OPTIMAL STABILIZATION OF THE DISTRIBUTED PARAMETER SYSTEMS

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The methods of investigating the optimal stabilization problems occupied an important place in the optimal control theory. The fundamentals of the methods for the finite-dimensional systems have been described by N.N.Krasovsky in the supplement to the monograph [1]. Some particular results for the distributed parameter systems have been obtained in [2,3]. They can be generalized and concretized with the help of Bellman's equations with the functional derivatives and on the basis of the functional derivatives the second Ljapunov's method can be employed.

In this article the method is used for controlling heat process. Other problems of controlling distributed parameter systems can be considered in the same way.

1. THE PROBLEM STATEMENT. Let the controlled process be described by the boundary-value problem

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left[a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right]+f\left(t,x,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n},\alpha\right),\ t>t_0,\ x\in\Omega, \tag{1}$$

$$a\frac{du}{d\nu}+cu=\varphi(t,x,u,\beta),\ x\in S,\ t>t_0 \tag{2}$$

$$u(t_0,x)=\psi(x),\ x\in\Omega+S, \tag{3}$$

where a_{ij} , f , c , φ and ψ are the given functions,

$$\sum_{i,j=1}^{n}a_{ij}\xi_i\xi_j\geqslant\gamma^2\sum_{i=1}^{n}\xi_i^2,\quad a_{ij}=a_{ji},\quad i,j=1,2,\dots n$$

for $x\in\Omega+S$, $t\geqslant t_0$ and

$$a=\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\cos(n,x_j)\right)^2\right]^{1/2}.$$

Ω is a limited area in E^n with partly-smooth boundary S , n is an external normal to S , and ν is a conormal. α and β are the scalar control parameters, which can take any real values.

The functions $\alpha(t,x)$ and $\beta(t,x)$ will be considered as admissible controls measured and bounded in respect to t if they satisfy the conditions

$$0 \leq \mathcal{J}_1(t) = \int_\Omega r(t,x)\alpha^2(t,x)d\Omega < \infty, \quad 0 \leq \mathcal{J}_2(t) = \int_S q(t,x)\beta^2(t,x)dS < \infty,$$

where r and q are given functions, such that $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$ are locally integrable in respect to t for the admissible controls.

According to [5] the only function $u(t,x) \in W_2^1(Q)$, $Q = \Omega \times [t_0,T]$ corresponds to every pair of the admissible controls α and β with some restrictions for the data of the problem. This function $u(t,x)$ is called a generalized solution and it meets both the integral identity

$$\int_\Omega [u\Phi]_{t_1}^{t_2} d\Omega = \int_{Q_1^2} \{u\Phi_t' - \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial \Phi}{\partial x_j} + f\Phi\} dQ - \int_{S_1^2} [cu - \varphi]\Phi dS \tag{4}$$

for almost all t and t_2 from $[t_0,T]$, and the condition

$$u(t,x) \xrightarrow{w} \Psi(x) \in L_2(\Omega), \text{ when } t \to t_0 . \tag{5}$$

Here $Q_1^2 = \Omega \times [t_1, t_2]$, and S_1^2 is the boundary of this cylinder; Φ is any function from $W_2^1(Q)$; T is an arbitary number, but always fixed $/t_0 \leq T/$.

Let the state of the object be measurable at any point $p \in Q$. The control functions $\alpha[t,x] = \alpha(t,x,u(t,x))$ and $\beta(t,x) = \beta(t,x,u(t,x))$ are formed on the basis of the information obtained. It is priori clear that for ensuring the optimal controls it is necessary to take α and β as the functionals determined on the $u \in L_2$. In this case the arguments t and x in the functions α and β are put into the square brackets.

Then let the functions $M(t,x)$, $M_{ij}(t,x)$ and $N(t,x)$ be such that

$$0 \leq \mathcal{J}_3(t) = \int_\Omega [Mu^2 + \sum_{i,j=1}^{n} M_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}] d\Omega + \int_S Nu^2 dS < \infty$$

on the solution of the problems (1)-(3), corresponding to any pair of the admissible controls, and the function $\mathcal{J}_3$ be locally integrable.

It is necessary to find such values $\alpha[t,x]$ and $\beta[t,x]$, that the functional

$$\mathcal{J}[t_0, u(t_0,x)] = \int_{t_0}^{\infty} [\mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t)] dt$$

have the least possible value.

