A GENERAL STOCHASTIC EQUATION FOR THE NON-LINEAR FILTERING PROBLEM

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1. Introduction.

A great deal of attention has been devoted in recent years to the theory of non-linear filtering, in particular, to the problem of deriving a stochastic differential equation for the filter. (see the bibliography in [2]). Perhaps the most general form of such an equation when the noise in the observation process model is the Wiener process is the one obtained in the paper of Fujisaki, Kallianpur and Kunita [2].

The work in [2] was motivated by applications in which the signal and observation processes are governed by an Ito stochastic differential equation or by a more general stochastic equation studied by Ito and Nisio (see [2]). However, the aim of the present paper is to show that the approach to filtering theory adopted in [2] is not limited to this kind of application and to give a generalization of the main result (Theorem 4.1) of the Fujisaki-Kallianpur-Kunita paper.

For reasons of brevity we shall consider real-valued observation processes but there is no difficulty whatever in making the appropriate changes to cover the vector-valued case.

2. Observation process model and the innovation process.

The system or signal process $x_t(\omega)$ taking values in a complete metric space S and the observation process $z_t(\omega)$ $(t \in [0,T])$ are assumed given on some complete probability space $(\Omega, \underline{\underline{A}}, P)$ and further related as follows.

$$(2.1) \qquad z_t(\omega) = \int_0^t h_u(\omega)\, du + w_t(\omega) ,$$

where

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(2.2) $w_t(\omega)$ is a real-valued standard Wiener process

(2.3) $h_t(\omega)$ is a (t,ω) measurable real-valued process such that $\int_0^T E(h_t^2)dt$ is finite.

Let us introduce the following family of σ-fields.

(2.4) $\underline{\underline{G}}_t = \sigma\{x_s, w_s, s \leq t\}$, $\underline{\underline{N}}_t^T = \sigma\{w_v - w_u, t \leq u \leq v \leq T\}$

and

$$\underline{\underline{F}} = \sigma\{z_s, s \leq t\} .$$

It will be assumed that the σ-fields $\underline{\underline{F}}_t$ and $\underline{\underline{G}}_t$ are augmented by adding to $\underline{\underline{F}}_o$ and $\underline{\underline{G}}_o$ all P-null sets. In the model (2.1) the information about the signal process is carried by (h_t) by means of the measurability assumption

(2.5) For each t , h_t is $\underline{\underline{G}}_t$ measurable, i.e. (h_t) is $(\underline{\underline{G}}_t)$ - adapted

In order to take into account applications involving stochastic control we make the further assumption that for every t the σ-fields

(2.6) $\underline{\underline{G}}_t$ and $\underline{\underline{N}}_t^T$ are stochastically independent.

Clearly (2.6) includes the case when the signal (x_t) and noise (w_t) are completely independent.

The derivation of the desired stochastic equation rests on two important results proved in Fujisaki-Kallianpur-Kunita [2]. We state them below without proof. From the assumptions made above on (h_t) it can be shown that one can work with a modification of the conditional expectation $E(h_t \mid \underline{\underline{F}}_t)$ which is jointly measurable and $(\underline{\underline{F}}_t)$ - adapted. This particular modification will be henceforth denoted by $\hat{h}_t$.

Let us now define the process (ν_t) by

$$(2.7) \qquad \nu_t = z_t - \int_0^t \hat{h}_s \, ds \, .$$

Proposition 1. $(\nu_t, \underline{\underline{F}}_t, P)$ is a Wiener martingale. Furthermore $\underline{\underline{F}}_t$ and $\sigma\{\nu_v - \nu_u;\ t < u < v \leq T\}$ are independent.

(ν_t) is called the innovation process.

Proposition 2. (A martingale representation theorem). Under conditions (2.1), (2.2), (2.3), (2.5) and (2.6) every separable square integrable martingale $(Y_t, \underline{\underline{F}}_t, P)$ is sample continuous and has the Ito stochastic integral representation

$$(2.8) \qquad Y_t - E(Y_o) = \int_o^t \Phi_s \, d\nu_s$$

where

$$(2.9) \qquad \int_o^T E(\Phi_s^2) \, ds < \infty$$

and (Φ_s) a jointly measurable and adapted to $(\underline{\underline{F}}_s)$.

