/\|z_{n_i}\|\} = +\infty$
   if $\|z_{n_i}\| \to +\infty$.

Then $w\text{-}\overline{\mathrm{Lim}}\, z_n \neq \phi$ and $w\text{-}\overline{\mathrm{Lim}}\, z_n \subset M[(12)-(13)]$.

If additionally the assumption

3. $(T_n, X_n) \to (T,X)$ with property (S_+) or $(k_n, X_n) \to (k,X)$ with property (F_+)
is fulfilled, then every weakly convergent subsequence of $\{z_n\}$ is also strongly convergent.

Proof. cf. [8]. For special cases see [4-7].

Remark. Analogous to the theorems 2 and 3 we may give theorems on the strong convergence of projection-iteration methods for (12)-(13) (see [4-8]).

5. Optimization with variational inequalities

We consider (7) where $S \in (U \to X)$ is the solution operator of the following problem

(15) $(Q(z,u), y-z) \geq k(z,u) - k(y,u)$, $y \in M[(16)]$,

(16) $(P(y,u), x-y) \geq l(y,u) - l(x,u)$, $x \in X$.

Theorem 4 can then be used to investigate the weak, strong or increased continuity of S (cf. [9,11]). As an approximation method for (7), (15), (16) we consider (8n) with the solution operator S_n of

(17n) $(R_n(z_n,u), x-z_n) \geq m_n(z_n,u) - m_n(x,u)$, $x \in X_n$, $n=1,2,\ldots$,

$R_n(x,u) = P_n(x,u) + e_n \cdot Q_n(x,u)$, $m_n(x,u) = l_n(x,u) + e_n \cdot k_n(x,u)$.

In (8n),(17n) as special cases combined Ritz-Galerkin (Ritz-projected penalty, Ritz-projected regularization) methods are contained. Theorem 4 gives results on the continuous, weakly continuous or increased continuous convergence of $(S_n, X_n) \to (S,X)$. Using these results and Theorem 1 the convergence of solutions of (8n),(17n) to solutions of (7), (15), (16) can be proved (cf.[9]). For special cases see KRAUSS [12] and YVON [15]. In some cases also iteration methods of the form (4) and (6) can be derived ([9]).

For applications of the results see [9] and [11] .

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