2. BELLMAN'S EQUATION. The designation

$$S[t,u(t,x)] = \min_{\alpha,\beta} \mathcal{Y}[t,u(t,x)]$$

is introduced in accordance with the general Bellman's method. Let's suppose that the function $S(t,u)$ is differentiable with respect to t and as a functional of u it is differentiable by Freshe on $L_2(\Omega)$. Then

$$S[t+\Delta t, u(t+\Delta t,x)] = S[t,u(t,x)] + \frac{\partial S[t,u]}{\partial t}\Delta t + \Phi(t,\Delta u) + o(\Delta t,\Delta u), \quad (6)$$

where Φ is the linear functional with respect to Δu in L_2, calculated at the point (t,u), $o(\Delta t,\Delta u)$ is small by Δt and $\|\Delta u\|$ is a magnitude of the order which is higher than the first one. In the case when $\Delta u \in L_2(\Omega)$ almost to all t there is the function $v(t,x) \in L_2(\Omega)$ and

$$\Phi(t,u) = \int_\Omega v(t,x)\Delta u(t,x)\,d\Omega = \int_\Omega v(t,x)(u)_t^{t+\Delta t}\,d\Omega. \quad (7)$$

So according to the Bellman's optimal principle we have

$$S[t,u(t,x)] = \min_{\alpha,\beta}\left\{\mathcal{Y}(t)\Delta t + S[t,u] + \frac{\partial S[t,u]}{\partial t} + \int_\Omega (u)_t^{t+\Delta t} v\,d\Omega + o\right\} \quad (8)$$

As

$$(u)_t^{t+\Delta t} v = (uv)_t^{t+\Delta t} - (v)_t^{t+\Delta t} u(t+\Delta t,x), \quad (9)$$

we suppose that $v(t,x) \in W_2^1(\Omega)$, and the integral in (8) can be substituted by its value from the formula (4) where $t=t_1$ and $t_2 = t+\Delta t$

From (8) at $\Delta t \to 0$ we obtain the Bellman's equation in the functional derivatives

$$-\frac{\partial S}{\partial t} = \min_{\alpha,\beta}\left\{\mathcal{Y}(t) - \int_\Omega\left[\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} - f v\right]d\Omega - \int[cu-\varphi]v\,dS\right\}. \quad (10)$$

This equation makes it possible to show the different procedure of the approximate solution of the problem. The solution of the problem for the linear objects with quadratic criterion of the optimality has been done by the author together with G.Bachoi and M.Rakhimov. The authors have received boundary value problems, which are the analogues of the Rikkarti's equation. The methods of the approximate solution are given too.

3. THE APPLICATION OF THE SECOND LJAPUNOV'S METHOD. Let $f(t,x,0,..,0)=0$ $\varphi(t,x,0,0)\equiv 0$ and, hence, $u\equiv 0$ be the solution of the boundary-value problem (1)-(3) for $\alpha=\beta=0$. This solution will be called stable according to Ljapunov in the metric $W_2^1(Q)$ for fixed admissible controls $\alpha(t,x)$ and $\beta(t,x)$, if there is such $\delta>0$ for any small $\varepsilon>0$, that from the inequality $\|u(t_0,x)\|_{L_2}<\delta$ for $t>t_0$ will follow, that $\|u(t,x)\|_{W_2^1}<\varepsilon$ for $t>t_0$, where $u(t,x)$ is the solution of the problem (1)-(3), defined from the integral identity (4) and condition (5). The solution is called asymptotically stable, if, besides, $\|u(t,x)\|_{W_2^1}\to 0$ for $t\to 0$. Therefore in accordance with the theory of stability the considered problem should be formulated as follows:

Both the admissible controls $\alpha^0[t,x]$ and $\beta^0[t,x]$ and the solution $u^0(t,x)$ of the problem (1)-(3), corresponding to them, must be determined in such a way, that the trivial solution $u\equiv 0$ would be asymptotically stable for $\alpha=\alpha^0$ and $\beta=\beta^0$, and the functional would have the least possible value.

The functional $V(t,x)$, determined on the elements $u\in L_2(\Omega)$, in which t is a numerical parameter, will be called the Ljapunov's functional (4), if it is differentiable in t and if we can show such a value $C_1>0$, that $|v(t,u)|<C_2$ for all $t\geq t_0$ and $\|u\|_{L_2}<C_1$.