3. A stochastic differential equation for the general non-linear filtering problem. First we replace the conditions defining the class $D(\widetilde{A})$ of [2] by a wider set of assumptions which make the theory applicable to more general types of signal processes (x_t). Let f be a real measurable function on S such that

$$(3.1) \qquad E[f(x_t)]^2 < \infty \quad \text{for all } t \text{ in } [0,T] \, .$$

The function f is said to belong to the class D if there exists a jointly measurable, real function $B_t[f](\omega)$ adapted to $(\underline{\underline{F}}_t)$ and having the following properties.

(3.2) Almost all trajectories of the process $(B_t[f])$ are right - continuous and of bounded variation over the interval $[0,T]$ with $B_o[f] = 0$.

(3.3) $E(\text{Var } B[f])^2 < \infty$

where

$\text{Var } B[f](\omega)$ is the total variation of the trajectory $B_t[f](\omega)$ $(0 \leq t \leq T)$.

(3.4) The process $M_t(f) \equiv f(x_t) - E[f(x_o) \mid \underline{\underline{F}}_o] - B_t[f]$ is a $(\underline{\underline{G}}_t, P)$ martingale.

Note that from conditions (3.1) and (3.3) it follows that $(M_t(f))$ is a square integrable martingale.

Theorem 1. Let the conditions of Section 2 and (3.1)-(3.4) hold. Then for every f in D there exists a jointly measurable process $(\overline{B}_t[f])$ adapted to the family $(\underline{\underline{F}}_t)$ such that almost all its trajectories are right-continuous and of bounded variation over the interval $[0,\tau]$ with $\overline{B}_o[f] = 0$. Furthermore,

(3.5) $E(\text{Var } \overline{B}[f])^2 < \infty$,

and the process

(3.6) $\overline{M}_t(f) \equiv E[f(x_t) \mid \underline{\underline{F}}_t] - E[f(x_o) \mid \underline{\underline{F}}_o] - \overline{B}_t[f]$ is a square-integrable $(\underline{\underline{F}}_t, P)$ martingale.

Proof. Since f is fixed we shall suppress it for the time being and write B_t for $B_t[f]$, etc. The existence of $(\overline{B}_t)$ follows from the ideas of C. Dellacherie [1] and P.A. Meyer [3] concerning the "dual projection" or "compensator" associated with an increasing integrable process. Write $B_t = U_t - V_t$ where (U_t) , (V_t) are increasing processes with right-continuous trajectories and such that $U_o = V_o = 0$. From (3.3) we also have $E(U_T^2) < \infty$ and $E(V_T^2) < \infty$. The process $\xi_t = E(U_T - U_t \mid \underline{\underline{F}}_{t+})$ is a positive supermartingale of class (D) . Hence it has a Doob decomposition $Y_t - \overline{U}_t$ where (Y_t) is a martingale and $\bar{U}_t$ is a (uniquely determined) predictable, integrable increasing process adapted to $(\underline{\underline{F}}_{t+})$ with $\bar{U}_o = 0$. Hence for $s < t$, we have $E(U_t - U_s \mid \underline{\underline{F}}_{s+}) = E(\bar{U}_t - \bar{U}_s \mid \underline{\underline{F}}_{s+})$ which gives

(3.7) $$E(U_t - U_s \mid \underline{\underline{F}}_s) = E(\bar{U}_t - \bar{U}_s \mid \underline{\underline{F}}_s) .$$

The predictability of $(\bar{U}_t)$ also implies that $\bar{U}_t$ is actually $\underline{\underline{F}}_t$ -measurable. Define the process $(\bar{V}_t)$ in a similar fashion and write $\bar{B}_t = \bar{U}_t - \bar{V}_t$. It is clear that $\bar{B}_o = 0$, almost all trajectories of $(\bar{B}_t)$ are right-continuous and of bounded variation over $[0, T]$ and further that $E(\text{Var } \bar{B}) = E(\bar{U}_T) + E(\bar{V}_T) < \infty$. However, the deduction of the square integrability of Var $\bar{B}$ from (3.3) is a bit more complicated and will not be given here. It, of course, implies the square integrability of $\bar{M}_t(f)$. The fact that $(\bar{M}_t(f), \underline{\underline{F}}_t, P)$ is a martingale follows easily. For $s < t$,

$$\begin{aligned} & E\{\bar{M}_t(f) - \bar{M}_s(f) \mid \underline{\underline{F}}_s\} \\ & = E\{f(x_t) - f(x_s) \mid \underline{\underline{F}}_s\} - E(\bar{B}_t - \bar{B}_s \mid \underline{\underline{F}}_s) \\ & = E\{M_t(f) - M_s(f) \mid \underline{\underline{F}}_s\} + E(B_t - B_s \mid \underline{\underline{F}}_s) - E(\bar{B}_t - \bar{B}_s \mid \underline{\underline{F}}_s) \\ & = 0 \quad \text{from (3.4) and (3.7)} . \end{aligned}$$

