A NUMERICAL METHOD FOR SOLVING LINEAR CONTROL PROBLEMS WITH MIXED RESTRICTIONS ON CONTROL AND PHASE COORDINATES

V.I.CHARNY
Institute of Control Sciences, Moscow, USSR

1. The proposed method is a result of the combination of the dual approach and the method of feasible directions. Three types of problems that can be solved by this method are listed below.

1°. Determination of feasible control in the system

$$\frac{dx(t)}{dt} = A(t)\ x(t) + B(t)u(t) + C(t)\ , \tag{1}$$

$$u(t) \geq 0,\ M(t)u(t) \leq N(t)\ x(t) + \rho(t)\ ,$$

$$x(0) = x_o,\quad Qx(T) \geq \bar{x}\ .$$

Matrices A, B, M, N, Q, vectors C, ρ, x_o, $\bar{x}$, and terminal time T are given here, with A, B, C, M, N, ρ being piece-wise continuous functions of t ; it is required to find a feasible control $u(t)$ belonging to the class of piece-wise continuous functions of t.

2°. Type (1) problem with delays in differential and finite relations of the system (delays may belong both to phase coordinates and to control).

3°. Finite-dimensional linear programming problem.

2. Let L_n^2 be the space of piece-wise continuous functions of t from L^2 , defined on $[0, T]$ and valued in R^n, $X^{nm} = L_n^2 \times R^m$ - prehilbertian space of pairs $a=\{W_a, V_a\}$ $(W_a \in L_n^2, V_a \in R^m,\ a \in X^{nm})$ with a scalar product $\langle a,b\rangle = \int_0^T (W_a, W_b)dt + (V_a, V_b)$ $(a,b \in X^{nm})$. Problems 1°-3° can be considered as a particular case of functional linear programming problem

$$Lu \leq h\ , \tag{2}$$

in which $u \in X_u$, $h \in X_h$, where $X_u = X^{nm}$, $X_h = X^{kr}$, n, m, k, r are given constants, L is a bounded linear operator transforming X_u into X_h.

The reduction of considered control problems to type (2) problem is a wellknown mode used by various authors, including [1-4] .

3. In application to the problem (2) the numerical method is constructed in the following way.

Let us consider the auxiliary problem

$$\max_{u,\beta} \beta \, : \, Lu \leq h(1+\beta), \quad \beta \leq 0 \tag{3}$$

with a scalar parameter β and apply the method of feasible directions [5,6] to it starting from the feasible solution $u = 0$, $\beta = -1$ of (3). It is not difficult to see that two cases are possible: 1) $\max\beta = 0$; then the first iteration of the method of feasible directions gives the feasible solution of (2) by the formula

$$u = \frac{g}{g_\beta} , \tag{4}$$

where the pair $\{g, g_\beta\}$ defines a feasible direction in the problem (3) for $u=0$, $\beta = -1$ (g, g_β) are defined by relations $Lg \leq hg_\beta , g_\beta > 0$); 2) $\sup \beta < 0$ or $\sup\beta = 0$,but the value $\beta = 0$ in (3) is not attained; in this case there is no feasible direction $\{g, g_\beta\}$ and the problem (2) has no solutions. To determine g, g_β in formula (4) we shall formulate a "best" direction problem [5](for case 1):

$$\max_{g, g_\beta} g_\beta : \, Lg \leq hg_\beta , \; g_\beta^2 + \langle g, g \rangle = 1. \tag{5}$$

The numerical method for the solution of the problem (2) is based on the application of dual approach [4] to the problem (5) with subsequent utilization of formula (4). In the end this method is reduced to the solution of the problem

$$\min_{\lambda \geq 0} \Phi \, : \, \Phi = (1+ \langle h, \lambda \rangle)^2 + \| L^* \lambda \|^2 , \tag{6}$$

where $\lambda \in X_h$, $\|a\| = \sqrt{\langle a, a \rangle}$, L^* is the operator conjugate with L . Solutions of problems (2) and (6) are related in the following way: 1) let $\lambda \in X_h$ be a solution of (6) and $\min \Phi > 0$; then

$$u = - \frac{L^* \lambda}{1+ \langle h, \lambda \rangle} \tag{7}$$

(the denominator in (7) is not equal to zero, because we can show that $\min \Phi = 1 + \langle h, \lambda \rangle$);

2) let $\inf \Phi = 0$; then the problem (2) is unsolvable;
3) let $\inf \Phi > 0$ be unattainable in X_h ; then we can construct a minimizing sequence $\lambda_n \in X_h$, such that for $u = u_n$, where u_n is defined by substituting $\lambda = \lambda_n$ into (7), the following condition is fulfilled:

$$\lim_{n \to \infty} \|(Lu_n - h)^+\| = 0, \tag{8}$$

in other words, for u_n positive lacks in (2) tend on the average to zero.
It is not difficult to see that case 2 corresponds to unsolvable conditions in Farkas' lemma.
4. The formulation of problem (6) for problem (1) follows:

$$\min_{\substack{\phi\geq 0,\\ \omega\geq 0,\\ \mu\geq 0}} \Phi : \Phi = \int_0^T (g, g)\, dt + g_\beta^2 , \tag{9}$$