Then let's determine the concept of full derivative with respect to t, made according to the boundary-value problem (1)-(3) by the rule

$$\frac{dV}{dt}=\lim_{\Delta t\to 0}\frac{V[t+\Delta t,u(t+\Delta t,x)]-V[t,u(t,x)]}{\Delta t},$$

where $u(t,x)$ meets the identity (4) and condition (5).

Taking into consideration the properties of V we have:

$$V[t+\Delta t,u(t+\Delta t,x)]-V[t,u(t,x)]=\frac{\partial V[t,u]}{\partial t}\Delta t+\Phi(t,\Delta u)+o(\Delta t,\Delta u),$$

where Φ is uniquely determined by the formula (7). As u meets the identity (4), then, according to the identity (4) and (9), we obtain

$$\Phi(t,\Delta u)=\iint_{Q_t^2}\Big\{u v_t-\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+fv\Big\}dQ-\int_{S_t}(cu-\varphi)v\,dS+ +\int_{\Omega}(v)_t^{t+\Delta t}u(t+\Delta t,x)d\Omega.$$

Taking into account, that

$$\frac{1}{\Delta t}\Big(\int_{Q_t^2}u v_t'\,dQ-\int_{\Omega}(v)_t^{t+\Delta t}u(t+\Delta t,x)d\Omega\Big)\to 0,\ \text{when } \Delta t\to 0$$

and proceeding from the last three formulas we obtain the formula of the full derivative

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} - \int_{\Omega} \Big(\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} - f v \Big) d\Omega - \int_{S} (cu - \varphi)\, v\, dS. \qquad (11)$$

Then we determine the Bellmań functional

$$B(V, t, u, \alpha, \beta) = \frac{\partial V}{\partial t} - \int_{\Omega} \Big(\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} - f v \Big) d\Omega - \int_{S} (cu - \varphi)\, v\, dS + \mathcal{I},$$

where $\mathcal{Y}$ is the functional introduced into the determination of the criterion of optimality, and $V \in W_2^1(Q)$.

We shall mark $U(t,x)$ as the solution of the boundary-value problems (1)-(3), according to Krasovsky [1]. This solution corresponds to the control $\alpha[t,x]$, $\beta[t,x]$.

THEOREM. Let the positive definite Ljapunovś functional V° for the boundary-value problems (1)-(3) be found in such a manner that the function $V(t,x)$ uniquely determined by its Fresheś differential, refers to $W_2^1(Q_T), Q_T = \Omega \times [t_0, T]$, where T is a positive arbitrary number, which is greater, than t .

Let such functionals $\alpha^{\circ}[t,x,u] \in L_2(Q_T)$ and $\beta^{\circ}[t,x,u] \in L_2(S_T)$ be for any $U \in W_2^1(Q_T)$, that

1) $B(V^{\circ}, t, u, \alpha^{\circ}[t,x,u]), \beta^{\circ}[t,x,u]) = 0$ for $u \in W_2^1$ and almost for all $t \in [t_0, T]$

2) $B(V^{\circ}, t, u, \alpha(x), \beta(x)) \geqslant 0$ for any $\alpha(x) \in L_2(\Omega)$, $\beta(x) \in L_2(S)$,

then $\alpha^{\circ}[t,x,u]$ and $\beta^{\circ}[t,x,u]$ solve the problem of the optimum stability. In this case

$$\int_{t_0}^{\infty} \{ \mathcal{Y}_1(\alpha^{\circ}[t,x]) + \mathcal{Y}_2(\beta^{\circ}[t,x]) + \mathcal{Y}_3(u^{\circ}[t,x]) \} dt = \min_{\alpha,\beta} \int_{t_0}^{\infty} \{ \mathcal{Y}_1(\alpha[t,x]) + \mathcal{Y}_2(\beta[t,x]) + \mathcal{Y}_3(u[t,x]) \} dt = V^{\circ}(t_0, u^{\circ}[t_0,x]).$$

The proof of this theorem can easily be obtained by the method, accounted in [1], p.p. 486-487, based on the properties of the functional V° and the theorem of the asymptotic stability [4].

REFERENCE. The formula (11) of the full derivative of the functional V gives concrete recommendation for investigating the problems of the stability of distributed parameter systems with the help of the second Ljapunovś method. From that, in particular , sufficient conditions can be easily obtained for the stability of the first approximation systems.

References

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