Since $(M_t(f), \underline{\underline{G}}_t, P)$ is a square integrable martingale there exists a unique sample continuous process $\langle M(f), w \rangle$ adapted to $(\underline{\underline{G}}_t)$ such that almost all of its trajectories are absolutely continuous with respect to Lebesgue measure in $[0, T]$. Furthermore there exists a modification of the Radon-Nikodym derivative which is (t,ω)-measurable and adapted to $(\underline{\underline{G}}_t)$ and which we shall denote by $\widetilde{D}_t f(\omega)$. Then it follows that [2] $\langle M(f),w \rangle_t = \int_o^t \widetilde{D}_s f \, ds$ a.s. where

$$\int_o^T E(\widetilde{D}_s f)^2 \, ds < \infty .$$

We now state the principal result which yields the stochastic equation of non-linear filtering. Conditions (3.3) and (3.5) are crucially used in the proof which will be presented in detail elsewhere. It will be understood that only separable versions of the martingales $M_t(f)$ and $\bar{M}_t(f)$ are considered. We shall also use the shorter notation $E^t(\cdot)$ for $E(\cdot \mid \underline{\underline{F}}_t)$.

Theorem 2. Assume conditions (2.1)-(2.3), (2.5), (2.6) and (3.1)-(3.4). Suppose f belongs to the class D and satisfies

$$\int_o^T E[f(x_t)\, h_t]^2\, dt < \infty . \tag{3.8}$$

Then $E^t[f(x_t)]$ satisfies the following stochastic differential equation.

$$\begin{aligned} E^t[f(x_t)] &= E^o[f(x_o)] + \bar{B}_t[f] \\ &+ \int_o^t [E^s(f(x_s)h_s) - E^s(f(x_s))E^s(h_s) + E^s(\tilde{D}_s f)]d\nu_s . \end{aligned} \tag{3.9}$$

Theorem 4.1 of Fujisaki-Kallianpur-Kunita [2] is a particular case of the above result. According to the assumptions in Section 4 of [2] if $f \in D$ (which is denoted by $D(\tilde{A})$ in that paper) there exists a (t,ω)-measurable real function $\tilde{A}_t f(\omega)$ adapted to $(\underline{\underline{G}}_t)$ such that $\int_0^T E(\tilde{A}_t f)^2 dt$ is finite and $f(x_t) - E[f(x_o) \mid \underline{\underline{F}}_o] - \int_o^t \tilde{A}_s f\, ds$ is a (necessarily square integrable) $(\underline{\underline{G}}_t, P)$-martingale. Hence conditions (3.2),(3.3) and (3.4) are satisfied with

$$B_t[f] = \int_o^t \tilde{A}_s f\, ds . \tag{3.10}$$

It is then easily verified that $\bar{B}_t[f] = \int_o^t E^s[\tilde{A}_s f]ds$ and that the stochastic equation (3.9) reduces to equation (4.12) of [2]. As explained in the Introduction the assumption (3.10) above made in [2] was suggested by applications of which the following is a typical example. The signal and observation processes form a Markov process (x_t, y_t) satisfying the stochastic differential equation

$$\begin{aligned} dx_t &= a_1(t,x_t,y_t)dt + b_1(t,x_t,y_t)dw_t \\ dy_t &= a_2(t,x_t,y_t)dt + b_2(t,y_t)dw_t \end{aligned} \tag{3.11}$$

where (w_t) is a (vector) Wiener process, $y_o = 0$ a.s., x_o is an arbitrary random variable independent of $\sigma\{w_s,\ 0 \le s \le T\}$ and the coefficients satisfy suitable conditions ensuring the existence and uniqueness of the solution of

(3.11). Let (A_t) , $t \in [0,T)$ be the extended generator of the Markov process $\eta_t = (x_t,y_t)$ as defined in [2] and let $D(A)$ be the set of all real functions f depending only on the first variable x, belonging to the domain of (A_t) and satisfying the conditions $E[f(\eta_t)]^2 < \infty$ for each t and $\int_0^T E[A_t f(\eta_t)]^2 dt < \infty$. It can easily be shown that in Theorem 2 we may take $D(A)$ for D and $B_t[f] = \int_0^t A_s f(\eta_s) ds$. For details see [2].

It is hoped that the equation obtained in Theorem 2 will prove useful in the study of new types of stochastic filtering and control problems.

References

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