$$g = B^T p - M^T \omega + \phi, \quad g_\beta = 1 + (\mu, \varphi_T) - \int_0^T (\phi, \varphi) dt,$$

$$\frac{dp}{dt} = - A^T p - N^T \omega, \quad p(T) = Q^T \mu.$$

Here $\varphi = N(t)x^o(t) + \rho(t)$, $\varphi_T = Qx^o(T) - \bar{x}$,where $x^o(t)$ is the solution of the Cauchy problem:

$$\frac{dx^o(t)}{dt} = A(t)x^o(t) + C(t), \; x^o(0) = x_o .$$

5. The following simple algorithm can be used to solve the problem (6). Let λ_n be n-th approximation to the problem (6) solution. Let us determine λ_n^* by solving the problem

$$\min_\alpha \Phi(\lambda_n + \alpha(- \operatorname{grad} \Phi(\lambda_n))^+) \tag{10}$$

$$(\lambda_n^* = \lambda_n + \alpha(- \operatorname{grad} \Phi (\lambda_n))^+ .$$

Then let us find λ_{n+1} by the formula $\lambda_{n+1} = \nu\lambda_n^*$, where ν is the solution of the problem

$$\min_\nu \Phi(\nu\lambda_n^*) . \tag{11}$$

The process termination can be controlled by the substitution of $\lambda=\lambda_n$ into (7) and by determination of the system lacks.
Finite formulae can be applied to the one- dimensional problems (10) and (11). The algorithm is proved in [4] . To accelerate its conver-

gence a combined algorithm can be used, in which formulae (10), (11) are used with formulae of the conjugate gradients method.

6. The numerical method permits the following modification. The problem

$$Lu = h, \quad u \geq 0 \tag{12}$$

(the canonical form of the linear programming problem) is considered instead of (2) and the problem

$$\min_{\substack{\omega \in X_h, \\ \lambda \in X_u, \lambda \geq 0}} \Phi: \; \Phi = (1 + \langle h,\omega \rangle)^2 + \| \lambda - L^*\omega \|^2 \tag{13}$$

is solved instead of (6). The relation of (13) with (12) is analoguos to that of (6) with (2); now instead of (7) the formula

$$u = \frac{\lambda - L^*\omega}{1 + \langle h,\omega \rangle} \tag{14}$$

is used.

The algorithm of consecutive minimization over ω and λ can be used for the solution of problem (13). Besides it is essential that there are no restrictions on ω in (13), and the minimization over λ is possible by finite formulae.

7. The numerical method has been used for the solution of a minimum time problem formulated for the Leontieff type dynamic input-output model (type (1) problem). The relations of this problem [4] are shown below:

$$\begin{aligned} &\frac{dV(t)}{dt} = u(t) , \\ &u(t) \geq 0, \quad M(t)u(t) \leq V(t) - V^o(t), \\ &V(0) = V_o, \qquad V(T) \geq \bar{V} , \end{aligned} \tag{15}$$

where matrix M and vectors V^o, V_o, $\bar{V}$ are given. A series of problems of feasible control search with the fixed value of T was solved with different values of T for a minimum time problem ($\min_u T$) . The combined algorithm based on formulae (10),(11) and formulae of the conjugate gradients method was used there. The type (15) problem for a twenty-nine industry model (vector V has 29 components) was solved in less than ten minutes on a third-generation computer (the program was written in ALGOL).

8. As follows from Section 3 the proposed method can be interpreted as a feasible directions method in which the direction is determined only once. On the other hand, this method in a way similar to methods of penalty functions is reduced to the problem of minimization of a quadratic functional. The basic difference of these methods is in the dependence of $\min \Phi$ on system parameters: in the proposed method $\min \Phi$ has a jump on the boundary of the region of parameters in which the system has solutions, provided $\|u\|$ doesn't grow to infinity in this region; in analogous case for methods with quadratic function of penalty the corresponding functional is continuous on the same boundary. This jump can be used effectively for solving minimum time problem: in this case $\min_u T$ must be treated as a limit point of set of values of T, for which the system has a feasible solution.

References

1 N.N.Krasovskii. The control theory of motion. (Russian).Izd. "Nauka", 1968.
2 M.V.Meerov, B.L.Litvak. Optimization of multi-connected control systems (Russian). Izd. "Nauka", 1972.
3 N.V.Gabashvili, N.N.Lominadzé, L.L.Chkhaidzé. An approximate solution of certain optimal control and discrete programming problems (Russian).Tekhnicheskaya Kibernetika, N6, 1972 .
4 V.I.Charny, V.A.Boĭkov. Numerical solution of linear dynamic problems in economic planning (Russian).Preprint, Izd.IAT, 1973.
5 G.Hadley. Nonlinear and Dynamic Programming. Addison-Wesley Pub. Co. Inc., Reading, Massachusetts, 1964.
6 A.V.Fiacco, G.P.Mc Cormick. Nonlinear Programming: Sequential Unconstrained Minimization Techniques. New York-London, 1968.